Topological Groupoid Quantales

Alessandra Palmigiano and Riccardo Re

Abstract

We associate a canonical unital involutive quantale to a topological groupoid. When the groupoid is also étale, this association is compatible with but independent from the theory of localic étale groupoids and their quantales [19] of P. Resende. As a motivating example, we describe the connection between the quantale and the C^*-algebra that both classify Penrose tilings, which was left as an open problem in [15].

Keywords: unital involutive quantale, regular frame, topological groupoid, étale groupoid, representation theorem, Penrose tilings.

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1 Introduction

Groupoids, i.e. small categories such that every morphism is an iso, have been first introduced by Brandt in 1926 as algebraic structures generalizing groups, by allowing the group product to be partially defined. Groupoid can be usefully seen as the 'categorification' of equivalence relations: indeed, since every morphism is an iso, any two objects joined by at least one arrow are equivalent in "as many ways" as there are arrows between them. Moreover, equivalence relations can often be meaningfully represented as the orbit equivalence relations of some nontrivial groupoids over their domains. This observation has led to important applications of groupoids in algebraic and non-commutative geometry: when an equivalence relation on a topological space induces a pathological quotient space, the equivalence relation itself can be studied as a groupoid, as was done for instance by Connes [2] with the classifying space K of Penrose tilings. The main role of groupoids in Connes' noncommutative geometry, particularly when they are étale (see Definition 2.5 below) is their giving rise to C^*-algebras, a fact of which the space K of Penrose tilings is also an interesting example: indeed the C^*-algebra $A(K)$ associated with K seen as a groupoid classifies the Penrose tilings up to isomorphism [2] (see also [17] and [16]). Thanks to their connection with C^*-algebras, when endowed with suitable topological or localic structure, groupoids can also be regarded as noncommutative spaces. Finally, in algebraic logic, discrete groupoids have been used in Jónsson and Tarski's representation theorems for certain classes of relation algebras [9].

Quantales were introduced [13] as the noncommutative generalizations of locales (i.e. point-free topologies) and have been investigated in close connection with C^*-algebras, in the context of a research program aimed at developing noncommutative extensions of the Gelfand-Naimark duality (which establishes a dual equivalence of categories between unital *commutative* C^*-algebras and compact Hausdorff spaces).

In this context, the hope was that the functor *Max*, associating every C^*-algebra with the quantale of its closed linear subspaces, would provide the correspondence from C^*-algebras to unital involutive quantales generalizing the commutative Gelfand duality in a natural way. However, although *Max* is a faithful complete invariant of C^*-algebras, it does not preserve limits, so it does not have a left adjoint, and hence it is not viable for establishing a dual equivalence between C^*-algebras and quantales.

In the quest for an alternative way of establishing a duality between quantales and C^*-algebras, Penrose tilings provided again an interesting case study: Mulvey and Resende [15] associated the classifying space K of Penrose tilings with a unital involutive quantale **Pen**, defined as the Lindenbaum-Tarski algebra of a logic of finite observations on certain geometric properties of the tilings, and canonically interpreted as a certain relational quantale *Pen*, thanks to the fact that any relational representation of **Pen** factors through *Pen*. In particular this holds for any irreducible relational representation of **Pen**, which is used in [15] to show that these representations classify the Penrose tilings of the plane up to isomorphism, exactly like Connes' C^*-algebra $A(K)$ does. However the precise relation between *Pen* and $A(K)$ was left as an open problem in [15]. Since *Pen* is a different quantale than $MaxA(K)$, and both $A(K)$ and *Pen* arise from the same étale groupoid K, this case study suggested the possibility of an alternative correspondence between C^*-algebras and quantales using *groupoids* as intermediate structures. This line of investigation was further developed by Resende [19] who established an equivalence on objects between localic étale groupoids and *inverse quantale frames*.

The aim of our own contribution is extending Resende's correspondence to non étale topological groupoids: in this paper, any topological groupoid is associated with a unital involutive quantale (its *topological groupoid quantale*) in a way that is alternative to [19] but compatible with it[1] when the groupoids are étale. As a case study, we show that *Pen* is the topological groupoid quantale associated with the classifying space of Penrose tilings, from which fact we derive the relation between *Pen* and $A(K)$[2].

2 Basic definitions and examples

A *quantale* $\mathcal{Q}$ (see [13], [20]) is a complete join-semilattice endowed with an associative binary operation $\cdot$ that is completely distributive in each coordinate, i.e.

[1]We will report about the comparison with the correspondence defined in [19] in a forthcoming paper.

[2]In private communication, P. Resende informed us that the relation between *Pen* and $A(K)$ was independently known to him and Mulvey, but was not officially communicated.

D1: $c \cdot \bigvee I = \bigvee\{c \cdot q : q \in I\}$
D2: $\bigvee I \cdot c = \bigvee\{q \cdot c : q \in I\}$
for every $c \in \mathcal{Q}$, $I \subseteq \mathcal{Q}$. Since it is a complete join-semilattice, $\mathcal{Q}$ is also a complete, hence bounded, lattice. Let $0, 1$ be the lattice bottom and top of $\mathcal{Q}$, respectively. Conditions $D1$ and $D2$ readily imply that $\cdot$ is order-preserving in both coordinates and, as $\bigvee \varnothing = 0$, that $c \cdot 0 = 0 = 0 \cdot c$ for every $c \in \mathcal{Q}$. $\mathcal{Q}$ is *unital* if there exists an element $e \in \mathcal{Q}$ for which
U: $e \cdot c = c = c \cdot e$ for every $c \in \mathcal{Q}$,
and is *involutive* if it is endowed with a unary operation $*$ such that, for every $c, q \in \mathcal{Q}$ and every $I \subseteq \mathcal{Q}$,
I1: $c^{**} = c$.
I2: $(c \cdot q)^* = q^* \cdot c^*$.
I3: $(\bigvee I)^* = \bigvee\{q^* : q \in I\}$.
Relevant examples of unital involutive quantales are:
1. The quantale $\mathcal{P}(R)$ of subrelations of a given equivalence relation $R \subseteq X \times X$.
2. The quantale $\mathcal{P}(G)$, for every group G.
3. Any frame $\mathcal{Q}$, setting $\cdot := \wedge$, $* := \mathrm{id}$ and $e := 1_{\mathcal{Q}}$.

Definition 2.1. *A* groupoid *is a tuple $\mathcal{G} = (G_0, G_1, m, d, r, u, ()^{-1})$, such that:*
G1. G_0 and G_1 are sets;
G2. $d, r : G_1 \to G_0$ and $u : G_0 \to G_1$ s.t. $d(u(p)) = p = r(u(p))$ for every $p \in G_0$;
G3. $m : (x, y) \mapsto xy$ is an associative map defined on $\{(x, y) \mid r(x) = d(y)\}$ and s.t. $d(xy) = d(x)$ and $r(xy) = r(y)$;
G4. $xu(r(x))) = x = u(d(x))x$ for every $x \in G_1$.;
G5. $()^{-1} : G_1 \to G_1$ is an operator such that $xx^{-1} = u(d(x))$, $x^{-1}x = u(r(x))$, $d(x^{-1}) = r(x)$ and $r(x^{-1}) = d(x)$ for every $x \in G_1$.

Let us list some relevant examples of groupoids:

Examples.
1. For any equivalence relation $R \subseteq X \times X$, the tuple $(X, R, \circ, \pi_1, \pi_2, \Delta, ()^{-1})$ defines a groupoid.
2. For any group $(G, \cdot, e, ()^{-1})$, the tuple $(\{e\}, G, \cdot, d, r, u, ()^{-1})$ is a groupoid, and the equalities G4 and G5 just restate the group axioms.
3. The following example is a special but important case of the first one: every topological space X can be seen as a groupoid by setting $G_1 = G_0 = X$ and identity structure maps. In this case, $G_1 \times_{G_0} G_1 = \{(x, x) \mid x \in X\}$ and $xx = x$ for every $x \in X$.
4. For any action $G \times X \to X$ of a group G on a set X, one can naturally associate the groupoid such that $G_1 = G \times X$, $G_0 = X$, its domain and range maps are defined by $d(g, x) = x$ and $r(g, x) = gx$, the unit map $u(x) = (e, x)$ with $e \in G$ is the identity element, and multiplication is defined by $(g, x) \cdot (h, y) = (hg, x)$ iff $y = gx$.
Let us report some easy to show but useful facts about groupoids:

Lemma 2.2. *For all $p \in G_0$, $x, y \in G_1$,*

1. $u(p)^{-1} = u(p)$,
2. $x = xx^{-1}x$ *and* $x^{-1} = x^{-1}xx^{-1}$,
3. *if* $xy^{-1}, x^{-1}y \in u[G_0]$ *then* $x = y$,
4. *if* $x = xyx$ *and* $yxy = y$, *then* $y = x^{-1}$,
5. $(x^{-1})^{-1} = x$,
6. $(xy)^{-1} = y^{-1}x^{-1}$.

For every groupoid $\mathcal{G}$, $\mathcal{P}(G_1)$ can be given the structure of a unital involutive quantale (see also [19] 1.1 for a more detailed discussion): indeed, the product and involution on G_1 can be lifted to $\mathcal{P}(G_1)$ as follows:

$$S \cdot T = \{x \cdot y \mid x \in S,\ y \in T \text{ and } r(x) = d(y)\} \qquad S^* = \{x^{-1} \mid x \in S\}.$$

Denoting by E the image of the structure map $u : G_0 \to G_1$, we get:

Fact 2.3. $\langle \mathcal{P}(G_1), \bigcup, \cdot, ()^*, E \rangle$ *is a unital involutive quantale.*

Definition 2.4. *A* topological groupoid *is a groupoid* $\mathcal{G}$ *such that* G_0 *and* G_1 *are topological spaces and the structure maps are continuous.*

Definition 2.5. *a topological groupoid* $\mathcal{G}$ *is* étale *if* $d : G_1 \to G_0$ *is a local homeomorphism (d is étale).*

Since the involution in G_1 swaps the roles of d and r, if d is étale, then so is r.

3 Topological groupoid quantales

In this section we will define a general procedure that associates a unital involutive quantale with *any* topological groupoid $\mathcal{G}$. Our method is based on the following simple fact:

Fact 3.1. *If* $\mathcal{Q} = (Q, \bigvee, \cdot, ^*, e)$ *is a unital involutive quantale and* $S \subseteq Q$ *contains* e *and is closed under* $\cdot$ *and* *, *then the sub-semilattice of* $(Q, \bigvee)$ *generated by* S *is a unital involutive subquantale of* $\mathcal{Q}$.

So we will define the quantale $\mathcal{Q}(\mathcal{G})$ associated with a topological groupoid $\mathcal{G}$ as the sub join-semilattice of $\langle \mathcal{P}(G_1), \bigcup \rangle$ generated by a suitable subset $S \subseteq \mathcal{P}(G_1)$. The following definition will provide us with the building blocks for this subset:

Definition 3.2. *A* local section of d *is a continuous map* $s : U \to G_1$ *defined on some open subset* U *of* G_0, *s.t.* $d \circ s = id_U$. *A local section* s *is a* local bisection *if* $r \circ s : U \to V$ *is a local homeomorphism for some open subset* V *of* G_0.

By G2, the structure map u is a local bisection. In the context of our first example, local bisections can be identified with the graphs $\Gamma_f = \{(x, f(x)) \mid x \in U\}$ of local homeomorphisms $f : U \to V$ such that $\Gamma_f \subseteq R$. In the context of our second example, local bisections can be identified with the elements of the group G. In the third example, the local bisections are the identity maps $i : U \to U$ on open subsets $U \subseteq X$.

The subset $\mathcal{S}(\mathcal{G}) \subseteq \mathcal{P}(G_1)$ of the images of the local bisections, besides containing $E = u[G_0]$, is also closed under product and involution on $\mathcal{P}(G_1)$: indeed we first define composition and involution on local bisections as follows: If $s : U \to G_1$ and $t : U' \to G_1$ are local bisections, then the composition $s \cdot t$ is defined by

$$(s \cdot t)(p) = s(p)t(r \circ s(p)) \tag{1}$$

on the open set $(r \circ s)^{-1}[U']$. Similarly, the involution of a local bisection $s : U \to G_1$ is defined on the open set $V = (r \circ s)[U]$ by

$$s^*(r \circ s(p)) = s(p)^{-1}. \tag{2}$$

Again, it is easy to verify that in the context of our first (second) example, compositions and involutions of local bisections respectively correspond to compositions[3] (products) and inverses of the associated local homeomorphisms (elements of the group G). It not difficult to show that:

Lemma 3.3. *The following properties hold for every local bisections s and t:*
1. $s \cdot t$ is a local bisection of d.
2. $s \cdot s^*$ and $s^* \cdot s$ coincide with u wherever defined.
3. s^* is a local bisection of d.

It is well known and easy to see that the collection $\mathcal{S}(\mathcal{G})$ of G-sets is closed under the product and involution of $\mathcal{P}(G_1)$:

Lemma 3.4. *For every local bisection $s : U \to G_1$ and $t : U' \to G_1$, let $S = s[U]$, $T = t[U']$, V be the domain of st and $W = (r \circ s)[U]$ be the domain of s^*. Then:*
1. $(st)[V] = S \cdot T = \{xy \mid x \in S, y \in T \text{ and } r(x) = d(y)\}$.
2. $s^*[W] = S^* = \{x^{-1} \mid x \in S\}$.

The following facts (cf. [17] chapter I, Definition 2.6, Lemma 2.7 and Proposition 2.8) will be used later on:

Fact 1. If G_0 is locally compact and $\mathcal{G}$ is étale, the G-sets form a basis for the topology of G_1. Then in particular $u[G_0]$ is open.
Fact 2. If, for a topological groupoid $\mathcal{G}$, G_0 is locally compact and there exists a base of G-sets for the topology of G_1, then $\mathcal{G}$ is étale.
Fact 3. Under the assumptions of Fact 2, every G-set is open in G_1.

We are ready to introduce our main definition, i.e. the unital involutive quantale that we will associate with any topological groupoid $\mathcal{G}$:

Definition 3.5. *The* topological groupoid quantale $\mathcal{Q}(\mathcal{G})$ *associated with $\mathcal{G}$ is the sub $\bigcup$-semilattice of $\mathcal{P}(G_1)$ generated by the collection $\mathcal{S}(\mathcal{G})$ of the G-sets. Composition and involution in $\mathcal{Q}(\mathcal{G})$ are defined as the lifted operations from G_1. The unit $e_{\mathcal{Q}(\mathcal{G})}$ is E.*

[3] Notation is treacherous here: if s and t respectively correspond to the local homeomorphisms f and g, then the *algebraic* product $s \cdot t$ as defined in (1) set-theoretically corresponds to the relational composition of the graphs $\Gamma_f \circ \Gamma_g$ (and thus to the functional composition $g \circ f$).

By Fact 3.1 and Lemmas 3.3 and 3.4, $\mathcal{Q}(\mathcal{G})$ is indeed a unital involutive subquantale of $\mathcal{P}(G_1)$. The three basic examples of unital involutive quantales given above can be retrieved as instances of topological groupoid quantales: If X is a discrete space and R is an equivalence relation on X, then singletons $\{(x,y)\} \subset R$ are local bisections and so $\mathcal{Q}(\mathcal{G}) = \mathcal{P}(R)$. Similarly, if G is a group, then $\mathcal{Q}(\mathcal{G}) = \mathcal{P}(G)$. As we remarked early on, for topological spaces X seen as groupoids, local bisections are the identity maps $i : U \to U$ on open subsets. So $\mathcal{Q}(\mathcal{G})$ is the frame $\Omega(X)$.

Example 1. Let (X, G) be as in example 4 section 2, with X a locally connected topological space and G a group with the discrete topology. Then the local bisections are the locally constant maps $U \to G$ with $U \subset X$ an open set. Hence $\mathcal{Q}(\mathcal{G})$ is given by the product topology on $G_1 = G \times X$ and it is obviously étale.

Example 2. On the other hand, let $R \subset X \times X$ be the equivalence relation induced by the action of G, i.e. xRy iff there exists $g \in G$ such that $y = gx$. If R is endowed with the quotient topology with respect to the map $(d,r) : G \times X \to R$, defined by $(g,x) \mapsto (x, gx)$, then the first projection $\pi_1 : R \to X$ is not necessarily étale. For example, let $X = \mathbb{C}$ and $G = \{z \in \mathbb{C} \mid z^n = 1\}$ the group of nth roots of unity, with $n \geq 2$. Consider the action of G on X given by the multiplication $(z,x) \mapsto zx$. Then the induced equivalence relation is $R = \{(x,y) \mid y = zx,\ z \in G\}$. Take any $z \neq w \in G$ and consider the two local bisections of the groupoid R defined respectively by $x \mapsto (x, zx)$ and $x \mapsto (x, wx)$. They have images intersecting only at $(0,0) \in R$, so $d : R \to X$ cannot be étale and the topological groupoid quantale $\mathcal{Q}(R)$ is not a frame.

In the following section we introduce the quantale of Penrose tilings, our motivating case study.

4 The quantale associated with Penrose tilings

Let $X \subseteq 2^{\aleph_0}$ be the set of *Penrose sequences* [2][15], i.e. the sequences $x = (x_k)_{k \in \mathbb{N}}$ such that $x_k = 1$ implies $x_{k+1} = 0$. X is a closed subset of $2^{\aleph_0}$ w.r.t. the Tychonoff topology, so it is homeomorphic to the Cantor space $2^{\aleph_0}$. Consider the equivalence relation R on X defined by xRy iff there exists some $n \in \mathbb{N}$ such that $x_k = y_k$ for every $k > n$. The equivalence classes of R classify the isomorphism classes of the Penrose tilings of the plane. In [15], Mulvey and Resende defined a quantale **Pen** by generators and relations and proved that its algebraically irreducible relational representations are in one-to-one correspondence with the equivalence classes of R. This quantale admits a concrete representation as a quantale *Pen* of subrelations of R, which is canonical, in the sense that every relational representation of **Pen** factors through *Pen*. Hence *Pen* classifies the isomorphism classes of Penrose tilings too. *Pen* is defined in [15] as the subquantale of $\mathcal{P}(R)$ generated by the following relations: for every $n \in \mathbb{N}$,

$$l_n = \{\langle x, y\rangle \in R \mid y_n = 0 \text{ and } x_k = y_k \text{ for any } k > n\}$$

$$s_n = \{\langle x, y\rangle \in R \mid y_n = 1 \text{ and } x_k = y_k \text{ for any } k > n\}$$

and their inverses l_n^{-1} and s_n^{-1}. The following theorem is the central result of this section, and its proof does not follow straightforwardly from the theory developed in [15].

Theorem 4.1. *The quantale Pen has the following properties:*

1. *Pen is $\bigcup$-generated by the graphs Γ_f of the local homeomorphisms of the form*
$$f(\varepsilon, x_{n+1}, x_{n+2}, \ldots) = (\eta, x_{n+1}, x_{n+2}, \ldots), \tag{3}$$
with $\varepsilon, \eta \in 2^n$.

2. *Pen is a frame, and, as a topology on R, is finer than the inherited product topology $R \subset X \times X$.*

3. *If R is endowed with Pen as a topology, (X, R) is an étale groupoid.*

4. *Pen is the topological groupoid quantale associated with (X, R).*

Proof. 1. Denoting $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ and $\eta = (\eta_1, \ldots, \eta_n)$, let us define $X_i = l_i^{-1}$ if $\varepsilon_i = 0$ and $X_i = s_i^{-1}$ if $\varepsilon_i = 1$. Similarly we define $Y_i = l_i$ if $\eta_i = 0$ and $Y_i = s_i$ if $\eta_i = 1$. It is now a straightforward calculation to check that $\Gamma_f = X_1 \circ \cdots \circ X_n \circ Y_n \circ \cdots \circ Y_1$, and so $\Gamma_f \in$*Pen*. Conversely, it is clear that s_n, l_n and their inverses are finite unions of graphs of homeomorphisms of the form (3), which proves the statement.
2. Let R_n be the subrelation of R defined by xR_ny if $x_k = y_k$ for any $k > n$. It is easy to see that graphs of homeomorphisms of the form (3) form a basis of the inherited product topology of $R_n \subseteq X \times X$. Moreover $R_0 \subset R_1 \subset \cdots R_n \subset \cdots$ is a chain of open inclusions. From this it easily follows that *Pen* is the limit topology $\lim R_n$ on R, that is, for $S \subseteq R$, $S \in$ *Pen* if and only if, for some n, $S \cap R_n$ is an open subset of R_n. Denoting, for any $\varepsilon \in 2^n$ the clopen $X_\varepsilon = \{x \in X \mid x_i = \epsilon_i,\ i = 1, \ldots, n\}$ one can denote the elements of a basis of the product topology for R by $B_{\varepsilon,\eta} = (X_\varepsilon \times X_\eta) \cap R$. One sees easily that $B_{\varepsilon,\eta} \cap R_n$ is open in R_n, hence the limit topology just defined is finer than the product topology for R. Finally, as a topology on R, *Pen* admits a basis of G-sets, notably the sets of the form (3), so, by Fact 2. of section 3, we conclude that R is étale over X.
3. Assume that $s : U \to R$ is a local bisection, with U open in X and R with the topology given by *Pen*. Then its image $S \subset R$ is a G-set, and by Fact 3 in section 3, S is generated by the given basis of G-sets as in (3), that is, $S \in$ *Pen*. Hence *Pen* is the topological groupoid quantale associated with $K = (X, R)$. $\square$

A question left open in [15] was to characterize the relation between **Pen** and the C^*-algebra $A(K)$ that Connes associates with the space $K = (X, R)$ of Penrose tilings. We can now answer this question by saying that $Pen = \lim R_n$, i.e. *Pen* is the limit topology on R that Connes used to construct $A(K)$ as the completion of a space of *continuous* functions $g : R \to \mathbb{C}$. Intuitively, this means that *Pen* encodes the purely topological content of $A(K)$. For sake of completeness, let us now remind the reader how $A(K)$ is defined in [2]. Using the limit topology $\lim R_n$, the ring of complex-valued continuous functions with compact support is introduced, the continuity of the functions in $\mathcal{C}_c(R)$ being assumed with respect to this limit topology. It is not difficult to see that $\mathcal{C}_c(R) = \bigcup_n \mathcal{C}(R_n)$. The convolution product of $f, g \in \mathcal{C}_c(R)$ is defined as

$$(f * g)(x, z) = \sum_{xRy} f(x, y)g(y, z),$$

the sum containing only finitely many non-zero summands by the hypothesis of compact support. Involution is defined as $f^*(x, y) = \overline{f(y, x)}$. With these operations $\mathcal{C}_c(R)$ becomes a C^*-algebra. The algebra $\mathcal{C}_c(R)$ admits canonical irreducible representations in the Hilbert spaces $l^2(R[x])$, $R[x]$ being the equivalence class of $x \in X$. If $v = (v_y)_{y\in R[x]} \in l^2(R[x])$ and $f \in \mathcal{C}_c(R)$, then $fv = ((fv)_z)_{z\in R[x]}$ is defined as $(fv)_z = \sum f(z, y)v_y$. Then $\mathcal{C}_c(R)$ inherits an operator norm from the given representation on $l^2(R[x])$. Then the *reduced norm* on $\mathcal{C}_c(R)$ is defined by taking the supremum of all the operator norms arising in this way from the distinct equivalence classes $R[x]$ (see [2], chapter II, section 3 and [16] pp. 105-109). The C^*-algebra $A(K)$ associated with the Penrose tilings is the norm-closure of $\mathcal{C}_c(R)$ with respect to this reduced norm. The fact that this C^*-algebra is limit of the $\mathcal{C}(R_n)$ was used in [2] to compute the group K_0 of the C^*-algebra of Penrose tilings.

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Alessandra Palmigiano
Institute for Logic, Language and Computation
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam, The Netherlands
e-mail: <a.palmigiano@uva.nl>
phone: +31 (0)20 525 53 60
fax: +31 20 525 52 06

Riccardo Re

Dipartimento di Matematica e Informatica
Università di Catania
Viale Andrea Doria, 6
95125 Catania, Italy
e-mail: <riccardo@dmi.unict.it>
phone: +39 095 738 30 75
fax: +39 095 33 00 94