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Relational Representation of Groupoid Quantales

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Abstract In Palmigiano and Re (J Pure Appl Algebra 215(8):1945–1957, 2011), spatial SGF-quantales are axiomatically introduced and proved to be representable as sub unital involutive quantales of quantales arising from set groupoids. In the present paper, spatial SGF-quantales of this class are shown to be optimally representable as unital involutive quantales of relations. The results of the present paper have several aspects in common with Jónsson and Tarski's representation theory for relation algebras (Jónsson and Tarski, Am J Math 74(2):127–162, 1952).

Keywords Unital involutive quantale · Strongly Gelfand quantale · Set groupoid · Representation theorem

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1 Introduction

Quantales were introduced [6] as noncommutative generalizations of locales, and have been investigated in close connection with groupoids, e.g. in [7–9, 11, 13]. In particular, in [8] a certain class of unital involutive quantales, called *spatial*

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SGF-quantales, is axiomatically introduced and proved to bijectively correspond to structures based on groupoids. Key to this result is the proof that spatial SGF-quantales are representable as sub unital involutive quantales of quantales arising from set groupoids.

The present paper was motivated by the question whether an optimal relational representation could be given for spatial SGF-quantales, i.e., whether any spatial SGF-quantale could be embedded into the powerset of some equivalence relation in such a way that joins are represented as unions, the product as standard relational composition, the product unit as the identity relation, and the quantale involution as relational converse. Relational representability for quantales has already been studied by some authors in the literature: for instance, in [1] it is shown that every quantale $\mathcal{Q}$ can be order-embedded into a quantale of relations in such a way that the noncommutative product is represented as relational composition, but this method does not extend to involutive quantales and the join of $\mathcal{Q}$ is not in general represented as the union. Similar limitations apply to the results in [16]. In [4], it is shown that any involutive quantale is embeddable into a quantale of join-semilattice endomorphisms, which give back the quantales of relations when the join-semilattice is a powerset.

Quantales and relation algebras are akin: it is well known that both kinds of algebras naturally arise from groupoids. Concrete relation algebras have been identified by Jónsson and Tarski as those embeddable into relation algebras arising from groupoids (cf. [3, Theorem 5.8]). In the light of these facts, the present paper was set out to extend the optimal representation theory of relation algebras, as is developed in [3], to spatial SGF-quantales. The Jónsson–Tarski representation theory is grounded in the theory of canonical extensions. In particular, it is crucial that any relation algebra can be canonically embedded into its perfect, or *canonical extension* [3, Theorem 4.21], which is an *atomistic* relation algebra (i.e. it is join-generated by its atoms). In the present paper, the role of atomistic relation algebras is played by *FA-quantales* (Definition 4.1), and the role of the canonical embedding is played by the map α in Theorem 3.5.

Before moving on, it is worth mentioning[1] that, besides the similarities between quantales and relation algebras, on which the present paper is based, similarities exist between the embedding of Corollary 4.15 and the Gelfand–Naimark embedding [2] of any C^*-algebra A into the algebra of bounded operators $\mathcal{B}(\mathcal{H})$ of some Hilbert space $\mathcal{H}$. For the sake of this analogy, the role of A is played by any SGF-quantale Q, which is associated with some groupoid $\mathcal{G}(\mathcal{Q}) = (G_1, G_0)$, and the role of the von Neumann algebra $\mathcal{B}(\mathcal{H})$ is played by the quantale $\mathcal{P}(G_1)$, into which Q is embedded (cf. Corollary 4.15). This analogy can be even pushed further: indeed, the role that the pure states of the C^* algebra play in the Gelfand–Naimark embedding via the Gelfand–Naimark–Segal construction is similar to the role played by the equivalence classes $[p, f]$ defined right before Definition 3.3 in the present paper. Moreover, the FA-quantales into which SGF-quantales embed, which are used in the present paper as intermediate structures to obtain the representation theorem, can be regarded as the quantale-theoretic counterparts of the enveloping von Neumann algebras of any C^*-algebra A.

[1] We wish to thank the anonymous referee for bringing this to our attention.

Finally, Proposition 4.17 provides a characterization of quantales of the form $\mathcal{P}(R)$ for some equivalence relation R on some set X. Any such quantale can be identified with the cartesian product of the *discrete pair quantales* $\mathcal{P}(X_i \times X_i)$, where $\{X_i : i \in I\}$ is the partition of X induced by R; in the light of this observation, the characterization of Proposition 4.17 establishes a similarity between quantales of the form $\mathcal{P}(R)$ and von Neumann algebras which decompose as direct integrals of type I factors (cf. [15]). Summing up, relative to the quantales and the C^*-algebras bijectively corresponding to groupoids, the following list of systematic analogies emerges:

(Groupoid) C^*-algebras	(Groupoid) quantales
C^*-algebras	SGF-quantales $\mathcal{Q}$
von Neumann algebras	discrete groupoid quantales $\mathcal{P}(\mathcal{G})$
direct integrals of type I factors	relational quantales $\mathcal{P}(R)$
type I factors	discrete pair quantales $\mathcal{P}(X \times X)$.

Developing these analogies further will be the object of future work, as well as the question whether quantale-theoretic counterparts exist to type II and III factors in the theory of von Neumann algebras.

The paper is organized as follows: in Section 2 we give preliminaries on unital involutive quantales and groupoids. In Section 3 we report on the relevant definitions and results from [8], up to Theorem 3.5. Section 4 collects the original contributions of the paper: in Section 4.1, we develop the theory of FA-quantales, and show that they are identifiable with the quantales $\mathcal{P}(\mathcal{G})$ for any groupoid $\mathcal{G}$ (Proposition 4.6); in Section 4.2 we provide a dual characterization of homomorphisms and embeddings between FA-quantales (Propositions 4.7, 4.9, and 4.10). In Sections 4.3 and 4.4, we give two applications of this dual characterization: the first one (Corollary 4.15) is an optimal representation theorem for spatial SGF-quantales as sub unital involutive quantales of quantales of relations; this result is the quantale-theoretic counterpart of Theorem 4.22 in [3]. The second one (Theorem 4.21) is a characterization of those unital involutive quantales which are isomorphic to quantales of the form $\mathcal{P}(R)$ for some equivalence relation R; this result is the quantale-theoretic counterpart of Theorem 4.30 in [3].

2 Preliminaries

2.1 Strongly Gelfand Quantales

A *quantale* $\mathcal{Q}$ [6, 14] is a complete join-semilattice endowed with an associative binary operation $\cdot$ that is completely distributive in each coordinate, i.e.:

D1: $c \cdot \bigvee I = \bigvee\{c \cdot q : q \in I\}$
D2: $\bigvee I \cdot c = \bigvee\{q \cdot c : q \in I\}$

for every $c \in \mathcal{Q}$ and every $I \subseteq \mathcal{Q}$. Since it is a complete join-semilattice, any quantale $\mathcal{Q}$ is also a complete, hence bounded, lattice. Let 0, 1 be the lattice bottom and top of $\mathcal{Q}$, respectively. Conditions D1 and D2 readily imply that $\cdot$ is order-preserving in

both coordinates and, as $\bigvee \varnothing = 0$, that $c \cdot 0 = 0 = 0 \cdot c$ for every $c \in \mathcal{Q}$. A quantale $\mathcal{Q}$ is *unital* if there exists some element $e \in \mathcal{Q}$ for which

U: $e \cdot c = c = c \cdot e$ for every $c \in \mathcal{Q}$,

and is *involutive* if it is endowed with a unary operation $*$ such that, for all $c, q \in \mathcal{Q}$ and every $I \subseteq \mathcal{Q}$,

I1: $c^{**} = c$;
I2: $(c \cdot q)^* = q^* \cdot c^*$;
I3: $(\bigvee I)^* = \bigvee \{q^* : q \in I\}$.

Examples of unital involutive quantales, which will turn out to arise from groupoids, are:

1. The quantale $\mathcal{P}(R)$ of subrelations of a given equivalence relation $R \subseteq X \times X$; the two projections define the maps d and r, composition is given by $(x, y) \cdot (y, z) = (x, z)$, the diagonal map $x \mapsto (x, x)$ gives the unit u, and $i(x, y) := (y, x)$.
2. The quantale $\mathcal{P}(G)$, for every group G. The structure maps are canonically given by the algebraic structure of G.
3. Any frame $\mathcal{Q}$, setting $\cdot := \wedge$, $* := \mathrm{id}$ and $e := 1_{\mathcal{Q}}$.

A *homomorphism* of (involutive) quantales is a map $\varphi : \mathcal{Q} \to \mathcal{Q}'$ that preserves $\bigvee$, $\cdot$ (and $*$). If $\mathcal{Q}$ and $\mathcal{Q}'$ are unital quantales, then φ is *unital* if $e' \leq \varphi(e)$ and is *strictly unital* if $\varphi(e) = e'$. Notice that since every homomorphism is completely join-preserving, then $\varphi(0) = \varphi(\bigvee \varnothing) = \bigvee \varnothing = 0$. However, a homomorphism of quantales does not need to preserve the lattice top. For example, if $R \subsetneq S$ are equivalence relations on X, then the inclusion $\mathcal{P}(R) \to \mathcal{P}(S)$ is a strictly unital homomorphism of quantales that does not preserve the lattice top. If $\varphi(1_{\mathcal{Q}}) = 1_{\mathcal{Q}'}$ then φ is *strong*.

Let $\mathcal{Q}$ be a unital involutive quantale. An element $f \in \mathcal{Q}$ is *functional* if $f^* \cdot f \leq e$ and is a *partial unit* if both f and f^* are functional.[2] The set of functional elements (resp. partial units) will be denoted by $\mathcal{F}(\mathcal{Q})$ (resp. $\mathcal{I}(\mathcal{Q})$). It is easy to verify that $e \in \mathcal{I}(\mathcal{Q})$ and $\mathcal{I}(\mathcal{Q})$ is closed under the composition and involution of $\mathcal{Q}$. Moreover, if $f \leq g \in \mathcal{I}(\mathcal{Q})$ then $f \in \mathcal{I}(\mathcal{Q})$.

Let $\mathcal{Q}_e = \{c \in \mathcal{Q} : c \leq e\}$. Then $\mathcal{Q}_e \subseteq \mathcal{I}(\mathcal{Q})$; moreover, $\mathcal{Q}_e$ is a unital involutive subquantale of $\mathcal{Q}$.

Definition 2.1 A unital involutive quantale $\mathcal{Q}$ is *strongly Gelfand* (or an *SG-quantale*) if

SG. $a \leq a \cdot a^* \cdot a$ for every $a \in \mathcal{Q}$.

Recall that $\mathcal{Q}$ is a *Gelfand* quantale (see also [14]) if $a = a \cdot a^* \cdot a$ for every right-sided element of $\mathcal{Q}$ ($a \in \mathcal{Q}$ being *right-sided* if $a = a \cdot 1$). It is immediate to see that every SG-quantale is Gelfand, and that $f = f \cdot f^* \cdot f$ for every SG-quantale $\mathcal{Q}$ and every $f \in \mathcal{F}(\mathcal{Q})$. We will simplify notation and write $a \cdot b$ as ab.

[2] If $\mathcal{Q} = \mathcal{P}(R)$ for some equivalence relation $R \subseteq X \times X$, then functional elements (partial units) are exactly the graphs of (invertible) partial maps f on X.

Proposition 2.2 *For every SG-quantale $\mathcal{Q}$,*

1. *the subquantale $\mathcal{Q}_e$ is a frame: in particular, involution $*$ coincides with the identity, and composition $\cdot$ with $\wedge$.*
2. *$\mathcal{I}(\mathcal{Q})$ is an inverse monoid*[3] *whose set of idempotents coincides with $\mathcal{Q}_e$, and whose natural order coincides with the order inherited from $\mathcal{Q}$.*

Proof

1. Let $d \leq e$. By SG, we have that $d \leq dd^*d \leq ed^*e = d^*$, and likewise, $d^* \leq d$, hence involution coincides with the identity on $\mathcal{Q}_e$. If $c \leq e$, then $cc = c$: indeed, $cc \leq ce = c$, and by SG and the fact that involution is identity on $\mathcal{Q}_e$, we get that $c = cc^*c = (cc)c \leq (cc)e = cc$. Let $d_1, d_2 \leq e$. Then $d_1d_2 \leq d_1e = d_1$ and $d_1d_2 \leq ed_2 = d_2$, so $d_1d_2 \leq d_1 \wedge d_2$. Conversely, if $c \leq d_1$ and $c \leq d_2$, then $c = cc \leq d_1d_2$, hence $d_1 \wedge d_2 \leq d_1d_2$.
2. By SG, we get that $ff^*f = f$ and $f^*ff^* = f^*$ for every $f \in \mathcal{I}(\mathcal{Q})$. Hence, it is enough to show that the restriction of the product to the idempotent elements of $\mathcal{I}(\mathcal{Q})$ is commutative. This follows from item 1 above and from the fact that for every $f \in \mathcal{I}(\mathcal{Q})$, it holds that $ff = f$ iff $f \leq e$: Indeed, if $f \leq e$, then by (1), we have that $ff = f \wedge f = f$. Conversely, if $ff = f$, then $f^* = (ff)^* = f^*f^*$, hence $ff^* = ff^*f^* \leq ef^* = f^*$, and so $f = ff^*f \leq f^*f \leq e$. Since $\mathcal{Q}_e \subseteq \mathcal{I}(\mathcal{Q})$, this also shows that the set of idempotent elements of $\mathcal{I}(\mathcal{Q})$ coincides with $\mathcal{Q}_e$. Hence, the natural order of the inverse monoid $\mathcal{I}(\mathcal{Q})$ is defined as follows: $f \leq g$ iff $f = gh$ for some $h \in \mathcal{Q}_e$, and therefore it coincides with the order inherited from $\mathcal{Q}$. □

2.2 Groupoids

Definition 2.3 A *set groupoid* is a tuple $\mathcal{G} = (G_0, G_1, m, d, r, u, i)$, s.t.:

G1. G_0 and G_1 are sets;
G2. $d, r : G_1 \to G_0$ and $u : G_0 \to G_1$ s.t. $d(u(p)) = p = r(u(p))$ for every $p \in G_0$;
G3. $m : (x, y) \mapsto xy$ is an associative map defined on $G_1 \times_{G_0} G_1 = \{(x, y) \mid r(x) = d(y)\}$ and s.t. $d(xy) = d(x)$ and $r(xy) = r(y)$;
G4. $xu(r(x))) = x = u(d(x))x$ for every $x \in G_1$;
G5. the map $i : G_1 \to G_1$ denoted by $i(x) = x^{-1}$ is s.t. $xx^{-1} = u(d(x))$, $x^{-1}x = u(r(x))$, $d(x^{-1}) = r(x)$ and $r(x^{-1}) = d(x)$ for every $x \in G_1$.

$$\begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "m"] & G_1 \arrow[loop above, "i"] \arrow[r, bend left, "d"] \arrow[r, bend right, "r"'] & G_0 \arrow[l, "u"'] \end{tikzcd}$$

Axioms G1-5 equivalently say that groupoids are small categories (G_0 being the set of objects, G_1 being the set of arrows, u being the injective map which identifies

[3] An *inverse semigroup* (cf. [10]) is a semigroup such that for every element x there exists a unique inverse, i.e. an element y such that $x = xyx$ and $y = yxy$. Equivalently, an inverse semigroup is a semigroup such that every element has some inverse and any two idempotent elements commute. An *inverse monoid* is an inverse semigroup with a multiplicative unit. The natural order on an inverse semigroup is defined by $x \leq y$ iff $x = e \cdot y$ for some idempotent e.

any object with its identity arrow, etc) such that every arrow is an iso. Then a *morphism* of groupoids $f : \mathcal{G} \to \mathcal{G}'$ ($\mathcal{G}'$ has structure maps d', r', etc) is a functor from $\mathcal{G}$ to $\mathcal{G}'$. In other words, a morphism f consists of a pair of functions $f_1 : G_1 \to G'_1$ and $f_0 : G_0 \to G'_0$ that preserve the structure maps of the groupoids in the appropriate way; that is, for every $p \in G_0$ and all $x, y \in G_1$:

$$f_0(d(x)) = d'(f_1(x)), \quad f_0(r(x)) = r'(f_1(x)),$$
$$f_1(u(p)) = u'(f_0(p)), \quad f_1(xy) = f_1(x)f_1(y).$$

Clearly, the map f_0 is completely determined by f_1, so f can be identified with f_1. Groupoid morphisms preserve inverses: $f(x^{-1}) = f(x)^{-1}$ (to see this, use Lemma 2.5.4 below). An isomorphism of groupoids $f : \mathcal{G} \to \mathcal{G}'$ is a morphism which has an inverse functor $f^{-1} : \mathcal{G}' \to \mathcal{G}$. The groupoids $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic if there is an isomorphism $f : \mathcal{G} \to \mathcal{G}'$.

Example 2.4

1. For any equivalence relation $R \subseteq X \times X$, the tuple $(X, R, \circ, \pi_1, \pi_2, \Delta, ()^{-1})$ defines a groupoid. Conversely, for every groupoid $\mathcal{G}$, if the product map $d \times r : G_1 \to G_0 \times G_0$ is injective, then $\mathcal{G}$ is isomorphic to the groupoid arising from the equivalence relation $(d \times r)[G_1] \subseteq G_0 \times G_0$ (cf. [12], Example I.1.3). Groupoids arising from the relation $R = X \times X$ for some set X are referred to as *pair groupoids*.
2. For any group $(G, \cdot, e, ()^{-1})$, the tuple $(\{e\}, G, \cdot, d, r, u, ()^{-1})$ is a groupoid, and the equalities G4 and G5 just restate the group axioms.
3. Every topological space X can be seen as a groupoid by setting $G_1 = G_0 = X$ and identity structure maps. In this case, $G_1 \times_{G_0} G_1 = \{(x, x) \mid x \in X\}$ and $xx = x$ for every $x \in X$.
4. A groupoid can be associated with any action[4] $G \times X \to X$ of a group G on a set X, by setting $G_1 = G \times X$, $G_0 = X$, and for all $g, h \in G$ and $x, y \in X$, $d(g, x) = x$, $r(g, x) = gx$, $u(x) = (e, x)$ ($e \in G$ being the identity element), and $(g, x) \cdot (h, y) = (hg, x)$ whenever $y = gx$.
5. To a group action as above, another groupoid can be associated, which is given by the equivalence relation $R \subseteq X \times X$ defined by xRy iff there exists some $g \in G$ such that $y = gx$.

Some useful well known facts about groupoids are reported in the following:

Lemma 2.5 *For every $p \in G_0$, and all $x, y \in G_1$,*

1. $u(p)^{-1} = u(p)$,
2. $x = xx^{-1}x$ *and* $x^{-1} = x^{-1}xx^{-1}$,
3. *if* $xy^{-1}, x^{-1}y \in u[G_0]$ *then* $x = y$,
4. *if* $x = xyx$ *and* $yxy = y$, *then* $y = x^{-1}$,
5. $(x^{-1})^{-1} = x$,
6. $(xy)^{-1} = y^{-1}x^{-1}$.

[4]For any group G, a *(left) action* of G on a set X is a function $\cdot : G \times X \to X$ s.t. for all $g, h \in G$ and every $x \in X$, the identities $(gh)x = g(hx)$ and $ex = x$ hold (e being the identity of G). For any topological group G and any topological space X, a *continuous action* of G on X is a continuous map $G \times X \to X$ which is an action of G as a discrete group G on the underlying set of X.

Proof The proof is routine; we just show item 2: indeed, y G5 and G4, $xx^{-1}x = u(d(x))x = x$. The second identity is shown analogously. □

Every set groupoid $\mathcal{G}$ can be naturally associated with a unital involutive quantale $\mathcal{P}(\mathcal{G})$, based on the complete join-semilattice $(\mathcal{P}(G_1), \bigcup)$ (see [12] and [13, Section 1.1] for a more detailed discussion): indeed, the product and involution on G_1 can be lifted to $\mathcal{P}(G_1)$ as follows:

$$S \cdot T = \{x \cdot y \mid x \in S,\ y \in T \text{ and } r(x) = d(y)\} \qquad S^* = \{x^{-1} \mid x \in S\}.$$

Let E be the image of the structure map $u : G_0 \to G_1$.

Definition 2.6 For every set groupoid $\mathcal{G}$, let $\mathcal{P}(\mathcal{G}) = (\mathcal{P}(G_1), \bigcup, \cdot, ()^*, E)$ be its associated *discrete groupoid quantale*.

Fact 2.7 *$\mathcal{P}(\mathcal{G})$ is an SG-quantale for every set groupoid $\mathcal{G}$.*

Proof The proof is routine: for instance let us show that $S \cdot E = S = E \cdot S$ for every $S \in \mathcal{P}(G_1)$: indeed, one inclusion immediately follows from G4, and conversely if $x \cdot u(p)$ is defined, then, by G2, we have that $r(x) = d(u(p)) = p$, and again by G4, we get $x \cdot u(p) = x \cdot u(r(x)) = x$.

The axiom SG follows from Lemma 2.5.2. □

3 Spatial SGF Quantales and their Representation

In [8], spatial SGF quantales are axiomatically introduced and proved to be representable as sub involutive quantales of $\mathcal{P}(\mathcal{G})$ for some groupoid $\mathcal{G}$. In this section, we are going to report on these results, referring to [8] for the proofs.

Definition 3.1 (cf. Definition 4.1 in [8]) A *strongly Gelfand and functionally-generated quantale* (abbreviated as *SGF quantale*) is a unital involutive quantale $\mathcal{Q}$ satisfying the following extra axioms:

SGF1. $\mathcal{Q}$ is $\bigvee$-generated by $\mathcal{I}(\mathcal{Q})$.
SGF2. $f = ff^*f$ for every $f \in \mathcal{I}(\mathcal{Q})$.
SGF3. For any $f, g \in \mathcal{I}(\mathcal{Q})$ and $h \in \mathcal{Q}_e$ if $f \leq h \cdot 1 \vee g$ then $f \leq h \cdot f \vee g$.

Axioms SGF1–2 readily imply that every SGF-quantale is SG. Let $\mathcal{Q}$ be an SGF-quantale. For every $f \in \mathcal{I}(\mathcal{Q})$ let $d(f) = ff^*$ and $r(f) = f^*f$. Let $\mathcal{P}_e$ be the set of the *prime* elements of $\mathcal{Q}_e$ (cf. [5]) i.e. those non-top elements $p \in \mathcal{Q}_e$ s.t. for every $h, k \in \mathcal{Q}_e$, if $h \wedge k \leq p$, then $h \leq p$ or $k \leq p$. Let

$$\mathcal{I} = \{(p, f) \in \mathcal{P}_e \times \mathcal{I}(\mathcal{Q}) \mid p \in \mathcal{P}_e,\ d(f) \not\leq p\}.$$

Informally, the prime elements of $\mathcal{Q}_e$ can be thought of as the points (or rather, as the closures of points) of a topological space of which $\mathcal{Q}_e$ is the topology; elements $f \in \mathcal{I}(\mathcal{Q})$ can be thought of as partial homeomorphisms $f : d(f) \to r(f)$ between open sets of this space. Under this interpretation, the element $e \in \mathcal{I}(\mathcal{Q})$ is the identity homeomorphism, any $h \in \mathcal{Q}_e$ is both an open set of the space and the restriction of

the identity homeomorphism to the open set h, and $f^* : r(f) \to d(f)$ is the inverse of f. Moreover, the inequality $d(f) \not\leq p$ means that the point p belongs to the domain of f, hence the set $\mathcal{I}$ is essentially the space in which the operation 'f applied to p' is defined. Lemma 4.7 and Proposition 4.8 in [8] describe the behaviour of these definitions, which, under the interpretation given above, is just the expected one. For instance, (cf. [8], Lemma 4.7.1) for every $(p, f) \in \mathcal{I}$, there exists a unique element $q \in \mathcal{P}_e$ s.t. $r(f) \not\leq q$ and $pf = fq$. This prime element is denoted by $f[p]$, and represents the point '$f(p)$' (or rather, the f-direct image of the closure of p). The following definition is meant to express, in the language of SGF-quantales, that '$f(p) = g(p)$' for two such given partial homeomorphisms (or rather, that f and g coincide over the intersection of some open neighbourhood h of p with the closure of p):

Definition 3.2 (cf. Definition 4.5 in [8]) The *incidence relation* $\sim$ on $\mathcal{I}$ is defined by setting

$$(p, f) \sim (q, g) \text{ iff } p = q \text{ and } h \not\leq p \text{ and } hf \leq pf \vee g \text{ for some } h \leq d(f) \wedge d(g).$$

We will also alternatively write $f \sim_p g$ (read: f and g are incident in p) in place of $(p, f) \sim (q, g)$.

The relation $\sim$ is an equivalence relation (cf. Proposition 4.6.1 in [8]), so we can consider the quotient set $\mathcal{I}/\!\sim$. Elements in $\mathcal{I}/\!\sim$ will be denoted by $[p, f]$, $[q, g]$, etc.

Definition 3.3 (cf. Definition 4.9 in [8]) For every SGF-quantale $\mathcal{Q}$, its associated set groupoid $\mathcal{G}(\mathcal{Q})$ is defined as follows (see discussion above for notation): $G_0 = \mathcal{P}_e$ and $G_1 = \mathcal{I}/\!\sim$; moreover, the structure maps of $\mathcal{G}(\mathcal{Q})$ are given by the following assignments:

$$d([p, f]) = p, \quad r([p, f]) = f[p], \quad u(p) = [p, e],$$

$$[p, f][q, g] = [p, fg] \quad \text{only if} \quad q = f[p]$$

$$[p, f]^{-1} = [f[p], f^*].$$

For every SGF-quantale $\mathcal{Q}$, the structure $\mathcal{G}(\mathcal{Q})$ is indeed a set groupoid (cf. Proposition 4.11 in [8]). The SGF axiomatization of Definition 3.1 is constructive, i.e., it does not imply the 'existence of enough points'; its point-set strengthening is given below:

Definition 3.4 (cf. Definition 5.1 in [8]) For every SGF-quantale $\mathcal{Q}$ and every $[p, f] \in \mathcal{I}/\!\sim$, let

$$\mathcal{I}_{[p,f]} = \{g \in \mathcal{I}(\mathcal{Q}) \mid d(g) \leq p \text{ or } (p, g) \not\sim (p, f)\} \quad \text{and} \quad I_{[p,f]} = \bigvee \mathcal{I}_{[p,f]}.$$

$\mathcal{Q}$ is *spatial* if:

SPQ1. $f \not\leq I_{[p,f]}$ for every $(p, f) \in \mathcal{I}$.
SPQ2. $a = \bigwedge\{I_{[p,f]} \mid a \leq I_{[p,f]}\}$ for every $a \in \mathcal{Q}$.

3.1 The Canonical Embedding

For every SGF-quantale $\mathcal{Q}$, let $\mathcal{P}(\mathcal{G}(\mathcal{Q})))$ be defined according to Definitions 3.3 and 2.6, and let $\alpha : \mathcal{Q} \to \mathcal{P}(\mathcal{G}(\mathcal{Q})))$ be defined by

$$\alpha(a) = \{[p, f] \mid a \not\leq I_{[p,f]}\}.$$

Theorem 3.5 (cf. Theorem 6.2 in [8]) *For every SGF-quantale $\mathcal{Q}$,*

1. $\alpha(\bigvee_i a_i) = \bigcup_i \alpha(a_i)$ *for any family* $\{a_i | i \in I\}$ *of elements of* $\mathcal{Q}$.
2. *if* $\mathcal{Q}$ *is spatial, then* α *is an embedding.*
3. $\alpha(ab) = \alpha(a)\alpha(b)$ *for any* $a, b \in \mathcal{Q}$.
4. $\alpha(a^*) = \alpha(a)^*$ *for any* $a \in \mathcal{Q}$.
5. $\alpha(1) = G_1$ *and* $\alpha(e) = u[G_0]$.

So, if $\mathcal{Q}$ is spatial, then α is a strong and strictly unital embedding of unital involutive quantales.

4 Representation Theory

As mentioned early on, the map α in Theorem 3.5 above plays the same crucial role as the embedding of a relation algebra into its perfect extension plays in [3]. The reason why this is important is that perfect extensions enjoy the additional property of being atomistic, i.e., join generated by their atoms. In the present paper, *FA-quantales* (Definition 4.1 below) are the quantale counterparts of atomistic relation algebras. In the two following subsections, we develop the theory of FA-quantales, show that they are identifiable with the quantales $\mathcal{P}(\mathcal{G})$ for any groupoid $\mathcal{G}$ (cf. Proposition 4.6), and provide a dual characterization of homomorphisms of FA-quantales and embeddings of FA-quantales which preserve finite meets (cf. Propositions 4.7, 4.9 and 4.10).

4.1 FA-quantales and Set Groupoids

For every quantale $\mathcal{Q}$, let $At(\mathcal{Q})$ denote the collection of the atoms[5] of $\mathcal{Q}$.

Definition 4.1 A *functionally atomic quantale* (abbreviated as *FA-quantale*) is an SG-quantale $\mathcal{L}$ s.t. its underlying complete join-semilattice is a complete atomic boolean algebra, and

FA. $At(\mathcal{L}) \subseteq \mathcal{F}(\mathcal{L})$.

It immediately follows from the definition that $a = aa^*a$ for every $a \in At(\mathcal{L})$. For every quantale $\mathcal{Q}$, if $a \in At(\mathcal{Q})$, then $a^* \in At(\mathcal{Q})$: indeed, the following chain $0^* = 0 \neq b \leq a^*$ implies that $0^{**} = 0 \neq b^* \leq a^{**} = a$, which implies that $b^* = 0$ (hence $b = 0$), or $b^* = a$ (hence $b = a^*$). This immediately implies that $At(\mathcal{L}) \subseteq \mathcal{I}(\mathcal{L})$ for every FA-quantale $\mathcal{L}$. For every $c \in \mathcal{L}$, let $At(c) = \{a \in At(\mathcal{L}) \mid a \leq c\}$.

[5]Recall that an *atom* $a \in \mathcal{Q}$ is a non-zero element such that, for every $b \in \mathcal{Q}$, if $0 \leq b \leq a$ then $b = 0$ or $b = a$.

Proposition 4.2 *For every groupoid $\mathcal{G}$, the quantale $\mathcal{P}(\mathcal{G})$ is an FA-quantale.*

Proof By Fact 2.7, the quantale $\mathcal{P}(\mathcal{G})$ is an SG-quantale, and its underlying complete join-semilattice is clearly a complete atomic boolean algebra. Atoms $a \in \mathcal{P}(\mathcal{G})$ are of the form $\{x\}$ for $x \in G_1$. By Definition 2.6 and G5, we have that $a^*a = \{x\}^* \cdot \{x\} = \{x^{-1}\} \cdot \{x\} = \{x^{-1}x\} = \{u(r(x))\} \subseteq u[G_0] = e_{\mathcal{P}(\mathcal{G})}$. □

Hence, the quantale $\mathcal{P}(\mathcal{G}(\mathcal{Q}))$ is an FA-quantale for every SGF-quantale $\mathcal{Q}$.

Proposition 4.3 *For every FA-quantale $\mathcal{L}$ and every $a \in At(\mathcal{L})$,*

1. *if $f \in \mathcal{I}(\mathcal{L})$, then either $af = 0$ or $af \in At(\mathcal{L})$;*
2. *if $b \in At(\mathcal{L})$, then either $ab = 0$ or $ab \in At(\mathcal{L})$;*
3. *$aa^*, a^*a \in At(\mathcal{L})$;*
4. *$At(\mathcal{L}_e) = \{aa^* \mid a \in At(\mathcal{L})\}$.*

Proof

1. If $af \neq 0$, then there exists some $b \in At(af)$ such that $bf^* \neq 0$: indeed, otherwise $aff^* = \bigvee\{cf^* \mid c \in At(af)\} = 0$ and so $af = aff^*f = 0$. Hence we get that $0 \neq bf^* \leq aff^* \leq a$; since $a \in At(\mathcal{L})$, this implies that $bf^* = a$; hence we get that $0 \neq af = bf^*f \leq b \in At(af)$, and so we conclude that $af = b \in At(\mathcal{L})$.
2. Immediate from item 1 and the observation that $At(\mathcal{L}) \subseteq \mathcal{I}(\mathcal{L})$.
3. Immediate from item 2 and the observation that either $a^*a = 0$ or $aa^* = 0$ imply that $a = aa^*a = 0$.
4. The right-to-left inclusion immediately follows from item 3, the assumption FA, and $At(\mathcal{L}_e) = At(\mathcal{L}) \cap \mathcal{L}_e$. The converse direction follows from the latter identity and the fact that, by Proposition 2.2.2, it follows that $aa^* = a$ for every $a \in \mathcal{L}_e$. □

Since the complete join-semilattice reduct of any FA-quantale is a complete atomic boolean algebra, the meet in FA-quantales is frame-distributive.[6] By frame-distributivity, it is easy to show that, for any FA-quantale $\mathcal{L}$, the atoms of $\mathcal{L}$, being completely join-irreducible, are completely join-prime.[7] This implies for instance that

Lemma 4.4 *For every FA-quantale $\mathcal{L}$, and all $x, y \in \mathcal{L}$,*

1. *$At(xy) = \{a'b' \mid a'b' \neq 0, a' \in At(x)$ and $b' \in At(y)\}$;*
2. *$At(x^*) = \{a^* \mid a \in At(x)\}$.*

Proof (1) By Proposition 4.3.2, if $a' \in At(x), b' \in At(y)$ and $a'b' \neq 0$, then $a'b' \in At(xy)$. As to the left-to-right inclusion, if $c \in At(\mathcal{L})$, then c is completely join-prime; so $c \leq xy = \bigvee At(x) \bigvee At(y) = \bigvee\{a'b' \mid a'b' \neq 0, a' \in At(x)$ and $b' \in At(y)\}$ implies that $0 \neq c \leq a'b'$ for some $a' \in At(x)$ and $b' \in At(y)$. Since $a'b'$ is an atom by Proposition 4.3.2, we conclude that $c = a'b'$. The proof of (2.) is similar. □

[6] In fact, every FA-quantale is an inverse quantal frame (cf. [13]); moreover, since inverse quantal frames are SGF-quantales (cf. [8]), FA-quantales form a special subclass of SGF-quantales.

[7] A non-bottom element $c \in \mathcal{Q}$ is *completely join-irreducible* if, for every $S \subseteq \mathcal{Q}$, if $c = \bigvee S$ then $c \in S$; such a c is *completely join-prime* if, for every $S \subseteq \mathcal{Q}$, if $c \leq \bigvee S$ then $c \leq s$ for some $s \in S$.

Definition 4.5 For every FA-quantale $\mathcal{L}$, let

$$\mathcal{G}(\mathcal{L}) = (At(\mathcal{L}_e), At(\mathcal{L}), \cdot, d, r, u, ()^{-1}),$$

where $d(a) = aa^*$ and $r(a) = a^*a$, the map u is the inclusion $At(\mathcal{L}_e) \subseteq At(\mathcal{L})$, the involution is given by $a^{-1} := a^*$ and $\cdot$ is the restriction of the quantale product, wherever it is non zero.

Proposition 4.6 *For every FA-quantale $\mathcal{L}$, and every groupoid $\mathcal{G}$,*

1. *The structure $\mathcal{G}(\mathcal{L})$ is a groupoid, and the map $c \mapsto At(c)$ defines a isomorphism $\mathcal{L} \cong \mathcal{P}(\mathcal{G}(\mathcal{L}))$.*
2. *The map $x \mapsto \{x\}$ defines an isomorphism $\mathcal{G} \to \mathcal{G}(\mathcal{P}(\mathcal{G}))$.*

Proof

1. The proof that Axioms G1-5 hold is straightforward. By atomicity of $\mathcal{L}$, join-primeness of the atoms, and Lemma 4.4, it is not difficult to show that the assignments $c \mapsto At(c)$ for every $c \in \mathcal{L}$ and $S \mapsto \bigvee S$ for every $S \subseteq At(\mathcal{L})$ are inverse to one another and define strictly unital homomorphisms of unital involutive quantales.
2. Immediate. □

4.2 Dual Characterization of Strictly Unital Homomorphisms of FA-quantales

The correspondence between FA-quantales and set groupoids given in the subsection above can be extended and framed in the context of a duality between the appropriate categories. In this section, we do not cover it in detail, and we limit ourselves to presenting those of its components which are relevant to the purposes of the present paper; the most important of these components is the dual characterization of the strictly unital homomorphisms of FA-quantales.

For any completely join-preserving map $h : \mathcal{L} \to \mathcal{L}'$, define the relation $R_h \subseteq At(\mathcal{L}') \times At(\mathcal{L})$ by setting $a' R_h a$ iff $a' \leq h(a)$, and let $R_h^{-1}[a] := \{a' \mid a' R_h a\}$.[8]

Proposition 4.7 *The following are equivalent for any completely join-preserving map $h : \mathcal{L} \to \mathcal{L}'$:*

1. *the map h is a strictly unital homomorphism of FA-quantales;*
2. *the relation R_h satisfies the following properties for all $a, b \in At(\mathcal{L})$:*

$$At(\mathcal{L}'_e) = \bigcup \left\{ R_h^{-1}[a] \mid a \in At(\mathcal{L}_e) \right\}, \quad R_h^{-1}\left[a^*\right] = \left\{ a'^* \mid a' \in R_h^{-1}[a] \right\},$$

$$R_h^{-1}[ab] = \left\{ a'b' \mid a'b' \neq 0,\ a' \in R_h^{-1}[a] \text{ and } b' \in R_h^{-1}[b] \right\}.$$

[8]This notation is standard, and is unrelated to the notions and notation of Section 3.

Proof The fact that h is completely join-preserving implies that h preserves products iff h preserves products restricted to atoms. Indeed, assuming the latter, we have, for all $x, y \in \mathcal{L}$:

$$\begin{aligned} h(x)h(y) &= \bigvee\{h(a)h(b) \mid a \in At(x), b \in At(y)\} \\ &= \bigvee\{h(ab) \mid ab \neq 0, a \in At(x), b \in At(y)\} \\ &= h\left(\bigvee\{ab \mid ab \neq 0, a \in At(x), b \in At(y)\}\right) \\ &= h\left(\bigvee At(xy)\right) = h(xy). \qquad \text{(Lemma 4.4.1)} \end{aligned}$$

Then by atomicity, for all $a, b \in At(\mathcal{L})$,

$$\begin{aligned} h(ab) &= h(a)h(b) \\ &\text{iff} \quad \forall c \in At(\mathcal{L}')[c \leq h(ab) \;\Leftrightarrow\; c \leq h(a)h(b)] \\ &\text{iff} \quad \forall c \in At(\mathcal{L}')[c \in R_h^{-1}[ab] \;\Leftrightarrow\; c = a'b' \text{ for some } a' \leq h(a), b' \leq h(b)] \\ &\text{iff} \quad R_h^{-1}[ab] = \{a'b' \mid a'b' \neq 0,\; a' \in R_h^{-1}[a] \text{ and } b' \in R_h^{-1}[b]\}. \end{aligned}$$

Likewise, using Lemma 4.4.2, one shows that h preserves involution iff h preserves involution restricted to atoms. Then by atomicity, for every $a \in At(\mathcal{L})$,

$$\begin{aligned} & h(a^*) = h(a)^* \\ \text{iff} \quad & \forall c \in At(\mathcal{L}')[c \leq h(a^*) \;\Leftrightarrow\; c \leq h(a)^*] \\ \text{iff} \quad & \forall c \in At(\mathcal{L}')[c \in R_h^{-1}[a^*] \;\Leftrightarrow\; c = a'^* \text{ for some } a' \leq h(a)] \\ \text{iff} \quad & R_h^{-1}[a^*] = \left\{a'^* \mid a' \in R_h^{-1}[a]\right\}. \end{aligned}$$

Finally, by atomicity,

$$\begin{aligned} e' = h(e) \quad \text{iff} \quad & \forall c \in At(\mathcal{L}')\left[c \leq e' \;\Leftrightarrow\; c \leq h(e) = \bigvee\{h(a) \mid a \in At(\mathcal{L}_e)\}\right] \\ \text{iff} \quad & \forall c \in At(\mathcal{L}')\left[c \in At(\mathcal{L}'_e) \;\Leftrightarrow\; c \in \bigcup\left\{R_h^{-1}[a] \mid a \in At(\mathcal{L}_e)\right\}\right] \\ \text{iff} \quad & At(\mathcal{L}'_e) = \bigcup\left\{R_h^{-1}[a] \mid a \in At(\mathcal{L}_e)\right\}. \end{aligned}$$

□

Conversely, if $\mathcal{G}'$ and $\mathcal{G}$ are groupoids, and $R \subseteq G'_1 \times G_1$, then the assignment $c \mapsto \bigvee\bigcup\{R^{-1}[a] \mid a \in At(c)\}$ defines a completely join-preserving map $h_R : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{G}')$.

The following proposition essentially says that the obvious defining assignments for the unit and the counit of the dual equivalence mentioned at the beginning of the present subsection yield indeed natural transformations, in the appropriate categories:

Proposition 4.8 *For every completely join-preserving map $h : \mathcal{L} \to \mathcal{L}'$, and every $R \subseteq G'_1 \times G_1$,*

1. *the isomorphisms $\mathcal{L} \cong \mathcal{P}(\mathcal{G}(\mathcal{L}))$ and $\mathcal{L}' \cong \mathcal{P}(\mathcal{G}(\mathcal{L}'))$ of Proposition* 4.6.1 *identify h with h_{R_h},*
2. *the isomorphisms $\mathcal{G} \cong \mathcal{G}(\mathcal{P}(\mathcal{G}))$ and $\mathcal{G}' \cong \mathcal{G}(\mathcal{P}(\mathcal{G}'))$ of Proposition* 4.6.2 *identify the relation R with R_{h_R}.*

Proof

1. Modulo identifying $\mathcal{P}(\mathcal{G}(\mathcal{L}))$ with $\mathcal{L}$ and $\mathcal{P}(\mathcal{G}(\mathcal{L}'))$ with $\mathcal{L}'$ (via the identification of $\{x\}$ with x, cf. Proposition 4.6.2),

$$\begin{aligned} h_{R_h}(c) &= \bigvee \bigcup \left\{ R_h^{-1}[x] \mid x \in At(c) \right\} \\ &= \bigvee \left\{ \bigvee R_h^{-1}[x] \mid x \in At(c) \right\} \\ &= \bigvee \{h(x) \mid x \in At(c)\} = h(c). \end{aligned}$$

2. Notice that if $c \in At(\mathcal{L})$, then $h_R(c) = R^{-1}[c]$. Modulo identifying $\mathcal{G}(\mathcal{P}(\mathcal{G}'))$ with $\mathcal{G}'$ and $\mathcal{G}(\mathcal{P}(\mathcal{G}))$ with $\mathcal{G}$ as before, we have:

$$x' R_{h_R} x \text{ iff } \{x'\} \subseteq h_R(\{x\}) = R^{-1}[x] \text{ iff } x' R x.$$

□

As a consequence of Propositions 4.7 and 4.8, we get:

Proposition 4.9 *The following are equivalent for $R \subseteq G'_1 \times G_1$:*

1. *$R \subseteq G'_1 \times G_1$ satisfies the following properties:*

   R1. $u'[G'_0] = \bigcup\{R^{-1}[u(p)] \mid p \in G_0\}$,
   R2. $R^{-1}[i(x)] = \{i'(x') \mid x' \in R^{-1}[x]\}$,
   R3. $R^{-1}[xy] = \{x'y' \mid x' \in R^{-1}[x]$ *and* $y' \in R^{-1}[y]\}$;

2. *$h_R : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{G}')$ is a strictly unital homomorphism of FA-quantales.*

Proposition 4.10 *The following are equivalent for $R \subseteq G'_1 \times G_1$:*

1. *R is the graph of a partial function onto G_1, i.e. $\pi_2[R] = G_1$, and if $a' R b$ and $a' R c$ then $b = c$;*
2. *$h_R : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{G}')$ is an order embedding which preserves finite meets.*

Proof In order to prove (2.) from (1.), it is enough to show that $h_R(x) \subseteq h_R(y)$ implies that $x \leq y$ for $x, y \in \mathcal{P}(\mathcal{G})$. By atomicity, it is enough to show that $At(x) \subseteq At(y)$. If $c \in At(x)$, then there exists some $a' \in G'_1$ such that $a' \in R^{-1}[c]$. The assumption $h_R(x) \subseteq h_R(y)$ implies that $a' \in R^{-1}[b]$ for some $b \in At(y)$. Then, by the second part of assumption (1.), we conclude that $c = b \in At(y)$. The remaining bit follows from the observation that inverse images of functions preserve intersections. In order

to prove (1.) from (2.), by injectivity we get that $R^{-1}[a] = h_R(a) \neq 0$ for every atom a, which proves that R is onto. If $a' Rb$ and $a' Rc$, then $a' \in R^{-1}[b] \cap R^{-1}[c]$, hence we have that $0 \neq h_R(b) \wedge h_R(c) = h_R(b \wedge c)$. By injectivity, this implies that $b \wedge c \neq 0$, and since b and c are atoms, this implies that $b = c$. □

Remark 4.11 Although this is not immediately relevant to the purposes of the present paper, it is worth mentioning that the category-theoretic perspective outlined in the present subsection can be further extended from a dual equivalence between appropriate categories of FA-quantales and set-groupoids to a dual adjunction between appropriate categories of SGF-quantales and set groupoids; this is due to the fact that, as we mentioned early on at the beginning of Section 4.1, FA-quantales form a subclass of SGF-quantales. The dual adjunction is defined on objects by the assignments $\mathcal{Q} \mapsto \mathcal{G}(\mathcal{Q})$ (cf. Definition 3.3) for every SGF-quantale $\mathcal{Q}$, and $\mathcal{G} \mapsto \mathcal{P}(\mathcal{G})$ for every set-groupoid $\mathcal{G}$. The counit of this adjunction is given by the canonical embedding $\alpha : \mathcal{Q} \mapsto \mathcal{P}(\mathcal{G}(\mathcal{Q}))$ of Theorem 3.5.

4.3 Relational Representations for Spatial SGF-quantales

The dual characterization of strictly unital homomorphisms of FA-quantales of the previous subsection can be used to obtain an optimal relational representation for spatial SGF-quantales (cf. Corollary 4.15).

Indeed, let $\mathcal{L}$ be an FA-quantale, let $\mathcal{G} = \mathcal{G}(\mathcal{L})$, and $\mathcal{G}'$ be the pair groupoid arising from $G_1 = At(\mathcal{L})$ (cf. Example 2.4.1); hence $G'_1 = G_1 \times G_1$ and $G'_0 = G_1$. Consider the relation $R \subseteq G'_1 \times G_1$ such that $R = graph(n)$, where $n : G_1 \times G_1 \to G_1$ is the partial map defined by $n(a, b) = ab^*$. Notice that, the SG condition implies that $c = cc^*c = ab^*$ with $a = c$ and $b = c^*c$ for every $c \in At(\mathcal{L})$, hence n is onto G_1.

Then we have that $R^{-1}[c] = n^{-1}[c] = \{(a, b) \mid ab^* = c\}$ for every $c \in G_1$. Identifying $\mathcal{L}$ with $\mathcal{P}(\mathcal{G})$, by Proposition 4.10, the completely join-preserving map $h_R : \mathcal{L} \to \mathcal{P}(G_1 \times G_1)$, defined by

$$h_R(x) = \{R^{-1}[c] \mid c \in At(x)\} = \{(a, b) \mid a, b \in At(\mathcal{L}) \text{ and } 0 \neq ab^* \leq x\},$$

is an order embedding.

Lemma 4.12 *for every FA-quantale $\mathcal{L}$, every $x \in \mathcal{L}$ and all $a, b \in At(\mathcal{L})$,*

$$0 \neq ab^* \leq x \text{ if and only if } a \leq xb.$$

Proof Assume that $0 \neq ab^* \leq x$. By SG, $0 \neq ab^* = ab^*bb^*$, hence $ab^*b \neq 0$. Then, the element b being functional implies that $0 \neq ab^*b \leq a$; since a is an atom, this implies that $a = ab^*b \leq xb$. Conversely, if $a \leq xb$, then $ab^* \leq xbb^* \leq x$. Moreover, we have that $0 \neq a = aa^*a$ implies $0 \neq aa^* \leq xba^*$, hence $xba^* \neq 0$, and so $ba^* \neq 0$, i.e. $ab^* \neq 0$. □

In view of the Lemma above, the embedding h_R can be equivalently defined by

$$h_R(x) = \{(a, b) \mid a, b \in At(\mathcal{L}) \text{ and } a \leq xb\}.$$

Proposition 4.13 *h_R is a strictly unital quantale homomorphism.*

Proof By Proposition 4.9, it is enough to show that the conditions R1–3 hold for $R = graph(n)$. As to R1, notice that Lemma 4.12 implies that for every $a, b \in At(\mathcal{L})$,

$$0 \neq ab^* \leq e \text{ iff } a \leq b \text{ iff } a = b.$$

Hence,

$$\bigcup\{R^{-1}[u(p)] \mid p \in G_0\} = \{(a, b) \mid ab^* \in At(\mathcal{L}_e)\}$$
$$= \{(a, a) \mid a \in At(\mathcal{L})\} = u'[G'_0].$$

As to R2, recall that the structure map i' on pair groupoids is defined by $i'(a, b) = (a, b)^{-1} = (b, a)$; hence

$$R^{-1}[i(c)] = R^{-1}[c^*] = \{(b, a) \mid ba^* = c^*\}$$
$$= \{(a, b)^{-1} \mid ab^* = c\}$$
$$= \{i'(x') \mid x' \in R^{-1}[c]\}.$$

As to R3, recall that the product on pair groupoids is defined by $(a, b) \cdot (b'c) = (a, c)$ if $b = b'$; notice preliminarily that for all $a, b, c, d \in At(\mathcal{L})$, if $cd \neq 0$ and $ab^* = cd$, then $ab'^* = c$ and $b'b^* = d$ for some $b' \in At(\mathcal{L})$: indeed, by Lemma 4.12, the chain $0 \neq ab^* = cd$ implies that $0 \neq a \leq cdb$, hence $a = cdb$. Hence we have that $0 \neq ab^* = cdbb^*$, which implies that $dbb^* \neq 0$. Since b is a partial unit, we have $0 \neq dbb^* \leq d$, which implies $dbb^* = d$. Take $b' = db$. From $dbb^* \neq 0$, we get $b' = db \neq 0$, so by Proposition 4.3.2, we conclude that $b' \in At(\mathcal{L})$. Since $a = cdb$, we have: $ab'^* = (cdb)(db)^* = cdbb^*d^* \leq cdd^* \leq c$. To finish the proof, it is enough to show that $ab'^* \neq 0$. Indeed, the chain $0 \neq a = cdb = cdb(cdb)^*cdb = (cdbb^*d^*)c^*cdb = ab'^*c^*cdb$ implies that $ab'^* \neq 0$. Therefore, the following chain of equalities holds:

$$R^{-1}[cd] = \{(a, b) \mid ab^* = cd\}$$
$$= \{(a, b')(b', b) \mid ab'^* = c \text{ and } b'b^* = d\}$$
$$= \{x'y' \mid x' \in R^{-1}[c] \text{ and } y' \in R^{-1}[d]\}.$$

□

Summing up, we proved the following

Theorem 4.14 *For every FA-quantale $\mathcal{L}$, the map $\mathcal{L} \to \mathcal{P}(At(\mathcal{L}) \times At(\mathcal{L}))$ defined by $x \mapsto F(x) = \{\langle a, b\rangle \mid a, b \in At(\mathcal{L}), a \leq xb\}$ is a strictly unital embedding of unital involutive quantales.*

Putting together Theorems 3.5 and 4.14, we obtain the quantale counterpart of Theorem 4.22 in [3]:

Corollary 4.15 *Every spatial SGF-quantale is isomorphic to a sub unital involutive quantale of some quantale of relations.*

Proof The composition of α of Theorem 3.5 and F of Theorem 4.14 is a strictly unital embedding of unital involutive quantales. □

Notice that, since the embedding α of Theorem 3.5 does not need to preserve meets, neither does the embedding of the corollary above.

4.4 Characterization of Quantales of Relations

As a second application of the dual characterization of strictly unital quantale embeddings given in Section 4.2, in the present subsection, a characterization is given of those quantales which are isomorphic to $\mathcal{P}(R)$ for some equivalence relation R (cf. Theorem 4.21).

We mentioned early on (cf. Example 2.4.1) that the groupoids arising from equivalence relations are characterized up to isomorphism as those groupoids $\mathcal{G}$ such that the product map $d \times r : G_1 \to G_0 \times G_0$ is injective. Now let $\mathcal{G}'$ be the pair groupoid of G_0 (i.e., $G'_1 = G_0 \times G_0$), and observe that $d \times r$ is injective if and only if the relation $R \subseteq (G_0 \times G_0) \times G_1$, defined as the converse of $graph(d \times r)$, is the graph of a partial map onto G_1, i.e., iff R satisfies the conditions in item 1 of Proposition 4.10. By the same proposition, this is equivalent to $h_R : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(G_0 \times G_0)$, given by

$$\begin{aligned} h_R(x) &= \bigcup\{R^{-1}[a] \mid a \in At(x)\} \\ &= \{(d(a), r(a)) \mid a \in At(x)\}, \end{aligned}$$

being a completely join-preserving embedding which preserves finite meets.

Proposition 4.16 *For every groupoid $\mathcal{G}$, the map h_R, where R is the converse of $graph(d \times r)$, is a strictly unital quantale homomorphism.*

Proof By Proposition 4.9, it is enough to show that the conditions R1–3 hold for R. As to R1, recall that, since $\mathcal{G}'$ is the pair groupoid of G_0, we have that $G'_0 = G_0$ and $u'[G'_0] = \{(a, a) \mid a \in G_0\}$. Hence,

$$\begin{aligned} \bigcup\{R^{-1}[u(p)] \mid p \in G_0\} &= \{(d(u(p)), r(u(p))) \mid p \in G_0\} \\ &= \{(p, p) \mid p \in G_0\} = u'[G'_0]. \end{aligned}$$

As to R2, recall that the structure map i' on pair groupoids is defined by $i'(a, b) = (a, b)^{-1} = (b, a)$; hence

$$\begin{aligned} R^{-1}[i(x)] &= \{(d(i(x)), r(i(x)))\} \\ &= \{(r(x), d(x))\} \\ &= \{(d(x), r(x))^{-1}\} \\ &= \{i'(x') \mid x' \in R^{-1}[x]\}. \end{aligned}$$

As to R3, recall that if $x, y \in G_1$, the product xy is defined iff $r(x) = d(y)$, and that the product on pair groupoids is defined by $(a, b) \cdot (b'c) = (a, c)$ if $b = b'$; therefore,

$$\begin{aligned} R^{-1}[xy] &= \{(d(xy), r(xy))\} \\ &= \{(d(x), r(y))\} \\ &= \{(d(x), r(x))(d(y), r(y))\} \\ &= \{x'y' \mid x' \in R^{-1}[x] \text{ and } y' \in R^{-1}[y]\}. \end{aligned}$$

□

Notice that the proposition above was proven for an arbitrary groupoid $\mathcal{G}$, without the assumption that $d \times r$ be injective. Therefore, by Proposition 4.6.1, we have the following

Proposition 4.17 *For every FA-quantale $\mathcal{L}$,*

1. *the map $\gamma : \mathcal{L} \to \mathcal{P}(At(\mathcal{L}_e) \times At(\mathcal{L}_e))$, defined by*

$$\gamma(x) = \{(aa^*, a^*a) \mid a \in At(x)\},$$

   is a strictly unital homomorphism of unital involutive quantales.
2. *γ is an embedding iff $\mathcal{L}$ is isomorphic to a quantale $\mathcal{P}(R)$ for some equivalence relation R.*

Proof Item 1 follows from Proposition 4.16. The right-to-left direction of item 2 is easy to verify. Conversely, setting $R = \gamma(1)$, the inequalities $e \leq 1$, $1 \cdot 1 \leq 1$ and $1^* = 1$ imply that R is an equivalence relation. If $S \subseteq \gamma(1)$, then $S = \{(aa^*, a^*a) \mid a \in X\}$ for some $X \subseteq At(\mathcal{Q})$. Then $\gamma(\bigvee X) = S$. □

Lemma 4.18 *The following are equivalent for any completely join-preserving map $g : \mathcal{L} \to \mathcal{L}'$ such that $g(c) \in At(\mathcal{L}')$ fore every $c \in At(\mathcal{L})$:*

1. *g is an order embedding;*
2. *for all $a, b \in At(\mathcal{L})$, if $g(a) = g(b)$ then $a = b$.*

Proof If g is an order embedding, then g is injective. Conversely, if $x, y \in \mathcal{L}$ and $g(x) \leq g(y)$, then, by the atomicity of $\mathcal{L}$, to show that $x \leq y$, it is enough to show that $At(x) \subseteq At(y)$. If $a \in At(x)$, then $g(a) \in At(g(x)) \subseteq At(g(y))$, so, since g is completely join-preserving, we get that $g(a) \leq g(y) = \bigvee\{g(b) \mid b \in At(y)\}$. Since $g(a)$ is completely join-prime, this implies that $0 \neq g(a) \leq g(b)$ for some $b \in At(y)$, which implies, since $g(b)$ is an atom, that $g(a) = g(b)$. Hence we conclude that $a = b \in At(y)$. □

For every FA-quantale $\mathcal{L}$ and every $c \in At(\mathcal{L})$, let

$$Stab(\mathcal{L}) = \{c \in At(\mathcal{L}) \mid cc^* = c^*c\}.$$

Remark 4.19 The notation $\mathrm{Stab}(\mathcal{L})$ is motivated by the existence of a partially defined action $At(\mathcal{L}) \times At(\mathcal{L}_e) \to At(\mathcal{L}_e)$ given by $(c, a) \mapsto c(a) = c^*ac$. One can show that $c(a) = a$ if and only if $h = cc^* = c^*c$. Hence $\mathrm{Stab}(\mathcal{L})$ can be regarded as the set of atoms stabilizing some atom in $\mathcal{L}_e$.

Lemma 4.20 *For every FA-quantale $\mathcal{L}$,*

1. $Stab(\mathcal{L}) = \{ab a^* \mid ab a^* \neq 0 \text{ and } a, b \in At(\mathcal{L})\}$.
2. $aa^* = bb^* \text{ and } a^*a = b^*b \text{ iff } ab^*, a^*b \in Stab(\mathcal{L})$.

Proof

1. If $a, b \in At(\mathcal{L})$ and $c = ab a^* \neq 0$ then $cc^* = ab a^* ab^* a^* \leq aa^*$, hence $cc^* = aa^*$. Likewise we have $c^*c \leq aa^*$, which implies $c^*c = aa^*$. This shows that $c \in Stab(\mathcal{L})$. Conversely, if $c \in Stab(\mathcal{L})$, then $0 \neq c = cc^*c = ccc^*$.
2. If $aa^* = bb^*$ and $a^*a = b^*b$, then $0 \neq a = aa^*a = ab^*b$, hence $ab^* \neq 0$. Therefore, by Proposition 4.3.2, we conclude that $ab^* \in At(\mathcal{L})$. Hence, by Proposition 4.3.3, we have: $0 \neq (ab^*)(ab^*)^* = ab^*ba^* \leq aa^*$, which implies $(ab^*)(ab^*)^* = aa^*$ and likewise $(ab^*)^*(ab^*) = bb^*$. Since $aa^* = bb^*$, we conclude that $ab^* \in Stab(\mathcal{L})$. Conversely, suppose that $ab^* \in Stab(\mathcal{L})$; then $bb^* \geq ba^*ab^* = (ab^*)^*(ab^*) = (ab^*)(ab^*)^* = ab^*ba^* \leq aa^*$ and by Proposition 4.3.3 this implies that $bb^* = aa^*$. □

We are ready to give the quantale counterpart of Theorem 4.30 in [3]:

Theorem 4.21 *The following are equivalent for every unital involutive quantale $\mathcal{Q}$:*

1. *$\mathcal{Q}$ is isomorphic to a quantale $\mathcal{P}(R)$ for some equivalence relation R;*
2. *$\mathcal{Q}$ is an FA-quantale such that $a1a^* \in At(\mathcal{Q})$ for every $a \in At(\mathcal{Q})$.*

Proof The '(1.) implies (2.)' direction is easy to check. For the converse, let $\mathcal{Q} = \mathcal{L}$ be an FA-quantale such that $a1a^* \in At(\mathcal{L})$ for every $a \in At(\mathcal{L})$. By Proposition 4.17 and Lemma 4.18, it is enough to show that the map γ, defined in Proposition 4.17, be injective over atoms. By Lemma 4.20.1, the assumptions imply that

$$Stab(\mathcal{L}) = \{ab a^* \mid ab a^* \neq 0 \text{ and } a, b \in At(\mathcal{L})\} = \{aa^* \mid a \in At(\mathcal{L})\} = At(\mathcal{L}_e).$$

Hence, by Lemma 4.20.2, $\gamma(b) = \gamma(a)$ implies that $ab^* \leq e$, and so $a = aa^*a = ab^*b \leq b$ which implies that $a = b$. □

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References

1. Brown, C., Gurr, D.: A representation theorem for quantales. J. Pure Appl. Algebra **85**(1), 27–42 (1993)
2. Gelfand, I.M., Naimark, M.A.: On the embedding of normed rings into the ring of operators in Hilbert space. Mat. Sb. **12**, 197–213 (1943)

3. Jónsson, B., Tarski, A.: Boolean algebras with operators, part II. Am. J. Math. **74**(2), 127–162 (1952)
4. Kruml, D., Paseka, J.: Embeddings of quantales into simple quantales. J. Pure Appl. Algebra **148**(2), 209–424 (2000)
5. Johnstone, P.T.: Stone Spaces. Cambridge Studies in Adv. Math., vol. 3. Cambridge University Press, Cambridge, UK (1982)
6. Mulvey, C.J.: &. Rend. Circ. Mat. Palermo **12**, 99–104 (1986)
7. Mulvey, C.J., Resende, P.: A noncommutative theory of Penrose tilings. Int. J. Theor. Phys. **44**, 655–689 (2005)
8. Palmigiano, A., Re, R.: Groupoid quantales: a non-étale setting. J. Pure Appl. Algebra **215**(8), 1945–1957 (2011)
9. Palmigiano, A., Re, R.: Topological Groupoid Quantales. In: Aerts, D., Smets, S., van Bendegem, J.P. (eds.) Special Issue: The Contributions of Logic to the Foundations of Physics. Studia Logica, vol. 95, pp. 125–137 (2010)
10. Paterson, A.L.T.: Groupoids, Inverse Semigroups, and their Operator Algebras. Progress in Mathematics, vol. 170. Birkhäuser, Boston (1999)
11. Protin, M.C., Resende, P.: Quantales of Open Groupoids. arXiv:0811.4539v2
12. Resende, P.: Lectures on Étale Groupoids, Inverse Semigroups and Quantales. Lecture Notes for the GAMAP IP Meeting, Antwerp (2006)
13. Resende, P.: Étale groupoids and their quantales. Adv. Math. **208**, 147–209 (2007)
14. Rosenthal, K.: Quantales an their Applications. Pitman Research Notes in Math., vol. 234. Longman (1990)
15. Takesaki, M.: Theory of Operator Algebras I, EMS Series, vol. 124. Springer, New York (2002)
16. Valentini, S.: Representation Theorems for Quantales. Math. Log. Q. **40**, 182–190 (1994)