Dualities for Intuitionistic Modal Logics

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Abstract

We present a duality for the intuitionistic modal logic IK introduced by Fischer Servi in [10, 11]. Unlike other dualities for IK, the dual structures of the duality presented here are ordered topological spaces endowed with just *one* extra relation, that is used to define the set-theoretic representation of both $\Box$ and $\Diamond$. We also give a parallel presentation of dualities for the intuitionistic modal logics $I_\Box$ and $I_\Diamond$. Finally, we turn to the intuitionistic modal logic MIPC, and give a very natural characterization of the dual spaces for MIPC introduced in [2] as a subcategory of the category of the dual spaces for IK introduced here.

1 Introduction and Preliminaries

Fischer Servi [10] proposed a general method for defining the intuitionistic analogue of a given classical modal system. Her method is based on a translation of formulas of the intuitionistic modal language $\mathcal{L}_{\Box\Diamond} = \{\wedge, \vee, \rightarrow, \neg, \Box, \Diamond\}$ into a classical modal language with two necessity operators, that extends the Gödel translation of the intuitionistic propositional calculus IPC into Lewis' classical modal logic S4. The intuitionistic modal logics K-IC (IK elsewhere) and S5-IC (I5 elsewhere) in the language $\mathcal{L}_{\Box\Diamond}$ are defined in the same paper according to this method as the intuitionistic counterparts of the classical modal logics K and S5, and are axiomatized in [11]. In particular I5, which coincides (see also [12]) with Prior's intuitionistic modal logic MIPC [16], is axiomatized by extending the set of axioms of IK with the following ones:

$$\begin{array}{lll} \Box p \rightarrow p & \Box p \rightarrow \Box\Box p & \Diamond p \rightarrow \Box\Diamond p \\ p \rightarrow \Diamond p & \Diamond\Diamond p \rightarrow \Diamond p & \Diamond\Box p \rightarrow \Box p, \end{array}$$

which shows that MIPC is to IK what S5 is to K, also from the point of view of axioms. A distinguished feature of IK is that the modal operators are not interdefinable but they are connected by the pair of "connecting axioms"(see (1) in subsection 1.1): In this respect, IK is similar to the negation- and implication-free modal logic PML [7], for which a duality was established in [3], relating PML-algebras with structures consisting of ordered topological spaces (indeed, Priestley spaces [6]) endowed with one extra relation, and called K^+-*spaces*. This paper presents a "duality for IK", i.e. a duality relating algebraic and topological semantics of IK. Like the duality in [3] and unlike other dualities for IK reported in the literature (see for example [17]), the dual structures of the duality presented here are ordered topological spaces (indeed, *Esakia spaces* [9], see also [5, 14]) endowed with just *one* extra relation, that is used to define the set-theoretic representation of both $\Box$ and $\Diamond$. Also, unlike the duality in [17], this duality naturally extends the definitions and techniques used by Fischer Servi in the proof of completeness for IK via canonical model construction [12]. The similarities between the duality presented here and the one in [3] confirm the intuition that PML and IK are akin, and indeed, the motivation of this work came from the project of extending to IK the results on PML presented in [14], where an equivalence of categories was established between K^+-spaces and the coalgebras of the Vietoris functor on Priestley spaces. This class of coalgebras is therefore as adequate a semantics for PML as the algebraic one (consisting of PML-algebras), and as the topological/relational one (consisting of K^+-spaces),

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and this gives a very concrete sense to the expression "PML is a coalgebraic logic", i.e. the logic of a category of coalgebras. The duality that we are going to present is a basic ingredient in the investigation on whether IK or MIPC can be declared coalgebraic logics as well.
In the remainder of this section, we are going to formally introduce the logic IK together with the other two intuitionistic modal logics $I_\Box$ and $I_\Diamond$, and the categories of algebras canonically associated with each of them. In section 2, we define the frames of these logics, and show that their associated complex algebras belong to the expected categories. In section 3 we define the general frames and p-morphisms for these logics. In sections 4 and 5 we establish back-and-forth functorial correspondences between general frames and algebras, and between p-morphisms and algebra homomorphisms, for each logic. In section 6 we establish the dualities between the algebras and suitable full subcategories of the general frames categories: For $\mathbf{L} \in \{I_\Box, I_\Diamond, \text{IK}\}$ the objects of these subcategories are called **L**-*spaces*, and play an analogous role to descriptive general frames for the classical modal logic K. In section 7 we characterize the topological semantics of MIPC within the category of IK-spaces.

1.1 The logics

From now on, we take $\mathcal{L} = \{\wedge, \vee, \rightarrow, \bot, \top\}$ as the intuitionistic propositional language, or equivalently, as the algebraic similarity type of Heyting algebras. For a non-empty set M of unary modal operators, let $\mathcal{L}_M$ be the intuitionistic propositional language augmented by the connectives in M. By an *intuitionistic modal logic* we understand any subset of $\mathcal{L}_M$ containing all the theorems of IPC and closed under modus ponens, substitution and the regularity rule $\varphi \rightarrow \psi / m\varphi \rightarrow m\psi$ for every $m \in M$.
The logic $I_\Box$, in the language $\mathcal{L}_\Box$, is axiomatized by adding the following axioms to IPC:

$$\Box(\phi \wedge \psi) = \Box\phi \wedge \Box\psi \quad \text{and} \quad \Box\top = \top.$$

The logic $I_\Diamond$, in the language $\mathcal{L}_\Diamond$, is axiomatized by adding the following axioms to IPC:

$$\Diamond(\phi \vee \psi) = \Diamond\phi \vee \Diamond\psi \quad \text{and} \quad \Diamond\bot = \bot.$$

The logic $I_{\Box\Diamond}$ is the smallest logic $\mathcal{S}$ in the language $\mathcal{L}_{\Box\Diamond}$ that contains $I_\Box$ and $I_\Diamond$. IK is the axiomatic extension of $I_{\Box\Diamond}$ obtained by adding the *connecting axioms*

$$\Diamond(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Diamond\psi) \quad \text{and} \quad (\Diamond\phi \rightarrow \Box\psi) \rightarrow \Box(\phi \rightarrow \psi). \tag{1}$$

1.2 Algebras for intuitionistic modal logics

An $\mathcal{L}_\Box$-algebra $\mathcal{A}$ is an $I_\Box$-*algebra* if its $\mathcal{L}$-reduct is a Heyting algebra and the following axioms are satisfied:

$$\Box(a \wedge b) = \Box a \wedge \Box b \quad \text{and} \quad \Box 1 = 1.$$

An $\mathcal{L}_\Diamond$-algebra $\mathcal{A}$ is an $I_\Diamond$-*algebra* if its $\mathcal{L}$-reduct is a Heyting algebra and the following axioms are satisfied:

$$\Diamond(a \vee b) = \Diamond a \vee \Diamond b \quad \text{and} \quad \Diamond 0 = 0.$$

An $\mathcal{L}_{\Box\Diamond}$-algebra $\mathcal{A}$ is an IK-*algebra* if its $\mathcal{L}$-reduct is a Heyting algebra and the following axioms are satisfied:

1. $\Box 1 = 1$	2. $\Diamond 0 = 0$
3. $\Box(a \wedge b) = \Box a \wedge \Box b$	4. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
5. $\Diamond(a \rightarrow b) \leq \Box a \rightarrow \Diamond b$	6. $\Diamond a \rightarrow \Box b \leq \Box(a \rightarrow b)$.

For $\mathbf{L} \in \{I_\Box, I_\Diamond, \text{IK}\}$, let $\mathbf{L}\mathsf{Alg}$ be the category of **L**-algebras and their homomorphisms. Clearly, $\mathbf{L}\mathsf{Alg}$ is closed under subalgebras.

2 Frames

For every relation $S \subseteq X \times X$ and every $Y \subseteq X$, let

$$S[Y] = \{x \in X \mid ySx \text{ for some } y \in Y\} \quad \text{and} \quad S^{-1}[Y] = \{x \in X \mid xSy \text{ for some } y \in Y\}.$$

We abbreviate $S[\{x\}]$ and $S^{-1}[\{x\}]$ with $S[x]$ and $S^{-1}[x]$ respectively. A *preorder* is a structure $\langle X, \leq\rangle$, such that $X \neq \emptyset$ and $\leq$ is a reflexive and transitive binary relation on X. For every preorder $\langle X, \leq\rangle$ and every $Y \subseteq X$, $Y{\uparrow} = {\leq}[Y]$ and $Y{\downarrow} = {\leq}^{-1}[Y]$. Y is an *up-set* (a *down-set*) if $Y = Y{\uparrow}$ ($Y = Y{\downarrow}$). We abbreviate $\{x\}{\uparrow}$ and $\{x\}{\downarrow}$ with $x{\uparrow}$ and $x{\downarrow}$ respectively. $\mathsf{Up}(X)$ ($\mathsf{Down}(X)$) is the collection of the up-sets (down-sets) of X.

A map $f : \langle X, \leq\rangle \longrightarrow \langle Y, \leq\rangle$ between preorders is *strongly isotone* if the following conditions are satisfied for every $x, y \in X$ and every $z \in Y$:

M1. If $x \leq y$ then $f(x) \leq f(y)$.

M2. If $f(x) \leq z$ then $f(x') = z$ for some $x' \in x{\uparrow}$.

An *intuitionistic frame* [4] is a poset, i.e. an antisymmetric preorder. For every $Y, Z \subseteq X$,

$$\begin{array}{rcl} \Diamond_S(Y) &=& \{x \in X \mid S[x] \cap Y \neq \emptyset\} \;=\; S^{-1}[Y] \\ \Box_S(Y) &=& \{x \in X \mid S[x] \subseteq Y\} \;=\; X \setminus S^{-1}[X \setminus Y] \\ Z \Rightarrow_S Y &=& \{x \in X \mid S[x] \cap Z \subseteq Y\} \;=\; \Box_S((X \setminus Z) \cup Y). \end{array}$$

We will always abbreviate $\Rightarrow_{\leq}$ with $\Rightarrow$. One can easily verify that for every poset $\langle X, \leq\rangle$ and every $A, B, C \in \mathsf{Up}(X)$, $A \Rightarrow B \in \mathsf{Up}(X)$ and $(A \cap C) \subseteq B$ iff $C \subseteq (A \Rightarrow B)$, which implies that $\langle \mathsf{Up}(X), \cap, \cup, \Rightarrow, \emptyset, X\rangle$ is a Heyting algebra.

Definition 2.1. (Frames) *Let $\mathcal{F} = \langle X, \leq, R\rangle$ be a relational structure such that $\leq$ is a preorder.*

1. *$\mathcal{F}$ is an* $\mathrm{I}_\Box$-frame *iff* $(\leq \circ R) \subseteq (R \circ \leq)$.
2. *$\mathcal{F}$ is an* $\mathrm{I}_\Diamond$-frame *iff* $(\geq \circ R) \subseteq (R \circ \geq)$.
3. *$\mathcal{F}$ is an* IK-frame *iff* $(\geq \circ R) \subseteq (R \circ \geq)$ *and* $(R \circ \leq) \subseteq (\leq \circ R)$.

If $\langle X, \leq\rangle$ is a poset, then $\langle X, \leq, \leq\rangle$ is an $\mathrm{I}_\Box$-frame, $\langle X, \leq, \geq\rangle$ is an $\mathrm{I}_\Diamond$-frame, and $\langle X, \leq, \geq \circ \leq\rangle$ is an IK-frame. The following lemma is easy to show by direct computation, and explains the meaning of the conditions in the definition of frames in algebraic terms:

Lemma 2.2. *For every preorder $\langle X, \leq\rangle$ and every binary relation S on X,*

1. $(\leq \circ S) \subseteq (S \circ \leq)$ *iff* $\mathsf{Up}(X)$ *is closed under* $\Box_S$.
2. $(\geq \circ S) \subseteq (S \circ \geq)$ *iff* $\mathsf{Up}(X)$ *is closed under* $\Diamond_S$.
3. $(S \circ \leq) \subseteq (\leq \circ S)$ *iff* $S[x{\uparrow}] \in \mathsf{Up}(X)$ *for every* $x \in X$.

Notice that for every every preorder $\langle X, \leq\rangle$ and every binary relation R on X, the following inclusion $(\leq \circ (\leq \circ R)) \subseteq ((\leq \circ R) \circ \leq)$ always holds, hence by applying 2.2 (1) we get:

Corollary 2.3. *For every preorder $\langle X, \leq\rangle$ and every binary relation R on X, $\mathsf{Up}(X)$ is closed under $\Box_{(\leq \circ R)}$.*

Proposition 2.4. *Let $\mathcal{F} = \langle X, \leq, R\rangle$ be a relational structure.*

1. *If $\mathcal{F}$ is an $\mathrm{I}_\Box$-frame, then $\mathcal{A}_\mathcal{F} = \langle \mathsf{Up}(X), \cap, \cup, \Rightarrow, \Box_R, \emptyset, X\rangle$ is an $\mathrm{I}_\Box$-algebra. Hence, every subalgebra $\mathcal{A}$ of $\mathcal{A}_\mathcal{F}$ is an $\mathrm{I}_\Box$-algebra.*
2. *If $\mathcal{F}$ is an $\mathrm{I}_\Diamond$-frame, then $\mathcal{A}_\mathcal{F} = \langle \mathsf{Up}(X), \cap, \cup, \Rightarrow, \Diamond_R, \emptyset, X\rangle$ is an $\mathrm{I}_\Diamond$-algebra. Hence, every subalgebra $\mathcal{A}$ of $\mathcal{A}_\mathcal{F}$ is an $\mathrm{I}_\Diamond$-algebra.*
3. *If $\mathcal{F}$ is an* IK*-frame, then $\mathcal{A}_\mathcal{F} = \langle \mathsf{Up}(X), \cap, \cup, \Rightarrow, \Box_{(\leq \circ R)}, \Diamond_R, \emptyset, X\rangle$ is an* IK*-algebra. Hence, every subalgebra $\mathcal{A}$ of $\mathcal{A}_\mathcal{F}$ is an* IK*-algebra.*

Proof. We only check axioms 5 and 6 for IK-algebras: Let us show that for every $U, V \in \mathsf{Up}(X)$

(a) $\Diamond_R(U \Rightarrow V) \subseteq (\Box_{(\leq \circ R)}U \Rightarrow \Diamond_R V)$ and (b) $(\Diamond_R U \Rightarrow \Box_{(\leq \circ R)}V) \subseteq \Box_{(\leq \circ R)}(U \Rightarrow V)$.

(a) Assume that $x \in \Diamond_R(U \Rightarrow V)$, let $x \leq z$ and $z \in \Box_{(\leq \circ R)}U$, and let us show that $z \in \Diamond_R V$, i.e. that $R[z] \cap V \neq \emptyset$. As $x \in \Diamond_R(U \Rightarrow V)$, then there exists $y \in R[x] \cap (U \Rightarrow V)$, hence $z \geq xRy$, and so, as $\mathcal{F}$ is an IK-frame, $zRv \geq y$ for some $v \in X$. As $v \in R[z] \subseteq (\leq \circ R)[z] \subseteq U$, $y \leq v$ and $y \in (U \Rightarrow V)$, then $v \in V$, and as $v \in R[z]$, then $R[z] \cap V \neq \emptyset$.
(b) Assume that $x \in (\Diamond_R U \Rightarrow \Box_{(\leq \circ R)}V)$, let $z \in (\leq \circ R)[x]$ and $z \leq y \in U$, and let us show that $y \in V$. As $z \in (\leq \circ R)[x]$, then $x \leq vRz \leq y$ for some $v \in X$, hence, as $\mathcal{F}$ is an IK-frame, $x \leq v \leq wRy$ for some $w \in X$. As $wRy \in U$, then $w \in \Diamond_R U$, and as $x \leq w$, then $w \in \Box_{(\leq \circ R)}V$, hence $y \in R[w] \subseteq (\leq \circ R)[w] \subseteq Y$. □

3 Topological semantics

Definition 3.1. (General frame) *A* general frame *is a structure $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ such that X is a nonempty set, $\leq$ is a partial order on X, R is a binary relation on X, and $\mathcal{A}$ is a subalgebra of $\langle \mathsf{Up}(X), \cap, \cup, \Rightarrow, \emptyset, X\rangle$. For every general frame $\mathcal{G}$, $\mathcal{F}_{\mathcal{G}} = \langle X, \leq, R\rangle$ is its associated frame, and its associated ordered topological space $\mathbf{X}_{\mathcal{G}} = \langle X, \leq, \tau_{\mathcal{A}}\rangle$ has the following subbase: $\{Y \mid Y \in \mathcal{A}\} \cup \{(X \setminus Y) \mid Y \in \mathcal{A}\}$.*

We recall that a Priestley space is a structure $\mathbf{X} = \langle X, \leq, \tau\rangle$ such that $\langle X, \leq\rangle$ is a partial order, $\langle X, \tau\rangle$ is a compact topological space which is *totally order-disconnected*, i.e. for every $x, y \in X$, if $x \not\leq y$ then $x \in U$ and $y \notin U$ for some clopen up-set U. An Esakia space $\mathbf{X} = \langle X, \leq, \tau\rangle$ is a Priestley space such that for every clopen subset U of $\mathbf{X}$, $U{\downarrow} = \{x \in X \mid x \leq u \text{ for some } u \in U\}$ is clopen. For every (preordered) topological space $\mathbf{X}$, $K(\mathbf{X})$ is the set of closed subsets of $\mathbf{X}$, and $K^{\uparrow}(\mathbf{X})$ is the set of closed up-sets of $\mathbf{X}$.
For sake of self-containment, we report the following basic fact on Priestley spaces, that will be needed onwards:

Lemma 3.2. *For every Priestley space $\mathbf{X} = \langle X, \leq, \tau\rangle$ and every $F \in K(\mathbf{X})$, $F{\uparrow}$ and $F{\downarrow}$ are closed subsets of $\mathbf{X}$. Hence, for every $x \in X$, $x{\uparrow}$ and $x{\downarrow}$ are closed subsets of $\mathbf{X}$.*

Proof. In order to show that $F{\uparrow} \in K(\mathbf{X})$, assume that $x \notin F{\uparrow}$, and show that $x \in A$ and $A \cap F{\uparrow} = \emptyset$ for some $A \in \tau$.
If $x \notin F{\uparrow}$, then for every $y \in F$, $x \notin y{\uparrow}$, i.e. $y \not\leq x$. Then by total order-disconnectedness, for every $y \in F$ there exists a clopen up-set U_y such that $y \in U_y$ and $x \notin U_y$. Therefore $F \subseteq \bigcup_{y \in F} U_y$, and as F is compact, for F is a closed subset of the compact space $\mathbf{X}$, then $F \subseteq \bigcup_{i=1}^{n} U_{y_i}$ for some $y_1, \ldots, y_n \in F$. Let $A = X \setminus \bigcup_{i=1}^{n} U_{y_i}$. A is an open down-set of X, $x \in A$ and $A \cap F = \emptyset$. Let us show that $A \cap F{\uparrow} = \emptyset$. Suppose that $z \in A \cap F{\uparrow}$ for some $z \in X$. Then $z \in A$ and $y_0 \leq z$ for some $y_0 \in F$, and as A is a down-set, then $y_0 \in A$, hence $y_0 \in A \cap F = \emptyset$, contradiction. The proof that $F{\downarrow} \in K(\mathbf{X})$ is similar. As for the second part of the statement, since $\mathbf{X}$ is Hausdorff, then $\{x\}$ is closed for every $x \in X$. □

Next, we are going to define, for $\mathbf{L} \in \{\mathrm{I}_{\Box}, \mathrm{I}_{\Diamond}, \mathrm{IK}\}$, the categories $\mathbf{L}\mathsf{GF}$ of general $\mathbf{L}$-frames and their p-morphisms.

3.3 General $\mathrm{I}_{\Box}$-frames and their morphisms

Definition 3.4. (General $\mathrm{I}_{\Box}$-frame) *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame. $\mathcal{G}$ is a* general $\mathrm{I}_{\Box}$-frame *iff the following list of conditions is satisfied:*

D1. $\mathbf{X}_{\mathcal{G}}$ is an Esakia space, and $\mathcal{A}$ is the collection of the clopen up-sets of $\mathbf{X}_{\mathcal{G}}$.
D2'. $\mathcal{A}$ is closed under $\Box_R$.
D3. For every $x \in X$, $R[x] \in K(\mathbf{X}_{\mathcal{G}})$.

Definition 3.5. (p-morphism of general $\mathrm{I}_\Box$-frames) *Let $\mathcal{G}_i = \langle X_i, \leq, R_i, \mathcal{A}_i\rangle$ be general* $\mathrm{I}_\Box$*-frames, $i = 1, 2$. A map $f : X_1 \to X_2$ is a* p-morphism *iff the following list of conditions is satisfied for every $x, x', y \in X_1$, $z \in X_2$:*

M1. if $x \leq y$ then $f(x) \leq f(y)$.
M2. If $f(x) \leq z$ then $f(x') = z$ for some $x' \in x\uparrow$.
M3. For every $Y \in \mathcal{A}_2$, $f^{-1}[Y] \in \mathcal{A}_1$.
M4. If xR_1y then $f(x)R_2f(y)$.
M5'. If $f(x)R_2z$ then $f(x') \leq z$ for some $x' \in R_1[x]$.

By M1 and M2, f is a strongly isotone map, and M3 says that $f : \mathbf{X}_{\mathcal{G}_1} \longrightarrow \mathbf{X}_{\mathcal{G}_2}$ is continuous.

3.6 General $\mathrm{I}_\Diamond$-frames and their morphisms

Definition 3.7. (General $\mathrm{I}_\Diamond$-frame) *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame. $\mathcal{G}$ is a* general $\mathrm{I}_\Diamond$-frame *iff the following list of conditions is satisfied:*

D1. $\mathbf{X}_\mathcal{G}$ is an Esakia space, and $\mathcal{A}$ is the collection of the clopen up-sets of $\mathbf{X}_\mathcal{G}$.
D2. $\mathcal{A}$ is closed under $\Diamond_R$.
D3. For every $x \in X$, $R[x] \in K(\mathbf{X}_\mathcal{G})$.

Definition 3.8. (p-morphism of general $\mathrm{I}_\Diamond$-frames) *Let $\mathcal{G}_i = \langle X_i, \leq, R_i, \mathcal{A}_i\rangle$ be general* $\mathrm{I}_\Diamond$*-frames, $i = 1, 2$. A map $f : X_1 \to X_2$ is a* p-morphism *iff the following list of conditions is satisfied for every $x, x', y \in X_1$, $z \in X_2$:*

M1. if $x \leq_1 y$ then $f(x) \leq_2 f(y)$.
M2. If $f(x) \leq_2 z$ then $f(x') = z$ for some $x' \in x\uparrow$.
M3. For every $Y \in \mathcal{A}_2$, $f^{-1}[Y] \in \mathcal{A}_1$.
M4. If xR_1y then $f(x)R_2f(y)$.
M5. If $f(x)R_2z$ then $z \leq_2 f(x')$ for some $x' \in R_1[x]$.

3.9 General IK-frames and their morphisms

Definition 3.10. (General IK-frame) *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame. $\mathcal{G}$ is a* general IK-frame *iff the following list of conditions is satisfied:*

D1. $\mathbf{X}_\mathcal{G}$ is an Esakia space, and $\mathcal{A}$ is the collection of the clopen up-sets of $\mathbf{X}_\mathcal{G}$.
D2. $\mathcal{A}$ is closed under $\Diamond_R$ and $\Box_{(\leq \circ R)}$.
D3. For every $x \in X$, $R[x] \in K(\mathbf{X}_\mathcal{G})$.
D4. For every $x \in X$, $R[x\uparrow] \in K^\uparrow(\mathbf{X}_\mathcal{G})$.

Example 3.11. *For every finite partial order $\langle X, \leq\rangle$, the general frame $\mathcal{G} = \langle X, \leq, (\geq \circ \leq), \mathsf{Up}(X)\rangle$ is a general* IK*-frame.*

Proof. By definition, the topology τ of $\mathbf{X}_\mathcal{G} = \mathbf{X}$ is generated by taking $\mathsf{Up}(X) \cup \mathsf{Down}(X)$ as a subbase. As X is finite, then $\mathbf{X} = \langle X, \leq, \tau\rangle$ is compact and τ is the discrete topology, i.e. every set is clopen. So it trivially follows that $\mathbf{X}$ is totally order-disconnected, hence it is a Priestley space, and that for every clopen subset U of $\mathbf{X}$, $U\downarrow$ is clopen, so $\mathbf{X}$ is an Esakia space. 2.2 (2) implies that $\mathsf{Up}(X)$ is closed under $\Diamond_{(\geq\circ\leq)}$, and by 2.3, $\mathsf{Up}(X)$ is closed under $\Box_{\leq\circ(\geq\circ\leq)}$. For every $x \in X$, $(\geq \circ \leq)[x] = x\downarrow\uparrow \in \mathsf{Up}(X)$ and $(\geq \circ \leq)[x\uparrow] = x\uparrow\downarrow\uparrow \in \mathsf{Up}(X)$, so they are clopen up-sets, therefore $(\geq \circ \leq)[x] \in K(\mathbf{X})$ and $(\geq \circ \leq)[x\uparrow] \in K^\uparrow(\mathbf{X})$. □

Definition 3.12. (p-morphism of general IK-frames) *Let $\mathcal{G}_i = \langle X_i, \leq, R_i, \mathcal{A}_i\rangle$ be general* IK*-frames, $i = 1, 2$. A map $f : X_1 \to X_2$ is a* p-morphism *iff the following list of conditions is satisfied for every $x, x', y \in X_1$, $z \in X_2$:*

M1. if $x \leq y$ then $f(x) \leq f(y)$.
M2. If $f(x) \leq z$ then $f(x') = z$ for some $x' \in x\uparrow$.
M3. For every $Y \in \mathcal{A}_2$, $f^{-1}[Y] \in \mathcal{A}_1$.
M4. If xR_1y then $f(x)R_2f(y)$.

M5. If $f(x)R_2 z$ then $z \leq f(x')$ for some $x' \in R_1[x]$.
M6. If $f(x)(\leq \circ R_2)z$ then $f(x') \leq z$ for some $x' \in R_1[x\uparrow]$.

4 From general frames to algebras

For every general frame $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$, let $\mathcal{G}^+ := \mathcal{A}$, and for every continuous $f : \mathbf{X}_{\mathcal{G}_1} \longrightarrow \mathbf{X}_{\mathcal{G}_2}$ let $f^+ : \mathcal{G}_2^+ \longrightarrow \mathcal{G}_1^+$ be given by the assignment $Y \longmapsto f^{-1}[Y]$ for every $Y \in \mathcal{A}_{\mathcal{G}_2}$.
In this section, we are showing that these assignments define functors $(\ )^+ : \mathbf{L}\mathsf{GF} \longrightarrow \mathbf{L}\mathsf{Alg}$ for $\mathbf{L} \in \{\mathrm{I}_\Box, \mathrm{I}_\Diamond, \mathrm{IK}\}$.

4.1 The action of $(\ )^+$ on objects

Let us recall that for every general frame $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$, $\mathcal{F}_{\mathcal{G}} = \langle X, \leq, R\rangle$ is the associated frame.

Lemma 4.2. *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame.*

1. *If $\mathcal{G}$ is a general $\mathrm{I}_\Box$-frame, then $\mathcal{F}_{\mathcal{G}}$ is an $\mathrm{I}_\Box$-frame.*
2. *If $\mathcal{G}$ is a general $\mathrm{I}_\Diamond$-frame, then $\mathcal{F}_{\mathcal{G}}$ is an $\mathrm{I}_\Diamond$-frame.*
3. *If $\mathcal{G}$ is a general IK-frame, then $\mathcal{F}_{\mathcal{G}}$ is an IK-frame.*

Proof. 1. Let us show that for every $x \in X$, $(\leq \circ R)[x] \subseteq (R \circ \leq)[x]$: Suppose that $z \in (\leq \circ R)[x]$ and $z \notin (R \circ \leq)[x] = R[x]\uparrow$ for some $z \in X$. As $z \notin R[x]\uparrow$, then $y \not\leq z$ for every $y \in R[x]$, hence, by D1, for every $y \in R[x]$ there exists a clopen up-set U_y of $\mathbf{X}_{\mathcal{G}}$ such that $y \in U_y$ and $z \notin U_y$, and so $R[x] \subseteq \bigcup_{y\in R[x]} U_y$, and as $\mathbf{X}_{\mathcal{G}}$ is compact and $R[x]$ is closed by D3, then $R[x] \subseteq \bigcup_{i=1}^n U_{y_i} = U$ for some $y_1, \ldots, y_n \in R[x]$. As U is a clopen up-set, then $U \in \mathcal{A}$, moreover, $z \notin U$ and $R[x] \subseteq U$.
As $z \in (\leq \circ R)[x]$, then $x \leq wRz$ for some $w \in X$. Since $z \in (R[w] \setminus U)$, then $w \notin \Box_R U \in \mathcal{A}$ by D2', so in particular $\Box_R U$ is an up-set, and as $x \leq w$, then $x \notin \Box_R U$, i.e. $R[x] \not\subseteq U$, contradiction.
2. Let us show that for every $x \in X$, $(\geq \circ R)[x] \subseteq (R \circ \geq)[x]$: Suppose that $z \in (\geq \circ R)[x]$ and $z \notin (R \circ \geq)[x] = R[x]\downarrow$ for some $z \in X$. As $z \notin R[x]\downarrow$, then $z \not\leq y$ for every $y \in R[x]$, hence, by D1, for every $y \in R[x]$ there exists a clopen down-set V_y of $\mathbf{X}_{\mathcal{G}}$ such that $y \in V_y$ and $z \notin V_y$, and so $R[x] \subseteq \bigcup_{y\in R[x]} V_y$, and as $\mathbf{X}_{\mathcal{G}}$ is compact and $R[x]$ is closed by D3, then $R[x] \subseteq \bigcup_{i=1}^n V_{y_i} = V$ for some $y_1, \ldots, y_n \in R[x]$. Let $U = (X \setminus V)$. As U is a clopen up-set, then $U \in \mathcal{A}$, moreover, $z \in U$ and $R[x] \cap U = \emptyset$.
As $z \in (\geq \circ R)[x]$, then $x \geq wRz$ for some $w \in X$. Since $z \in R[w] \cap U$, then $w \in \Diamond_R U \in \mathcal{A}$ by D2, so in particular $\Diamond_R U$ is an up-set, and as $w \leq x$, then $x \in \Diamond_R U$, i.e. $R[x] \cap U \neq \emptyset$, contradiction.
3. Let us show that for every $x \in X$, $(R \circ \leq)[x] \subseteq (\leq \circ R)[x]$: Suppose that $z \in (R \circ \leq)[x]$ and $z \notin (\leq \circ R)[x] = R[x\uparrow]$ for some $z \in X$. As $z \notin R[x\uparrow]$ which is a closed up-set of $\mathbf{X}_{\mathcal{G}}$ by D4, then $y \not\leq z$ for every $y \in R[x\uparrow]$, hence, by D1, for every $y \in R[x\uparrow]$ there exists a clopen up-set U_y of $\mathbf{X}_{\mathcal{G}}$ such that $y \in U_y$ and $z \notin U_y$, and so $R[x\uparrow] \subseteq \bigcup_{y\in R[x]} U_y$, and as $\mathbf{X}_{\mathcal{G}}$ is compact and $R[x]$ is closed by D3, then $R[x] \subseteq \bigcup_{i=1}^n U_{y_i} = U$ for some $y_1, \ldots, y_n \in R[x]$. As U is a clopen up-set, then $U \in \mathcal{A}$, moreover, $z \notin U$ and $R[x\uparrow] \subseteq U$.
As $z \in (R \circ \leq)[x]$, then $xRw \leq z$ for some $w \in X$. Since $w \in R[x] \subseteq R[x\uparrow] \subseteq U$, then $w \in U$ which is an up-set, and as $w \leq z$, then $z \in U$, contradiction. The proof of the other condition is the same as the proof of (2). □

Proposition 4.3. *Let $\mathbf{L} \in \{\mathrm{I}_\Box, \mathrm{I}_\Diamond, \mathrm{IK}\}$. For every general $\mathbf{L}$-frame $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$, $\mathcal{A}$ is an $\mathbf{L}$-algebra.*

Proof. It immediately follows from 2.4 and 4.2. □

4.4 The action of $(\ )^+$ on arrows

Proposition 4.5. *Let* $\mathbf{L} \in \{\mathrm{I}_\Box, \mathrm{I}_\Diamond, \mathrm{IK}\}$. *For every p-morphism* $h : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ *of general* $\mathbf{L}$*-frames,* $h^+ : {\mathcal{G}_2}^+ \longrightarrow {\mathcal{G}_1}^+$ *is a homomorphism of* $\mathbf{L}$*-algebras.*

Proof. If $h : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ is a p-morphism of general $\mathbf{L}$-frames, then in particular it is a continuous and strongly isotone map between the Esakia spaces $\mathbf{X}_{\mathcal{G}_1}$ and $\mathbf{X}_{\mathcal{G}_2}$, hence from the duality for Heyting algebras, h^+ is a homomorphism between the Heyting algebra reducts of ${\mathcal{G}_2}^+$ and ${\mathcal{G}_1}^+$. Let us show that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are general $\mathrm{I}_\Box$-frames, then for every $Y \in \mathcal{A}_2$,

$$h^{-1}[\Box_{R_2} Y] = \Box_{R_1} h^{-1}[Y].$$

For every $x \in X_1$, $x \in h^{-1}[\Box_{R_2} Y]$ iff $R_2[h(x)] \subseteq Y$, and $x \in \Box_{R_1} h^{-1}[Y]$ iff $R_1[x] \subseteq h^{-1}[Y]$.
($\subseteq$) Assume that $z \in R_1[x]$ and show that $z \in h^{-1}[Y]$: As xR_1z, then, by M4, $h(x)R_2h(z)$, i.e. $h(z) \in R_2[h(x)] \subseteq Y$, hence $z \in h^{-1}[Y]$.
($\supseteq$) Assume that $z \in R_2[h(x)]$ and show that $z \in Y$: If $h(x)R_2z$, then, by M5', there exists $y \in R_1[x] \subseteq h^{-1}[Y]$ such that $h(y) \leq z$. As $h(y) \in Y$ and Y is an up-set, then $z \in Y$.
Let us show that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are general $\mathrm{I}_\Diamond$-frames, then for every $Y \in \mathcal{A}_2$,

$$h^{-1}[\Diamond_{R_2} Y] = \Diamond_{R_1} h^{-1}[Y].$$

For every $x \in X_1$, $x \in h^{-1}[\Diamond_{R_2} Y]$ iff $R_2[h(x)] \cap Y \neq \emptyset$, and $x \in \Diamond_{R_1} h^{-1}[Y]$ iff $R_1[x] \cap h^{-1}[Y] \neq \emptyset$.
($\subseteq$) Assume that $z \in R_2[h(x)] \cap Y$. As $h(x)R_2z$, then, by M5, there exists $y \in R_1[x]$ such that $z \leq_2 h(y)$. As $z \in Y$ and Y is an up-set, then $h(y) \in Y$. Hence $y \in R_1[x] \cap h^{-1}[Y] \neq \emptyset$.
($\supseteq$) Assume that $z \in R_1[x] \cap h^{-1}[Y]$, hence $h(z) \in Y$ and xR_1z, so, by M4, $h(x)R_2h(z)$, i.e. $h(z) \in R_2[h(x)]$, and so $h(z) \in R_2[h(x)] \cap Y \neq \emptyset$.
Let us show that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are general IK-frames, then for every $Y \in \mathcal{A}_2$,

$$h^{-1}[\Box_{(\leq \circ R_2)} Y] = \Box_{(\leq \circ R_1)} h^{-1}[Y].$$

For every $x \in X_1$, $x \in h^{-1}[\Box_{(\leq \circ R_2)} Y]$ iff $(\leq \circ R_2)[h(x)] \subseteq Y$, and $x \in \Box_{(\leq \circ R_1)} h^{-1}[Y]$ iff $(\leq \circ R_1)[x] \subseteq h^{-1}[Y]$.
($\subseteq$) Assume that $z \in (\leq \circ R_1)[x]$ and show that $z \in h^{-1}[Y]$: As $x \leq wR_1z$ for some $w \in X_1$, then, by M1 and M4, $h(x) \leq h(w)R_2h(z)$, i.e. $h(z) \in (\leq \circ R_2)[h(x)] \subseteq Y$, hence $z \in h^{-1}[Y]$.
($\supseteq$) Assume that $z \in (\leq \circ R_2)[h(x)]$ and show that $z \in Y$: If $h(x)(\leq \circ R_2)z$, then, by M5, there exists $y \in (\leq \circ R_1)[x] \subseteq h^{-1}[Y]$ such that $h(y) \leq z$. As $h(y) \in Y$ and Y is an up-set, then $z \in Y$.
The proof that $h^{-1}[\Diamond_{R_2} Y] = \Diamond_{R_1} h^{-1}[Y]$ goes as in the $\mathrm{I}_\Diamond$-case. □

5 From algebras to general frames

Let $\mathbf{L} \in \{\mathrm{I}_\Box, \mathrm{I}_\Diamond, \mathrm{IK}\}$. For every $\mathbf{L}$-algebra $\mathcal{A}$ let $Pr(\mathcal{A})$ be the collection of the prime filters of the $\mathcal{L}$-reduct of $\mathcal{A}$. Let us define $\mathcal{A}_+ := \langle Pr(\mathcal{A}), \subseteq, \mathcal{R}_\mathcal{A}, \boldsymbol{A}\rangle$, where for every $P, Q \in Pr(\mathcal{A})$:
R1. If $\mathcal{A}$ is an $\mathrm{I}_\Box$-algebra, $P\mathcal{R}_\mathcal{A}Q$ iff $\Box^{-1}[P] \subseteq Q$.
R2. If $\mathcal{A}$ is an $\mathrm{I}_\Diamond$-algebra, $P\mathcal{R}_\mathcal{A}Q$ iff $Q \subseteq \Diamond^{-1}[P]$.
R3. If $\mathcal{A}$ is an IK-algebra, $P\mathcal{R}_\mathcal{A}Q$ iff $\Box^{-1}[P] \subseteq Q \subseteq \Diamond^{-1}[P]$.
As for the definition of $\boldsymbol{A}$, for every $a \in A$, let $\overline{a} = \{P \in Pr(\mathcal{A}) \mid a \in P\}$. Then the carrier of $\boldsymbol{A}$ is $\mathbf{A} = \{\overline{a} \mid a \in \mathcal{A}\}$. For every n-ary operation $*$ in the signature of $\mathcal{A}$, $*^{\boldsymbol{A}}(\overline{a_1}, \ldots, \overline{a_n}) = \overline{*(a_1, \ldots, a_n)}$. The operations $*^{\boldsymbol{A}}$ are well-defined, for if $\overline{a_i} = \overline{c_i}$, then for every $P \in Pr(\mathcal{A})$, $a_i \in P$ iff $c_i \in P$, and by Birkhoff-Stone theorem, this implies that $a_i = c_i$. From the Priestley duality restricted to Heyting algebras it follows that the $\mathcal{L}$-reduct of $\boldsymbol{A}$ is a subalgebra of the Heyting algebra $\langle \mathsf{Up}(Pr(\mathcal{A})), \cap, \cup, \Rightarrow, \emptyset, Pr(\mathcal{A})\rangle$.
For every homomorphism $f : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ let $f_+ : \mathcal{A}_{2+} \longrightarrow \mathcal{A}_{1+}$ be given by the assignment $P \longmapsto f^{-1}[P]$ for every $P \in Pr(\mathcal{A}_2)$.
In this section, we are showing that these assignments define functors $(\ )_+ : \mathbf{L}\mathsf{Alg} \longrightarrow \mathbf{L}\mathsf{GF}$ for $\mathbf{L} \in \{\mathrm{I}_\Box, \mathrm{I}_\Diamond, \mathrm{IK}\}$.

5.1 Properties of $\mathcal{R}_{\mathcal{A}}$

Lemma 5.2. *For every* $\mathbf{L}$*-algebra* $\mathcal{A}$*,* $\mathcal{R}_{\mathcal{A}}$ *is a closed subset of* $\mathbf{X}_{\mathcal{A}_+} \times \mathbf{X}_{\mathcal{A}_+}$*.*

Proof. Assume that $\mathcal{R}_{\mathcal{A}}$ is defined like in R1. If $\langle P, Q\rangle \notin \mathcal{R}_{\mathcal{A}}$, then $\Box^{-1}[P] \not\subseteq Q$, i.e. $\Box a \in P$ and $a \notin Q$ for some $a \in \mathcal{A}$. Hence $P \in \overline{(\Box a)}$ and $Q \notin \overline{a}$. Let us consider $\mathcal{U} = \overline{(\Box a)} \times (Pr(\mathcal{A}) \setminus \overline{a})$. $\mathcal{U}$ is an open subset of $\mathbf{X}_{\mathcal{A}_+} \times \mathbf{X}_{\mathcal{A}_+}$, for both $\overline{(\Box a)}$ and $Pr(\mathcal{A}) \setminus \overline{a}$ are open subsets of $\mathbf{X}_{\mathcal{A}_+}$, moreover $\langle P, Q\rangle \in \mathcal{U}$. Let us show that $\mathcal{R}_{\mathcal{A}} \cap \mathcal{U} = \emptyset$: If $\langle S, T\rangle \in \mathcal{U}$, then $\Box a \in S$ and $a \notin T$, hence $\Box^{-1}[S] \not\subseteq T$, i.e. $\langle S, T\rangle \notin \mathcal{R}_{\mathcal{A}}$.
Assume that $\mathcal{R}_{\mathcal{A}}$ is defined like in R2. If $\langle P, Q\rangle \notin \mathcal{R}_{\mathcal{A}}$, then $Q \not\subseteq \Diamond^{-1}[P]$, i.e. $a \in Q$ and $\Diamond a \notin P$ for some $a \in \mathcal{A}$. Hence $Q \in \overline{a}$ and $P \notin \overline{(\Diamond a)}$. Let us consider $\mathcal{U} = (Pr(\mathcal{A}) \setminus \overline{(\Diamond a)}) \times \overline{a}$. $\mathcal{U}$ is an open subset of $\mathbf{X}_{\mathcal{A}_+} \times \mathbf{X}_{\mathcal{A}_+}$, for both $Pr(\mathcal{A}) \setminus \overline{(\Diamond a)}$ and $\overline{a}$ are open subsets of $\mathbf{X}_{\mathcal{A}_+}$, moreover $\langle P, Q\rangle \in \mathcal{U}$. Let us show that $\mathcal{R}_{\mathcal{A}} \cap \mathcal{U} = \emptyset$: If $\langle S, T\rangle \in \mathcal{U}$, then $\Diamond a \notin S$ and $a \in T$, hence $T \not\subseteq \Diamond^{-1}[S]$, i.e. $\langle S, T\rangle \notin \mathcal{R}_{\mathcal{A}}$.
Assume that $\mathcal{R}_{\mathcal{A}}$ is defined like in R3. Then $\mathcal{R}_{\mathcal{A}}$ is the intersection of two sets that, by the previous cases, are closed, and so $\mathcal{R}_{\mathcal{A}}$ is closed. $\Box$

Corollary 5.3. *For every* $\mathbf{L}$*-algebra* $\mathcal{A}$*, if* $\mathcal{X}$ *is a closed subset of* $\mathbf{X}_{\mathcal{A}_+}$*, then* $\mathcal{R}_{\mathcal{A}}[\mathcal{X}]$ *is a closed subset of* $\mathbf{X}_{\mathcal{A}_+}$*.*

Proof. For $i = 1, 2$ let $\pi_i : \mathbf{X}_{\mathcal{A}_+} \times \mathbf{X}_{\mathcal{A}_+} \longrightarrow \mathbf{X}_{\mathcal{A}_+}$ be the canonical projections. For every closed subset $\mathcal{X}$ of $\mathbf{X}_{\mathcal{A}_+}$,

$$\begin{aligned} \mathcal{R}_{\mathcal{A}}[\mathcal{X}] &= \{Q \in Pr(\mathcal{A}) \mid P\mathcal{R}_{\mathcal{A}}Q \text{ for some } P \in \mathcal{X}\} \\ &= \pi_2[\mathcal{R}_{\mathcal{A}} \cap (\mathcal{X} \times Pr(\mathcal{A}))]. \end{aligned}$$

By 5.2 $\mathcal{R}_{\mathcal{A}}$ is closed, hence so is $\mathcal{R}_{\mathcal{A}} \cap (\mathcal{X} \times Pr(\mathcal{A}))$, and as π_2 is a closed map, for it is a continuous map between compact spaces, then $\pi_2[\mathcal{R}_{\mathcal{A}} \cap (\mathcal{X} \times Pr(\mathcal{A}))]$ is closed. $\Box$

Lemma 5.4.

1. *For every* $\mathrm{I}_{\Box}$*-algebra* $\mathcal{A}$*,* $\mathcal{R}_{\mathcal{A}} = (\subseteq \circ \mathcal{R}_{\mathcal{A}} \circ \subseteq)$*.*
2. *For every* $\mathrm{I}_{\Diamond}$*-algebra* $\mathcal{A}$*,* $\mathcal{R}_{\mathcal{A}} = (\supseteq \circ \mathcal{R}_{\mathcal{A}} \circ \supseteq)$*.*
3. *For every* IK*-algebra* $\mathcal{A}$*,* $\mathcal{R}_{\mathcal{A}} = (\subseteq \circ \mathcal{R}_{\mathcal{A}}) \cap (\mathcal{R}_{\mathcal{A}} \circ \supseteq)$*.*

Proof. We prove only the inclusions from right to left, being the converse inclusions immediate.
1. If $\langle P, Q\rangle \in (\subseteq \circ \mathcal{R}_{\mathcal{A}} \circ \subseteq)$, then $P \subseteq S_1\mathcal{R}_{\mathcal{A}}S_2 \subseteq Q$, for some $S_1, S_2 \in Pr(\mathcal{A})$, hence $\Box^{-1}[P] \subseteq \Box^{-1}[S_1] \subseteq S_2 \subseteq Q$, i.e. $\langle P, Q\rangle \in \mathcal{R}_{\mathcal{A}}$.
2. If $\langle P, Q\rangle \in (\supseteq \circ \mathcal{R}_{\mathcal{A}} \circ \supseteq)$, then $P \supseteq S_1\mathcal{R}_{\mathcal{A}}S_2 \supseteq Q$, for some $S_1, S_2 \in Pr(\mathcal{A})$, hence $Q \subseteq S_2 \subseteq \Diamond^{-1}[S_1] \subseteq \Diamond^{-1}[P]$, i.e. $\langle P, Q\rangle \in \mathcal{R}_{\mathcal{A}}$.
3. If $\langle P, Q\rangle \in (\subseteq \circ \mathcal{R}_{\mathcal{A}}) \cap (\mathcal{R}_{\mathcal{A}} \circ \supseteq)$, then $P \subseteq S_1\mathcal{R}_{\mathcal{A}}Q$ and $P\mathcal{R}_{\mathcal{A}}S_2 \supseteq Q$ for some $S_1, S_2 \in Pr(\mathcal{A})$, then $\Box^{-1}[P] \subseteq \Box^{-1}[S_1] \subseteq Q$ and $Q \subseteq S_2 \subseteq \Diamond^{-1}[P]$, hence $P\mathcal{R}_{\mathcal{A}}Q$. $\Box$

For every $\mathcal{L}_M$-algebra $\mathcal{A}$ and every subset B of $\mathcal{A}$, let $Fi(B)$ ($Id(B)$) be the filter (ideal) of the lattice reduct of $\mathcal{A}$ that is generated by B. B^c is the complement of B in $\mathcal{A}$.

Lemma 5.5. *For every* IK*-algebra* $\mathcal{A}$ *and every* $P, Q \in Pr(\mathcal{A})$*,*

1. $\langle P, Q\rangle \in (\mathcal{R}_{\mathcal{A}} \circ \supseteq)$ *iff* $Q \subseteq \Diamond^{-1}[P]$*.*
2. $\langle P, Q\rangle \in (\subseteq \circ \mathcal{R}_{\mathcal{A}})$ *iff* $\Box^{-1}[P] \subseteq Q$*.*

Proof. 1. ‘if’: Assume that $Q \subseteq \Diamond^{-1}[P]$, and let us show that there exists $S \in Pr(\mathcal{A})$ such that $P\mathcal{R}_{\mathcal{A}}S \supseteq Q$, i.e. such that $Q \cup \Box^{-1}[P] \subseteq S$ and $S \cap \Diamond^{-1}[P]^c = \emptyset$. Let us consider $Fi(Q \cup \Box^{-1}[P])$: If we show that

$$Fi(Q \cup \Box^{-1}[P]) \cap \Diamond^{-1}[P]^c = \emptyset,$$

then the statement will follow by Birkhoff-Stone theorem. Suppose that $Fi(Q \cup \Box^{-1}[P]) \cap \Diamond^{-1}[P]^c \neq \emptyset$. Then there exists $c \in A$ such that $\Diamond c \notin P$ and $a \wedge b \leq c$ for some $a \in \Box^{-1}[P]$ and $b \in Q$. Then $b \leq a \to c$, hence $\Diamond b \leq \Diamond(a \to c) \leq (\Box a \to \Diamond c)$. As $b \in Q \subseteq \Diamond^{-1}[P]$, then $\Diamond b \in P$, hence $\Box a \to \Diamond c \in P$, and as $\Box a \in P$, then $\Diamond c \in P$, contradiction.
'only if': If $P\mathcal{R}_{\mathcal{A}}S \supseteq Q$ for some $S \in Pr(\mathcal{A})$, then $Q \subseteq S \subseteq \Diamond^{-1}[P]$.
2. 'if': Assume that $\Box^{-1}[P] \subseteq Q$, and let us show that there exists $S \in Pr(\mathcal{A})$ such that $P \subseteq S$ and $\Box^{-1}[P] \subseteq Q \subseteq \Diamond^{-1}[P]$, i.e. such that $P \cup \Diamond[P] \subseteq S$ and $S \cap \Box[Q^c] = \emptyset$. Let us consider $Fi(P \cup \Diamond[Q])$: If we show that

$$Fi(P \cup \Diamond[Q]) \cap \Box[Q^c] = \emptyset,$$

then the statement will follow by Birkhoff-Stone theorem. Suppose that $Fi(P \cup \Diamond[Q]) \cap \Box[Q^c] \neq \emptyset$. Then there exist $a \in Q^c$, $b \in P$ and $c \in Q$ such that $b \wedge \Diamond c \leq \Box a$. Then $b \leq \Diamond c \to \Box a \leq \Box(c \to a)$. As $b \in P$, then $\Box(c \to a) \in P$, hence $c \to a \in \Box^{-1}[P] \subseteq Q$, and as $c \in Q$, then $a \in Q$, contradiction.
'only if': If $P \subseteq S\mathcal{R}_{\mathcal{A}}Q$ for some $S \in Pr(\mathcal{A})$, then $\Box^{-1}[P] \subseteq \Box^{-1}[S] \subseteq Q$. □

Corollary 5.6.

1. *For every* $\mathrm{I}_\Box$*-algebra* $\mathcal{A}$ *and every* $P \in Pr(\mathcal{A})$*, if* $\Box a \notin P$*, then* $a \notin Q$ *and* $P\mathcal{R}_{\mathcal{A}}Q$ *for some* $Q \in Pr(\mathcal{A})$.
2. *For every* $\mathrm{I}_\Diamond$*-algebra* $\mathcal{A}$ *and every* $P \in Pr(\mathcal{A})$*, if* $\Diamond a \in P$*, then* $a \in Q$ *and* $P\mathcal{R}_{\mathcal{A}}Q$ *for some* $Q \in Pr(\mathcal{A})$.
3. *For every* IK*-algebra* $\mathcal{A}$*, if* $\Box a \notin P$*, then* $a \notin Q$ *and* $P \subseteq S\mathcal{R}_{\mathcal{A}}Q$ *for some* $Q, S \in Pr(\mathcal{A})$.
4. *For every* IK*-algebra* $\mathcal{A}$ *and every* $P \in Pr(\mathcal{A})$*, if* $\Diamond a \in P$*, then* $a \in S$ *and* $P\mathcal{R}_{\mathcal{A}}S$ *for some* $S \in Pr(\mathcal{A})$.

Proof. 1. If $\Box a \notin P$, then $Id(a) \cap \Box^{-1}[P] = \emptyset$, for if not, then $c \leq a$ for some c such that $\Box c \in P$, hence $\Box c \leq \Box a$, and so $\Box a \in P$, contradiction. By Birkhoff-Stone theorem, there exists $Q \in Pr(\mathcal{A})$ such that $\Box^{-1}[P] \subseteq Q$, i.e. $P\mathcal{R}_{\mathcal{A}}Q$, and $a \notin Q$.
2. If $\Diamond a \in P$, then $Fi(a) \cap \Diamond^{-1}[P^c] = \emptyset$, for if not, then $a \leq c$ for some c such that $\Diamond c \notin P$, hence $\Diamond a \leq \Diamond c$, and so $\Diamond c \in P$, contradiction. By Birkhoff-Stone theorem, there exists $Q \in Pr(\mathcal{A})$ such that $a \in Q$ and $Q \subseteq \Diamond^{-1}[P]$, i.e. $P\mathcal{R}_{\mathcal{A}}Q$.
3. If $\Box a \notin P$, then $Id(a) \cap \Box^{-1}[P] = \emptyset$, so by Birkhoff-Stone theorem, there exists $Q \in Pr(\mathcal{A})$ such that $\Box^{-1}[P] \subseteq Q$ and $a \notin Q$. By 5.5 (2), $P \subseteq S\mathcal{R}_{\mathcal{A}}Q$ for some $S \in Pr(\mathcal{A})$.
4. If $\Diamond a \in P$, then $Fi(a) \cap \Diamond^{-1}[P^c] = \emptyset$, so by Birkhoff-Stone theorem, there exists $Q \in Pr(\mathcal{A})$ such that $a \in Q$ and $Q \subseteq \Diamond^{-1}[P]$. By 5.5 (1), $P\mathcal{R}_{\mathcal{A}}S \supseteq Q$ for some $S \in Pr(\mathcal{A})$, and as $a \in S \subseteq Q$, then $a \in S$. □

Corollary 5.7.

1. *For every* $\mathrm{I}_\Box$*-algebra* $\mathcal{A}$*,* $(\subseteq \circ \mathcal{R}_{\mathcal{A}}) \subseteq (\mathcal{R}_{\mathcal{A}} \circ \subseteq)$.
2. *For every* $\mathrm{I}_\Diamond$*-algebra* $\mathcal{A}$*,* $(\supseteq \circ \mathcal{R}_{\mathcal{A}}) \subseteq (\mathcal{R}_{\mathcal{A}} \circ \supseteq)$.
3. *For every* IK*-algebra* $\mathcal{A}$*,* $(\supseteq \circ \mathcal{R}_{\mathcal{A}}) \subseteq (\mathcal{R}_{\mathcal{A}} \circ \supseteq)$ *and* $(\mathcal{R}_{\mathcal{A}} \circ \subseteq) \subseteq (\subseteq \circ \mathcal{R}_{\mathcal{A}})$.

Proof. 1. If $P \subseteq S\mathcal{R}_{\mathcal{A}}Q$, then $\Box^{-1}[P] \subseteq \Box^{-1}[S] \subseteq Q$, hence $P\mathcal{R}_{\mathcal{A}}Q \subseteq Q$.
2. If $P \supseteq S\mathcal{R}_{\mathcal{A}}Q$, then $Q \subseteq \Diamond^{-1}[S] \subseteq \Diamond^{-1}[P]$, hence $P\mathcal{R}_{\mathcal{A}}Q \supseteq Q$.
3. If $P\mathcal{R}_{\mathcal{A}}S \subseteq Q$, then $\Box^{-1}[P] \subseteq S \subseteq Q$, hence by 5.5 (2), $\langle P, Q\rangle \in (\subseteq \circ \mathcal{R}_{\mathcal{A}})$.
If $P \supseteq S\mathcal{R}_{\mathcal{A}}Q$, then $Q \subseteq \Diamond^{-1}[S] \subseteq \Diamond^{-1}[P]$, hence by 5.5 (1), $\langle P, Q\rangle \in (\mathcal{R}_{\mathcal{A}} \circ \supseteq)$. □

5.8 The action of $(\ )_+$ on objects

Proposition 5.9. *Let* $\mathbf{L} \in \{\mathrm{I}_\Box, \mathrm{I}_\Diamond, \mathrm{IK}\}$. *For every* $\mathbf{L}$*-algebra* $\mathcal{A}$, $\mathcal{A}_+ = \langle Pr(\mathcal{A}), \subseteq, \mathcal{R}_\mathcal{A}, \boldsymbol{A}\rangle$ *is a general* $\mathbf{L}$*-frame.*

Proof. The Priestley duality restricted to Heyting algebras yields that $\mathbf{X}_{\mathcal{A}_+}$ is an Esakia space, and $\mathcal{A}$ is the collection of the clopen up-sets of $\mathbf{X}_{\mathcal{A}_+}$, which is D1. As $\mathbf{X}_{\mathcal{A}_+}$ is an Esakia space, then in particular it is Hausdorff, hence for every $P \in Pr(\mathcal{A})$, $\{P\}$ is closed in $\mathbf{X}_{\mathcal{A}_+}$, and so by 5.3, $\mathcal{R}_\mathcal{A}[P]$ is closed in $\mathbf{X}_{\mathcal{A}_+}$, which is D3.

Let us show that if $\mathcal{A}$ is an $\mathrm{I}_\Box$-algebra, then $\Box^{\boldsymbol{A}} = \Box_{\mathcal{R}_\mathcal{A}}$, i.e. that for every $a \in \mathcal{A}$,

$$\overline{(\Box a)} = \Box_{\mathcal{R}_\mathcal{A}} \overline{a}.$$

($\subseteq$) If $P \in \overline{(\Box a)}$, then $\Box a \in P$, i.e. $a \in \Box^{-1}[P]$ so, for every $Q \in Pr(\mathcal{A})$, if $P\mathcal{R}_\mathcal{A}Q$, then $a \in \Box^{-1}[P] \subseteq Q$.

($\supseteq$) If $P \notin \overline{(\Box a)}$, then by item 1 of 5.6, $a \notin Q$ and $P\mathcal{R}_\mathcal{A}Q$ for some $Q \in Pr(\mathcal{A})$, so $P \notin \Box_{\mathcal{R}_\mathcal{A}}\overline{a}$.

Let us show that if $\mathcal{A}$ is an $\mathrm{I}_\Diamond$-algebra (an IK-algebra), then $\Diamond^{\boldsymbol{A}} = \Diamond_{\mathcal{R}_\mathcal{A}}$, i.e. that for every $a \in \mathcal{A}$,

$$\overline{(\Diamond a)} = \Diamond_{\mathcal{R}_\mathcal{A}} \overline{a}.$$

($\subseteq$) If $P \in \overline{(\Diamond a)}$, then $\Diamond a \in P$, then by item 2 (item 4) of 5.6, $a \in Q$ and $P\mathcal{R}_\mathcal{A}Q$ for some $Q \in Pr(\mathcal{A})$, hence $P \in \Diamond_{\mathcal{R}_\mathcal{A}}\overline{a}$.

($\supseteq$) If $P \in \Diamond_{\mathcal{R}_\mathcal{A}}\overline{a}$, then $a \in Q$ and $P\mathcal{R}_\mathcal{A}Q$ for some $Q \in Pr(\mathcal{A})$, i.e. $Q \subseteq \Diamond^{-1}[P]$, hence $\Diamond a \in P$.

Let us show that if $\mathcal{A}$ is an IK-algebra, then $\Box^{\boldsymbol{A}} = \Box_{(\subseteq \circ \mathcal{R}_\mathcal{A})}$, i.e. that for every $a \in \mathcal{A}$,

$$\overline{(\Box a)} = \Box_{(\subseteq \circ \mathcal{R}_\mathcal{A})} \overline{a}.$$

($\subseteq$) If $P \in \overline{(\Box a)}$, then $\Box a \in P$, i.e. $a \in \Box^{-1}[P]$ so, for every $Q \in Pr(\mathcal{A})$, if $P\mathcal{R}_\mathcal{A}Q$, then $a \in \Box^{-1}[P] \subseteq Q$.

($\supseteq$) If $P \notin \overline{(\Box a)}$, then by item 3 of 5.6, $a \notin Q$ and $P \subseteq S\mathcal{R}_\mathcal{A}Q$ for some $Q, S \in Pr(\mathcal{A})$, so $P \notin \Box_{(\subseteq \circ \mathcal{R}_\mathcal{A})}\overline{a}$.

This is enough for proving that $\boldsymbol{A}$ is closed in each case under the appropriate operations.

If $\mathcal{A}$ is an IK-algebra, then as $\mathbf{X}_{\mathcal{A}_+}$ is an Esakia space, then in particular it is a Priestley space, so by 3.2 for every $P \in Pr(\mathcal{A})$, $P\!\uparrow = \{Q \in Pr(\mathcal{A}) \mid P \subseteq Q\}$ is a closed subset of $\mathbf{X}_{\mathcal{A}_+}$, hence by 5.3, $\mathcal{R}_\mathcal{A}[P\!\uparrow]$ is closed in $\mathbf{X}_{\mathcal{A}_+}$.

Let us show that $\mathcal{R}_\mathcal{A}[P\!\uparrow]$ is an up-set: If $Q \in \mathcal{R}_\mathcal{A}[P\!\uparrow]$ and $Q \subseteq T$, then $P \subseteq S\mathcal{R}_\mathcal{A}Q \subseteq T$, hence by 5.7 (3), $P \subseteq S \subseteq Q'\mathcal{R}_\mathcal{A}T$, and so $T \in \mathcal{R}_\mathcal{A}[P\!\uparrow]$. This proves D4. □

5.10 The action of $(\ )_+$ on arrows

Proposition 5.11. *Let* $\mathbf{L} \in \{\mathrm{I}_\Box, \mathrm{I}_\Diamond, \mathrm{IK}\}$. *For every* $\mathbf{L}$*-algebra homomorphism* $h : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$, $h_+ : \mathcal{A}_{2+} \longrightarrow \mathcal{A}_{1+}$ *is a p-morphism of general* $\mathbf{L}$*-frames.*

Proof. The Priestley duality restricted to Heyting algebras yields that h_+ is a continuous and strongly isotone map between $\mathbf{X}_{\mathcal{A}_{2+}}$ and $\mathbf{X}_{\mathcal{A}_{1+}}$, which is equivalent to conditions M1–M3.

Let us show that if $P, Q \in Pr(\mathcal{A}_2)$ and $\Box^{-1}[P] \subseteq Q$, then $\Box^{-1}[h^{-1}[P]] \subseteq h^{-1}[Q]$: For every $a \in \mathcal{A}_2$,

$$\begin{aligned} a \in \Box^{-1}[h^{-1}[P]] \quad &\Leftrightarrow \quad \Box a \in h^{-1}[P] \\ &\Leftrightarrow \quad h(\Box a) \in P \\ &\Leftrightarrow \quad \Box h(a) \in P \\ &\Leftrightarrow \quad h(a) \in \Box^{-1}[P] \subseteq Q \\ &\Rightarrow \quad a \in h^{-1}[Q]. \end{aligned}$$

Let us show that if $P, Q \in Pr(\mathcal{A}_2)$ and $Q \subseteq \Diamond^{-1}[P]$, then $h^{-1}[Q] \subseteq \Diamond^{-1}[h^{-1}[P]]$: For every $a \in \mathcal{A}_2$,

$$\begin{aligned} a \in h^{-1}[Q] &\Leftrightarrow h(a) \in Q \subseteq \Diamond^{-1}[P] \\ &\Rightarrow \Diamond h(a) \in P \\ &\Leftrightarrow h(\Diamond a) \in P \\ &\Leftrightarrow \Diamond a \in h^{-1}[P] \\ &\Leftrightarrow a \in \Diamond^{-1}[h^{-1}[P]]. \end{aligned}$$

This is enough for proving that for $\mathbf{L} \in \{\mathrm{I}_\Box, \mathrm{I}_\Diamond, \mathrm{IK}\}$ and for every $P, Q \in Pr(\mathcal{A}_2)$, if $P\mathcal{R}_{\mathcal{A}_2}Q$, then $h_+(P)\mathcal{R}_{\mathcal{A}_1}h_+(Q)$, which is M4.

Let us show M5 for $\mathrm{I}_\Diamond$-algebras, i.e. that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathrm{I}_\Diamond$-algebras and $P \in Pr(\mathcal{A}_2)$, $Q \in Pr(\mathcal{A}_1)$ are such that $h^{-1}[P]\mathcal{R}_{\mathcal{A}_1}Q$, then there exists $S \in \mathcal{R}_{\mathcal{A}_2}[P]$ such that $Q \subseteq h^{-1}[S]$. We need to find a prime filter S of $\mathcal{A}_2$ such that $S \subseteq \Diamond^{-1}[P]$, i.e. $S \cap \Diamond^{-1}[P]^c = \emptyset$ and $Q \subseteq h^{-1}[S]$, i.e. $h[Q] \subseteq S$. It holds that

$$Fi(h[Q]) \cap \Diamond^{-1}[P]^c = \emptyset,$$

for if not, then there are $a \in Q$ and $\Diamond b \notin P$ such that $h(a) \leq b$, hence $\Diamond h(a) \leq \Diamond b$. As $a \in Q \subseteq \Diamond^{-1}[h^{-1}[P]]$, then $\Diamond h(a) \in P$, and so $\Diamond b \in P$, contradiction.

By Birkhoff-Stone theorem, there exists $S \in Pr(\mathcal{A}_2)$ such that $h[Q] \subseteq S$ (i.e. $Q \subseteq h^{-1}[S]$) and $S \cap \Diamond^{-1}[P]^c = \emptyset$, i.e. $S \subseteq \Diamond^{-1}[P]$, i.e. $P\mathcal{R}_{\mathcal{A}_2}S$.

Let us show M5 for IK-algebras: Like before, it holds that $Fi(h[Q]) \cap \Diamond^{-1}[P]^c = \emptyset$, so by Birkhoff-Stone theorem, $h[Q] \subseteq T$ (i.e. $Q \subseteq h^{-1}[T]$) and $T \cap \Diamond^{-1}[P]^c = \emptyset$ for some $T \in Pr(\mathcal{A}_2)$. As $T \subseteq \Diamond^{-1}[P]$, then by 5.5 (1), $\langle P, T\rangle \in (\mathcal{R}_{\mathcal{A}_2} \circ \supseteq)$, i.e. $P\mathcal{R}_{\mathcal{A}_2}S \supseteq T$ for some $S \in Pr(\mathcal{A}_2)$, so $S \in \mathcal{R}_{\mathcal{A}_2}[P]$ and $Q \subseteq h^{-1}[T] \subseteq h^{-1}[S]$.

Let us show M5', i.e. that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathrm{I}_\Box$-algebras and $P \in Pr(\mathcal{A}_2)$, $Q \in Pr(\mathcal{A}_1)$ are such that $h^{-1}[P]\mathcal{R}_{\mathcal{A}_1}Q$, then there exists $S \in \mathcal{R}_{\mathcal{A}_2}[P]$ such that $h^{-1}[S] \subseteq Q$: what we need to hold is that $\Box^{-1}[P] \subseteq S$ and $S \subseteq h[Q]$, i.e. $S \cap h[Q^c] = \emptyset$. If we show that

$$h[Q^c] \cap Fi(\Box^{-1}[P]) = \emptyset,$$

then the statement will follow from Birkhoff-Stone theorem. Suppose that there are $a \notin Q$ and $\Box b \in P$ such that $b \leq h(a)$, hence $\Box b \leq \Box h(a) = h(\Box a)$. As $\Box b \in P$, then $h(\Box a) \in P$, hence $a \in \Box^{-1}[h^{-1}[P] \subseteq Q$, contradiction.

Let us show M6, i.e. that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are IK-algebras and $P \in Pr(\mathcal{A}_2)$, $Q \in Pr(\mathcal{A}_1)$ are such that $h^{-1}[P](\subseteq \circ \mathcal{R}_{\mathcal{A}_1})Q$, then there exists $S \in (\subseteq \circ \mathcal{R}_{\mathcal{A}_1})[P]$ such that $h^{-1}[S] \subseteq Q$: By 5.5 (2), we need that $\Box^{-1}[P] \subseteq S$, moreover, we need that $S \subseteq h[Q]$, i.e. $S \cap h[Q^c] = \emptyset$. The proof goes like in the case treated before. □

6 Duality

In this section we are going to introduce the full subcategories $\mathbf{L}\mathsf{Sp}$ of $\mathbf{L}\mathsf{GF}$ for $\mathbf{L} \in \{\mathrm{I}_\Box, \mathrm{I}_\Diamond, \mathrm{IK}\}$, and show that the functors $(\ )^+ : \mathbf{L}\mathsf{Alg} \longrightarrow \mathbf{L}\mathsf{Sp}$ and $(\ )_+ : \mathbf{L}\mathsf{Sp} \longrightarrow \mathbf{L}\mathsf{Alg}$ establish a duality.

For every general $\mathbf{L}$-frame $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$, let us consider the assignment $\varepsilon_\mathcal{G} : X \longrightarrow \mathcal{P}(\mathcal{A})$ that maps every $x \in X$ to the set $\varepsilon_\mathcal{G}(x) = \{Y \in \mathcal{A} \mid x \in Y\}$. The Priestley duality restricted to Heyting algebras yields that this assignment defines a map $\varepsilon_\mathcal{G} : \mathbf{X}_\mathcal{G} \longrightarrow \mathbf{X}_{(\mathcal{G}^+)_+}$ that is an iso in the category Es of Esakia spaces and continuous and strongly isotone maps (which is the dual of the category of Heyting algebras and their homomorphisms).

So we will only need to show that the objects in $\mathbf{L}\mathsf{Sp}$ are exactly those general $\mathbf{L}$-frames $\mathcal{G}$ such that $\varepsilon_\mathcal{G}$ is an iso in $\mathbf{L}\mathsf{GF}$.

6.1 L-spaces

Definition 6.2. ($\mathrm{I}_\Box$-space) *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame. $\mathcal{G}$ is an* $\mathrm{I}_\Box$-space *iff the following list of conditions is satisfied:*

D1. $\mathbf{X}_\mathcal{G}$ is an Esakia space, and $\mathcal{A}$ is the collection of the clopen up-sets of $\mathbf{X}_\mathcal{G}$.
D2'. $\mathcal{A}$ is closed under $\Box_R$.
D3. For every $x \in X$, $R[x] \in K^\uparrow(\mathbf{X}_\mathcal{G}) = \{F \in K(\mathbf{X}_\mathcal{G}) \mid F = F{\uparrow}\}$.

So $\mathrm{I}_\Box$-spaces are those general $\mathrm{I}_\Box$-frames such that $R[x]$ is an up-set for every $x \in X$. Let $\mathrm{I}_\Box\mathsf{Sp}$ be the category of the $\mathrm{I}_\Box$-spaces and their p-morphisms.

Example 6.3. *For every finite partial order $\langle X, \leq \rangle$, the general frame $\mathcal{G} = \langle X, \leq, \leq, \mathsf{Up}(X)\rangle$ is an $I_\Box$-space.*

Proof. By definition, the topology τ of $\mathbf{X}_\mathcal{G} = \mathbf{X}$ is generated by taking $\mathsf{Up}(X) \cup \mathsf{Down}(X)$ as a subbase. In 3.11, we saw that $\mathbf{X}$ is an Esakia space and that $\mathsf{Up}(X)$ is the collection of the clopen up-sets of $\mathbf{X}$. 2.2 (1) implies that $\mathsf{Up}(X)$ is closed under $\Box_\leq$. For every $x \in X$, $x\uparrow \in \mathsf{Up}(X)$ is a clopen up-set, so in particular $x\uparrow \in K^\uparrow(\mathbf{X})$. □

Definition 6.4. ($\mathbf{I}_\Diamond$-space) *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame. $\mathcal{G}$ is an* $\mathrm{I}_\Diamond$-space *iff the following list of conditions is satisfied:*

D1. $\mathbf{X}_\mathcal{G}$ is an Esakia space, and $\mathcal{A}$ is the collection of the clopen up-sets of $\mathbf{X}_\mathcal{G}$.
D2. $\mathcal{A}$ is closed under $\Diamond_R$.
D3. For every $x \in X$, $R[x] \in K^\downarrow(\mathbf{X}_\mathcal{G}) = \{F \in K(\mathbf{X}_\mathcal{G}) \mid F = F\downarrow\}$.

So $\mathrm{I}_\Diamond$-spaces are those general $\mathrm{I}_\Diamond$-frames such that $R[x]$ is a down-set for every $x \in X$. Let $\mathrm{I}_\Diamond\mathsf{Sp}$ be the category of the $\mathrm{I}_\Diamond$-spaces and their p-morphisms.

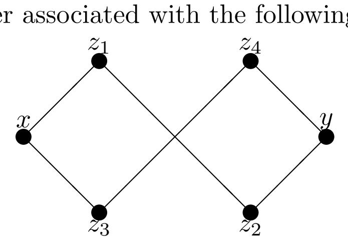
Example 6.5. *For every finite partial order $\langle X, \leq \rangle$, the general frame $\mathcal{G} = \langle X, \leq, \geq, \mathsf{Up}(X)\rangle$ is an $I_\Diamond$-space.*

Proof. By definition, the topology τ of $\mathbf{X}_\mathcal{G} = \mathbf{X}$ is generated by taking $\mathsf{Up}(X) \cup \mathsf{Down}(X)$ as a subbase. In 3.11, we saw that $\mathbf{X}$ is an Esakia space and that $\mathsf{Up}(X)$ is the collection of the clopen up-sets of $\mathbf{X}$. 2.2 (2) implies that $\mathsf{Up}(X)$ is closed under $\Diamond_\geq$. For every $x \in X$, $x\downarrow \in \mathsf{Down}(X)$ is a clopen down-set, so in particular $x\downarrow \in K^\downarrow(\mathbf{X})$. □

Definition 6.6. (IK-space) *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame. $\mathcal{G}$ is an* IK-space *iff the following list of conditions is satisfied:*

D1. $\mathbf{X}_\mathcal{G}$ is an Esakia space, and $\mathcal{A}$ is the collection of the clopen up-sets of $\mathbf{X}_\mathcal{G}$.
D2. $\mathcal{A}$ is closed under $\Diamond_R$ and $\Box_{(\leq \circ R)}$.
D3. For every $x \in X$, $R[x] \in K(\mathbf{X}_\mathcal{G})$.
D4. For every $x \in X$, $R[x\uparrow] \in K^\uparrow(\mathbf{X}_\mathcal{G})$.
D5. For every $x \in X$, $R[x] = R[x\uparrow] \cap R[x]\downarrow$.

Let IKSp be the category of the IK-spaces and their p-morphisms. Conditions D4 and D5 together imply that for every $x \in X$, $R[x]$ is the intersection of an up-set and a down-set, hence $R[x]$ is *convex*, i.e. $R[x] = R[x]\uparrow \cap R[x]\downarrow$[1]. So if $\mathcal{G}$ is an IK-space, then $\mathcal{G}$ is a general IK-frame such that $R[x]$ is convex for every $x \in X$. However, not all the general IK-frames such that $R[x]$ is convex for every $x \in X$ are IK-spaces: Indeed, we saw already that, given a finite partial order $\langle X, \leq \rangle$, the general frame $\mathcal{G} = \langle X, \leq, (\geq \circ \leq), \mathsf{Up}(X)\rangle$ is general IK-frame, and moreover for every $x \in X$, $(\geq \circ \leq)[x] = x\downarrow\uparrow$ is an up-set, hence it is convex. However, $\mathcal{G}$ is not an IK-space in general. Consider the partial order associated with the following Hasse diagram:



The relation $(\geq \circ \leq)$ does not satisfy D5: It holds that $x \leq z_1 \geq z_2 \leq y$, so $y \in (\geq \circ \leq)[x\uparrow]$, and $x \geq z_3 \leq z_4 \geq y$, so $y \in (\geq \circ \leq)[x]\downarrow$, but $y \notin (\geq \circ \leq)[x]$.

[1] It is very easy to show that for every partial order $\langle X, \leq \rangle$ and every $Y \subseteq X$, $Y = Y\uparrow \cap Y\downarrow$ (i.e. Y is *convex*) iff $Y = U \cap V$ for some $U \in \mathsf{Up}(X)$ and $V \in \mathsf{Down}(X)$.

Example 6.7. *For every finite linear order $\langle X, \leq \rangle$, the general* IK*-frame $\mathcal{G} = \langle X, \leq, (\geq \circ \leq), \mathsf{Up}(X) \rangle$ is an* IK*-space.*

Proof. Since $\leq$ is a linear order, then for every $x \in X$, $X = x{\uparrow} \cup x{\downarrow} \subseteq (\geq \circ \leq)[x]$, hence $(\geq \circ \leq)[x] = (\geq \circ \leq)[x{\uparrow}] \cap (\geq \circ \leq)[x]{\downarrow}$, which is D5. □

Proposition 6.8. *For every* **L***-algebra $\mathcal{A}$, $\mathcal{A}_+$ is an* **L***-space.*

Proof. By 5.9, $\mathcal{A}_+$ is a general **L**-frame. By 5.4 (1), if $\mathcal{A}$ is an $\mathrm{I}_\Box$-algebra, then $\mathcal{R}_\mathcal{A}[P] = (\subseteq \circ \mathcal{R}_\mathcal{A} \circ \subseteq)[P]$ is an up-set for every $P \in Pr(\mathcal{A})$. Analogously, 5.4 (2) and (3) respectively imply that if $\mathcal{A}$ is an $\mathrm{I}_\Diamond$-algebra, then $\mathcal{R}_\mathcal{A}[P]$ is a down-set for every $P \in Pr(\mathcal{A})$, and if $\mathcal{A}$ is an IK-algebra, then $\mathcal{R}_\mathcal{A}[P] = (\subseteq \circ \mathcal{R}_\mathcal{A})[P] \cap (\mathcal{R}_\mathcal{A} \circ \supseteq)[P]$ for every $P \in Pr(\mathcal{A})$. □

Lemma 6.9. *For every general* **L***-frame $\mathcal{G} = \langle X, \leq, R, \mathcal{A} \rangle$ and every $x, y \in X$,*

1. *$x \leq y$ iff $\varepsilon_\mathcal{G}(x) \subseteq \varepsilon_\mathcal{G}(y)$.*
2. *If xRy then $\varepsilon_\mathcal{G}(x) \mathcal{R}_\mathcal{A} \varepsilon_\mathcal{G}(y)$.*

Proof. 1. If $x \leq y$ then, as $\mathcal{A} \subseteq \mathsf{Up}(X)$, for every $Y \in \mathcal{A}$, if $x \in Y$ then $y \in Y$.
If $x \not\leq y$ then, as $\mathbf{X}_\mathcal{G}$ is totally order-disconnected and $\mathcal{A}$ is the collection of the clopen up-sets of $\mathbf{X}_\mathcal{G}$, $x \in Y$ and $y \notin Y$ for some $Y \in \mathcal{A}$, hence $Y \in (\varepsilon_\mathcal{G}(x) \setminus \varepsilon_\mathcal{G}(y))$, and so $\varepsilon_\mathcal{G}(x) \not\subseteq \varepsilon_\mathcal{G}(y)$.
2. Let us show that if $y \in R[x]$, then a) $\Box_R^{-1}[\varepsilon_\mathcal{G}(x)] \subseteq \varepsilon_\mathcal{G}(y)$, b) $\varepsilon_\mathcal{G}(y) \subseteq \Diamond_R^{-1}[\varepsilon_\mathcal{G}(x)]$ and c) $\Box_{(\leq \circ R)}^{-1}[\varepsilon_\mathcal{G}(x)] \subseteq \varepsilon_\mathcal{G}(y)$:
a) For every $Y \in \mathcal{A}$, $\Box_R Y \in \varepsilon_\mathcal{G}(x)$ iff $x \in \Box_R Y$, iff $R[x] \subseteq Y$, and so $y \in Y$, i.e. $Y \in \varepsilon_\mathcal{G}(y)$.
b) For every $Y \in \mathcal{A}$, $Y \in \varepsilon_\mathcal{G}(y)$ iff $y \in Y$, and as $y \in R[x]$, then $R[x] \cap Y \neq \emptyset$, i.e. $\Diamond_R Y \in \varepsilon_\mathcal{G}(x)$, i.e. $Y \in \Diamond_R^{-1}[\varepsilon_\mathcal{G}(x)]$.
c) For every $Y \in \mathcal{A}$, $\Box_{(\leq \circ R)} Y \in \varepsilon_\mathcal{G}(x)$ iff $x \in \Box_{(\leq \circ R)} Y$, iff $(\leq \circ R)[x] \subseteq Y$, and so $y \in R[x] \subseteq (\leq \circ R)[x] \subseteq Y$, i.e. $Y \in \varepsilon_\mathcal{G}(y)$.
a) proves the statement if $\mathcal{A}$ is an $\mathrm{I}_\Box$-algebra, b) proves the statement if $\mathcal{A}$ is an $\mathrm{I}_\Diamond$-algebra, and a) and c) together prove the statement if $\mathcal{A}$ is an IK-algebra. □

The following lemma says that the additional conditions that **L**-spaces satisfy are exactly what is needed in each situation in order for ε to be an iso.

Lemma 6.10.

1. *The following are equivalent for every general* $\mathrm{I}_\Box$*-frame:*
   (a) *For every $x \in X$, $R[x] = R[x]{\uparrow}$.*
   (b) *For every $x, y \in X$, if $\varepsilon(x) \mathcal{R}_\mathcal{A} \varepsilon(y)$ then xRy.*
2. *The following are equivalent for every general* $\mathrm{I}_\Diamond$*-frame:*
   (a) *For every $x \in X$, $R[x] = R[x]{\downarrow}$.*
   (b) *For every $x, y \in X$, if $\varepsilon(x) \mathcal{R}_\mathcal{A} \varepsilon(y)$ then xRy.*
3. *The following are equivalent for every general* IK*-frame:*
   (a) *For every $x \in X$, $R[x] = R[x{\uparrow}] \cap R[x]{\downarrow}$.*
   (b) *For every $x, y \in X$, if $\varepsilon(x) \mathcal{R}_\mathcal{A} \varepsilon(y)$ then xRy.*

Proof. 1. (a $\Rightarrow$ b) Suppose that $x, y \in X$ are such that $\varepsilon(x) \mathcal{R}_\mathcal{A} \varepsilon(y)$ but $y \notin R[x] = R[x]{\uparrow}$. Then $R[x] \subseteq U$ and $y \notin U$ for some $U \in \mathcal{A}$, hence $x \in \Box_R U$. As $\varepsilon(x) \mathcal{R}_\mathcal{A} \varepsilon(y)$, then $\Box_R^{-1}[\varepsilon(x)] \subseteq \varepsilon(y)$, i.e. for every $U \in \mathcal{A}$, if $x \in \Box_R U$, then $y \in U$, contradiction.
(b $\Rightarrow$ a) ($\supseteq$) If $y \in R[x]{\uparrow}$, then $xRz \leq y$ for some $z \in X$, hence, by 6.9, $\varepsilon(x) \mathcal{R}_\mathcal{A} \varepsilon(z) \subseteq \varepsilon(y)$, i.e. $\Box_R^{-1}[\varepsilon(x)] \subseteq \varepsilon(z) \subseteq \varepsilon(y)$, hence $\varepsilon(x) \mathcal{R}_\mathcal{A} \varepsilon(y)$, and so by assumption it follows that xRy.

2. (a $\Rightarrow$ b) Suppose that $x, y \in X$ are such that $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$ but $y \notin R[x] = R[x]{\downarrow}$. Then $y \in U$ and $R[x] \cap U = \emptyset$ for some clopen up-set subset U, hence $x \notin \Diamond_R U$. As $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$, then $\varepsilon(y) \subseteq \Diamond_R^{-1}[\varepsilon(x)]$, i.e. for every $U \in \mathcal{A}$, if $y \in U$ then $x \in \Diamond_R U$, contradiction.
(b $\Rightarrow$ a) ($\supseteq$) If $y \in R[x]{\downarrow}$, then $xRz \geq y$ for some $z \in X$, hence, by 6.9, $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(z) \supseteq \varepsilon(y)$, i.e. $\varepsilon(y) \subseteq \varepsilon(z) \subseteq \Diamond_R^{-1}[\varepsilon(x)]$, hence $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$, and so by assumption it follows that xRy.
3. (a $\Rightarrow$ b) Suppose that $x, y \in X$ are such that $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$ but $y \notin R[x] = R[x{\uparrow}] \cap R[x]{\downarrow}$. Then either $y \notin R[x{\uparrow}]$ or $y \notin R[x]{\downarrow}$. If $y \notin R[x{\uparrow}] = R[x{\uparrow}]{\uparrow}$ Then $R[x{\uparrow}] \subseteq U$ and $y \notin U$ for some $U \in \mathcal{A}$, hence $x \in \Box_{(\leq \circ R)}U$. As $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$, then $\Box^{-1}_{(\leq \circ R)}[\varepsilon(x)] \subseteq \varepsilon(y)$, i.e. for every $U \in \mathcal{A}$, if $x \in \Box_{(\leq \circ R)}U$, then $y \in U$, contradiction. If $y \notin R[x]{\downarrow}$ the proof is analogous to the proof of (2), (a $\Rightarrow$ b).
(b $\Rightarrow$ a) ($\supseteq$) If $y \in R[x{\uparrow}] \cap R[x]{\downarrow}$, then $x \leq z_1 R y$ and $xRz_2 \geq y$ for some $z_1, z_2 \in X$, hence, by 6.9, $\varepsilon(x) \subseteq \varepsilon(z_1)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$ and $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(z_2) \supseteq \varepsilon(y)$, and so $\Box_R^{-1}[\varepsilon(x)] \subseteq \Box_R^{-1}[\varepsilon(z_1)] \subseteq \varepsilon(y)$ and $\varepsilon(y) \subseteq \varepsilon(z_2) \subseteq \Diamond_R^{-1}[\varepsilon(x)]$, hence $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$, and so by assumption it follows that xRy. ☐

Proposition 6.11. *For every* $\mathbf{L}$*-space* $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$*,* $\varepsilon_{\mathcal{G}} : \mathcal{G} \longrightarrow (\mathcal{G}^+)_+$ *is a p-morphism of* $\mathbf{L}$*-spaces, hence it is an iso in* $\mathbf{L}\mathsf{GF}$*.*

Proof. From the duality for Heyting algebras, we know that $\varepsilon_{\mathcal{G}} : \mathbf{X}_{\mathcal{G}} \longrightarrow \mathbf{X}_{(\mathcal{G}^+)_+}$ is an iso in Es, hence it is bijective and satisfies M1–M3. M4 holds by 6.9 (2). The surjectivity of $\varepsilon_{\mathcal{G}}$ and 6.10 imply M5', M5 and M6. Let us show M6: If $\varepsilon_{\mathcal{G}}(x)(\subseteq \circ \mathcal{R}_{\mathcal{A}})P = \varepsilon_{\mathcal{G}}(y)$, then $\varepsilon_{\mathcal{G}}(x) \subseteq \varepsilon_{\mathcal{G}}(z)\mathcal{R}_{\mathcal{A}}\varepsilon_{\mathcal{G}}(y)$ for some $z \in X$, hence, by item 1 of 6.9 and 6.10, $x \leq zRy$, i.e. $y \in (\leq \circ R)[x]$. ☐

Theorem 6.12. *For every* $\mathbf{L} \in \{\mathrm{I}_\Box, \mathrm{I}_\Diamond, \mathrm{IK}\}$*, the category* $\mathbf{L}\mathsf{Alg}$ *of* $\mathbf{L}$*-algebras and their homomorphisms is dually equivalent to the category* $\mathbf{L}\mathsf{Sp}$ *of* $\mathbf{L}$*-spaces and their p-morphisms.*

Proof. It follows from 4.3, 4.5, 5.11, 6.8, and 6.11. ☐

7 Characterizing topological semantics of MIPC

Bezhanishvili [1, 2] introduced a topological semantics for MIPC, given by the category TPSOE of *perfect augmented Kripke frames* and their morphisms (see 7.6 and 7.10 below), and proved that TPSOE is dually equivalent to the category of *monadic Heyting algebras* and their homomorphisms, that is the class of algebras canonically associated with MIPC (see [2]). In this section, we will show that − as was to be expected − TPSOE is isomorphic to the full subcategory MIPCSp of IKSp whose objects are the MIPC-spaces defined below:

Definition 7.1. (MIPC-space) *An* MIPC-space *is an* IK*-space* $\mathcal{G} = \langle X, \leq, E, \mathcal{A}\rangle$ *such that* E *is an equivalence relation.*

Definition 7.2. (Augmented Kripke frame) (cf. Def 2.1 of [2]) *A relational structure* $\langle X, \leq, E\rangle$ *is an* augmented Kripke frame *iff* $\langle X, \leq\rangle$ *is a partial order and* E *is an equivalence relation on* X *such that* $(E \circ \leq) \subseteq (\leq \circ E)$*.*

Lemma 7.3. *For every relational structure* $\mathcal{F} = \langle X, \leq, E\rangle$*,* $\mathcal{F}$ *is an augmented Kripke frame iff* $\mathcal{F}$ *is an* IK*-frame such that* $\leq$ *is a partial order and* E *is an equivalence relation.*

Proof. The 'if' direction is immediate. As for the converse, if $\mathcal{F}$ is an augmented Kripke frame we only need to show that $(\geq \circ E) \subseteq (E \circ \geq)$: if $x, y, z \in X$ and $x \geq yEz$, then, as E is symmetric, $zEy \leq x$, and so $z \leq vEx$ for some $v \in X$, hence $xEv \geq z$. ☐

Definition 7.4. (Perfect Kripke frame) (cf. Section 3.1 of [2]) *A preordered Stone space* $\mathbf{X} = \langle X, \leq, \tau\rangle$ *is a* perfect Kripke frame *iff* $x{\uparrow} \in K(\mathbf{X})$ *for every* $x \in X$ *and for every clopen subset* U *of* $\mathbf{X}$*,* $U{\downarrow} = \leq^{-1}[U]$ *is clopen.*

Lemma 7.5. *For every perfect Kripke frame* $\mathbf{X} = \langle X, \leq, \tau\rangle$*,*
1. $\mathbf{X}$ *is totally order-disconnected. Hence, a perfect Kripke frame* $\mathbf{X} = \langle X, \leq, \tau\rangle$ *is an Esakia space iff* $\leq$ *is a partial order.*
2. For every $F \in K(\mathbf{X})$*,* $F{\uparrow}$ *and* $F{\downarrow} \in K(\mathbf{X})$*. Hence, for every* $x \in X$*,* $x{\uparrow}$ *and* $x{\downarrow} \in K(\mathbf{X})$*.*

Proof. 1. If $x, y \in X$ and $x \not\leq y$, then $y \notin x\uparrow \in K(\mathbf{X})$. As $\mathbf{X}$ is a Stone space, then $y \in U$ and $U \cap x\uparrow = \emptyset$ for some clopen $U \subseteq X$. Then $U\downarrow$ is clopen, $y \in U\downarrow$ and $U\downarrow \cap x\uparrow = \emptyset$, which proves total order-disconnectedness.

2. Analogous to the proof of 3.2. □

Definition 7.6. (Perfect augmented Kripke frame) (cf. Section 3.1 of [2]) *A* perfect augmented Kripke frame *is a structure $\mathcal{X} = \langle X, \leq, E, \tau\rangle$ such that the following list of conditions is satisfied:*

P1. $\langle X, \leq, E\rangle$ is an augmented Kripke frame (hence $\leq$ is a partial order).

P2. $\mathbf{X}_{\mathcal{X}} = \langle X, \leq, \tau\rangle$ and $\langle X, (\leq \circ E), \tau\rangle$ are perfect Kripke frames.

P3. For every clopen up-set U of $\mathbf{X}_{\mathcal{X}}$, $E[U]$ is clopen.

Lemma 7.7. *For every augmented Kripke frame $\langle X, \leq, E\rangle$, and every up-set Y, $E[Y]$ is an up-set.*

Proof. Let $x \in E[Y]$ and $x \leq z$ and let us show that $z \in E[Y]$: as $x \in E[Y]$ then yEx for some $y \in Y$, so $z \geq xEy$, hence, as $(\geq \circ E) \subseteq (E \circ \geq)$ by 7.3, $zEv \geq y$ for some $v \in X$, i.e. $y \leq vEz$, and as Y is an up-set and $y \in Y$, then $v \in Y$ and so $z \in E[Y]$. □

Lemma 7.8. (cf. Lemma 3.1 (1) of [2]) *For every perfect augmented Kripke frame $\mathcal{X} = \langle X, \leq, E, \tau\rangle$ and every $x \in X$, $E[x] = (\leq \circ E)[x] \cap (E \circ \geq)[x]$.*

For every perfect augmented Kripke frame $\mathcal{X} = \langle X, \leq, E, \tau\rangle$ let us define $\mathcal{G}_{\mathcal{X}} = \langle X, \leq, E, \mathcal{A}_\tau\rangle$, where $\mathcal{A}_\tau$ is the collection of the clopen up-sets of $\langle X, \leq, \tau\rangle$. Notice that $\mathbf{X}_{\mathcal{G}_{\mathcal{X}}} = \mathbf{X}_{\mathcal{X}}$.
For every IK-space $\mathcal{G} = \langle X, \leq, E, \mathcal{A}\rangle$ such that E is an equivalence relation let us consider its associated space $\mathbf{X}_{\mathcal{G}} = \langle X, \leq, \tau\rangle$ and define $\mathcal{X}_{\mathcal{G}} = \langle X, \leq, E, \tau\rangle$. Clearly, $\mathbf{X}_{\mathcal{X}_{\mathcal{G}}} = \langle X, \leq, \tau\rangle = \mathbf{X}_{\mathcal{G}}$.

Proposition 7.9.

1. *For every perfect augmented Kripke frame $\mathcal{X} = \langle X, \leq, E, \tau\rangle$, $\mathcal{G}_{\mathcal{X}} = \langle X, \leq, E, \mathcal{A}_\tau\rangle$ is an* MIPC*-space.*

2. *For every* MIPC*-space $\mathcal{G} = \langle X, \leq, E, \mathcal{A}\rangle$, $\mathcal{X}_{\mathcal{G}} = \langle X, \leq, E, \tau\rangle$ is a perfect augmented Kripke frame.*

Proof. 1. By P2, it holds that $\mathbf{X}_{\mathcal{G}_{\mathcal{X}}} = \mathbf{X}_{\mathcal{X}} = \langle X, \leq, \tau\rangle$ is a perfect Kripke frame and $\leq$ is a partial order, so by 7.5 (1) $\mathbf{X}_{\mathcal{G}_{\mathcal{X}}}$ is an Esakia space, and $\mathcal{A}_\tau$ is the algebra of the clopen up-sets of $\mathbf{X}_{\mathcal{G}_{\mathcal{X}}}$, hence D1 holds.

The assumption that $\langle X, (\leq \circ E), \tau\rangle$ is a perfect Kripke frame implies that: a) by 7.5 (2), for every $x \in X$ $E[x\uparrow] \in K(\mathbf{X}_{\mathcal{G}_{\mathcal{X}}})$, which is D4; and b) for every $U \in \mathcal{A}_\tau$ (U is a clopen up-set of $\mathbf{X}_{\mathcal{G}_{\mathcal{X}}}$, hence $(X \setminus U)$ is clopen, therefore), $(\leq \circ E)^{-1}[X \setminus U]$ is clopen, hence so is its complement $X \setminus (\leq \circ E)^{-1}[X \setminus U] = \Box_{(\leq \circ E)}U$. $\Box_{(\leq \circ E)}U$ is also an up-set, for if $z \in \Box_{(\leq \circ E)}U$ and $z \leq y$, then $y\uparrow \subseteq z\uparrow$, and so $E[y\uparrow] \subseteq E[z\uparrow] \subseteq U$, hence $y \in \Box_{(\leq \circ E)}U$. Let us show that for every $U \in \mathcal{A}_\tau$, $\Diamond_E U \in \mathcal{A}_\tau$: Since E is symmetric, then for every clopen up-set U, $\Diamond_E U = E^{-1}[U] = E[U]$, which is clopen by 7.6 P3, and it is also an up-set by 7.7. So this completes the proof of D2. By 7.8, for every $x \in X$, $E[x] = (\leq \circ E)[x] \cap (E \circ \geq)[x] = E[x\uparrow] \cap E[x]\downarrow$, which is D5. Since $\langle X, (\leq \circ E), \tau\rangle$ is a perfect Kripke frame, then by 7.5 (2), we get that for every $x \in X$, $E[x]\downarrow = (\leq \circ E)^{-1}[x] \in K(\mathbf{X}_{\mathcal{G}_{\mathcal{X}}})$. Then, (recall that D4 holds) $E[x] = E[x\uparrow] \cap E[x]\downarrow$ is the intersection of two closed sets, so it is closed, which is D3.

2. By 4.2 (3) it holds in particular that $(E \circ \leq) \subseteq (\leq \circ E)$, so $\langle X, \leq, E\rangle$ is an augmented Kripke frame, which is P1. D2 implies that for every clopen up-set U, $E[U] = \Diamond_E U$ is clopen, which is P3. By D1, $\langle X, \leq, \tau\rangle$ is an Esakia space, hence it is a perfect Kripke frame. So the only thing we need to show is that $\langle X, (\leq \circ E), \tau\rangle$ is a perfect Kripke frame. Clearly, $\langle X, (\leq \circ E), \tau\rangle$ is an ordered Stone space. By D4, it holds that $(\leq \circ E)[x] = E[x\uparrow] \in K^{\uparrow}(\mathbf{X}_{\mathcal{G}})$ for every $x \in X$, so we are left to show that for every V clopen, $(\leq \circ E)^{-1}[V]$ is clopen. For the remainder of this proof, we rely on definitions and facts that can be found in the appendix of this section. By D4, the assignment $x \longmapsto (\leq \circ E)[x] = E[x\uparrow] \in K^{\uparrow}(\mathbf{X}_{\mathcal{G}})$ defines a map $\rho : \mathbf{X}_{\mathcal{G}} \longrightarrow \mathbf{K}^{\uparrow}(\mathbf{X}_{\mathcal{G}})$ such that for every clopen subset U of $\mathbf{X}_{\mathcal{G}}$ $(\leq \circ E)^{-1}[U] = \rho^{-1}[m(U) \cap K^{\uparrow}(\mathbf{X}_{\mathcal{G}})]$, and since $m(U) \cap K^{\uparrow}(\mathbf{X}_{\mathcal{G}})$ is a clopen subset of $\mathbf{K}^{\uparrow}(\mathbf{X}_{\mathcal{G}})$, it is enough to show that ρ is continuous. Since

$$\mathcal{B}^{\uparrow} = \{t(U) \cap K^{\uparrow}(\mathbf{X}_{\mathcal{G}}) \mid U \text{ clopen up-set } \} \cup \{m(V) \cap K^{\uparrow}(\mathbf{X}_{\mathcal{G}}) \mid V \text{ clopen down-set}\}$$

is a subbase of $\mathbf{K}^{\uparrow}(\mathbf{X}_{\mathcal{G}})$, and the elements in the right-hand side of the union above are just the complements of the elements in its left-hand side, it is enough to show that for every clopen up-set U of $\mathbf{X}_{\mathcal{G}}$, $\rho^{-1}[t(U) \cap K^{\uparrow}(\mathbf{X}_{\mathcal{G}})] = \rho^{-1}[t(U)]$ is a clopen subset of $\mathbf{X}_{\mathcal{G}}$. For every clopen up-set U of $\mathbf{X}_{\mathcal{G}}$, $\rho^{-1}[t(U)] = \{x \in X \mid (\leq \circ E)[x] \subseteq U]\} = \Box_{(\leq \circ E)}U$, which is a clopen up-set by D2. □

Definition 7.10. (Morphism of perfect augmented Kripke frames) (cf. Section 3.1 of [2]) *Let $\mathcal{X}_i = \langle X_i, \leq_i, E_i, \tau_i \rangle$ be perfect augmented Kripke frames, $i = 1, 2$. A continuous map $f : \langle X_1, \tau_1 \rangle \longrightarrow \langle X_2, \tau_2 \rangle$ is a* morphism *iff the following list of conditions is satisfied for every $x, x', y \in X_1$, $z \in X_2$:*

M1. if $x \leq_1 y$ then $f(x) \leq_2 f(y)$.
M2. If $f(x) \leq_2 z$ then $f(x') = z$ for some $x' \in x\uparrow$.
M4'. If $x(\leq_1 \circ E_1)y$ then $f(x)(\leq_2 \circ E_2)f(y)$.
M6'. If $f(x)(\leq_2 \circ E_2)z$ then $z = f(x')$ for some $x' \in (\leq_1 \circ E_1)[x]$.
M5. If $f(x)E_2z$ then $z \leq_2 f(x')$ for some $x' \in E_1[x]$.

Proposition 7.11.

1. *For every morphism $f : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ of perfect augmented Kripke frames, f is a p-morphism between the associated* MIPC*-spaces $\mathcal{G}_{\mathcal{X}_1}$ and $\mathcal{G}_{\mathcal{X}_2}$.*

2. *For every p-morphism $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ of* MIPC*-spaces, f is a morphism between the associated perfect augmented Kripke frames $\mathcal{X}_{\mathcal{G}_1}$ and $\mathcal{X}_{\mathcal{G}_2}$.*

Proof. 1. We have to show the conditions M3, M4 and M6 in 3.12 hold: M3 is equivalent to the continuity of f, and M6' immediately implies M6. Let us show M4, i.e. assume that xE_1y and show that $f(x)E_2f(y)$: By 7.8, it is enough to show that $f(x)(\leq \circ E_2)f(y)$ and $f(x)(E_2 \circ \geq)f(y)$. As $x \leq xE_1y$, then by M4', $f(x)(\leq \circ E_2)f(y)$. As $xE_1y \geq y$, then $x \in (\leq \circ E_1)[y]$ so, by M4', $f(x) \in (\leq \circ E_2)[f(y)]$ i.e. $f(y) \in (\leq \circ E_2)^{-1}[f(x)] = (E_2 \circ \geq)[f(x)]$.
2. We have to show that f is continuous and that M4', M6' in 7.10 hold: M3 is equivalent to continuity, and M4' is easily implied by M1 and M4. Let us show M6': assume that $f(x)(\leq_2 \circ E_2)z$, and show that $z = f(x')$ for some $x' \in (\leq_1 \circ E_1)[x]$. By M6, $f(y) \leq_2 z$ for some $y \in (\leq \circ E_1)[x]$, hence, by M2, $z = f(x')$ for some $x' \in y\uparrow$, and as $y \in (\leq \circ E_1)[x]$, then $x' \in (\leq \circ E_1 \circ \leq)[x] = (\leq \circ E_1)[x]$, the last equality being implied by $(E_1 \circ \leq) \subseteq (\leq \circ E_1)$. □

Putting 7.9 and 7.11 together yields:

Theorem 7.12. MIPCSp *(*MIPC*-spaces and their p-morphisms) and* TPSOE *(perfect augmented Kripke frames and their morphisms) are isomorphic categories.*

7.13 Appendix

The proofs of the facts mentioned in this appendix will appear in a forthcoming paper. Recall that for every $\mathbf{X} = \langle X, \tau \rangle$, the *Vietoris topology* (cf. [13]) τ_v on $K(\mathbf{X})$ is the one generated by taking the following collection as a subbase:

$$\{t(A) \mid A \in \tau\} \cup \{m(A) \mid A \in \tau\},$$

where for every $A \in \tau$, $t(A) = \{F \in K(\mathbf{X}) \mid F \subseteq A\}$ and $m(A) = \{F \in K(\mathbf{X}) \mid F \cap A \neq \emptyset\}$. $\mathbf{K}(\mathbf{X}) = \langle K(\mathbf{X}), \tau_v \rangle$ is the *Vietoris space* of $\mathbf{X}$. It is well-known that for every Stone space $\mathbf{X}$, $\mathbf{K}(\mathbf{X})$ is a Stone space, moreover for every clopen subset U of $\mathbf{X}$, $m(X \setminus U) = K(\mathbf{X}) \setminus t(U)$ and $t(X \setminus U) = K(\mathbf{X}) \setminus m(U)$, hence $t(U)$ and $m(U)$ are clopen subsets of $\mathbf{K}(\mathbf{X})$.

Proposition 7.14. *Esakia spaces are exactly those Priestley spaces $\mathbf{X} = \langle X, \leq, \tau \rangle$ such that $K^{\uparrow}(\mathbf{X})$ is a closed subset of the Vietoris space of $\langle X, \tau \rangle$.*

This implies that, for every Esakia space $\mathbf{X} = \langle X, \leq, \tau \rangle$, the collection $K^{\uparrow}(\mathbf{X})$ of the closed up-sets of $\mathbf{X}$, endowed with the inherited Vietoris topology τ'_v, is a Stone space: So, for every Esakia space $\mathbf{X} = \langle X, \leq, \tau \rangle$, the space $\mathbf{K}^{\uparrow}(\mathbf{X}) = \langle K^{\uparrow}(\mathbf{X}), \supseteq, \tau'_v \rangle$ is an ordered Stone space, that can be shown to be also totally order-disconnected (i.e. a Priestley space). As a byproduct of the proof of this last fact one gets:

Proposition 7.15. *For every Esakia space* $\mathbf{X} = \langle X, \leq, \tau \rangle$*,*

$$\mathcal{B}^{\uparrow} = \{t(U) \cap K^{\uparrow}(\mathbf{X}) \mid U \text{ clopen up-set } \} \cup \{m(V) \cap K^{\uparrow}(\mathbf{X}) \mid V \text{ clopen down-set}\}$$

is a subbase of $\mathbf{K}^{\uparrow}(\mathbf{X})$.

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