JÓNSSON-STYLE CANONICITY FOR ALBA-INEQUALITIES

ALESSANDRA PALMIGIANO, SUMIT SOURABH, ZHIGUANG ZHAO

Abstract. The theory of canonical extensions typically considers extensions of maps $\mathbb{A} \to \mathbb{B}$ to maps $\mathbb{A}^\delta \to \mathbb{B}^\delta$. In the present paper, the theory of canonical extensions of maps $\mathbb{A} \to \mathbb{B}^\delta$ to maps $\mathbb{A}^\delta \to \mathbb{B}^\delta$ is developed, and is applied to obtain a new canonicity proof for those inequalities in the language of Distributive Modal Logic (DML) on which the algorithm ALBA [9] is successful.

Keywords: Modal logic, Sahlqvist theory, canonical extensions, algorithmic correspondence, canonicity, distributive lattices.

Math. Subject Class. 03B45, 06D50, 06D10, 03G10, 06E15.

Contents

1. Introduction 1
2. Preliminaries 3
3. Jónsson-style canonicity for Sahlqvist inequalities 11
4. Generalized canonical extensions of maps 18
5. Generalized canonical extensions of $\mathcal{L}^{++}$-terms 24
6. Jónsson-style canonicity for ALBA-inequalities 31
7. Comparing the two approaches to canonicity 33
References 36
Appendix A. The algorithm ALBA 37
Appendix B. Algebraic Ackermann lemmas 39

1. Introduction

Canonicity. Canonicity is a fundamental notion in classical modal logic and other logics for which semantics based on relational structures are available, since it provides the main proof-path towards completeness results. Thanks to duality theory, canonicity can be investigated both in an algebraic and in a frame-theoretic setting. In each of these settings, suitable syntactic identifications on given classes of formulas or inequalities are sought for, which guarantee the following preservation condition to hold for each element φ in the class:

$$\mathbb{S} \models \varphi \quad \Rightarrow \quad \mathbb{S}^\delta \models \varphi.$$

On the algebraic side of this account, $\mathbb{S}$ is an algebra and $\mathbb{S}^\delta$ is its *canonical extension* (cf. [22]), and on its relational structure side, $\mathbb{S}$ is a frame with topological structure (e.g. a descriptive general frame as in [2]), and $\mathbb{S}^\delta$ is its underlying frame.

The research of the first and third author has been made possible by the NWO Vidi grant 016.138.314, by the NWO Aspasia grant 015.008.054, and by a Delft Technology Fellowship awarded in 2013.

Two approaches to canonicity. The existing canonicity results have been obtained by means of one of the following two approaches, here respectively referred to as the *canonicity-via-correspondence* approach, and the *Jónsson-style* approach.

The *canonicity-via-correspondence* approach is grounded in [24, 27]. In its present form it appears for the first time in [25], and has also been pursued in e.g. [20, 9]. The strategy of this approach follows an argument best illustrated by the following U-shaped diagram:

$$
\begin{array}{ccc}
\mathbb{G} \Vdash \varphi & & \mathbb{F} \Vdash \varphi \\
\Updownarrow & & \\
\mathbb{F} \Vdash_{\mathbb{G}} \varphi & & \Updownarrow \\
\Updownarrow & & \\
\mathbb{F} \models_{\mathbb{G}} \mathrm{FO}(\varphi) & \Leftrightarrow & \mathbb{F} \models \mathrm{FO}(\varphi).
\end{array}
$$

In the diagram above, $\mathbb{G} = (\mathbb{F}, \tau)$ is a descriptive general frame, $\mathbb{F}$ is the underlying Kripke frame of $\mathbb{G}$, and τ is its additional topological structure. The U-shaped argument relies on the existence of a first-order sentence $\mathrm{FO}(\varphi)$, the *first-order correspondent* of φ, which holds of the Kripke frame $\mathbb{F}$ regarded as a first-order model iff φ is valid on $\mathbb{F}$, as shown in the right-hand side of the diagram. On its left-hand side, by definition, φ is valid on $\mathbb{G}$ iff φ is satisfied on $\mathbb{F}$ with respect to every admissible valuation (as the notation $\Vdash_{\mathbb{G}}$ intends to represent). The proof succeeds if an analogous correspondence-type result holds restricted to admissible valuations, as represented by the vertical equivalence in the lower left-hand side of the diagram. Indeed, the bottom equivalence always holds, since the fact that $\mathrm{FO}(\varphi)$ holds does not depend on admissible valuations: in other words, $\mathrm{FO}(\varphi)$ cannot distinguish between $\mathbb{F}$ and $\mathbb{G}$.

As outlined by the U-shaped argument, the canonicity results obtained following this approach are in essence a byproduct of *correspondence theory*. The main contributions to this line of research have been the design of algorithms, such as the Sahlqvist-van Benthem algorithm [24, 27], and more recently SQEMA [7], for computing the first-order correspondents of large classes of formulas.

The core of the Sahlqvist-van Benthem algorithm is the well-known *minimal valuation argument* (see [2] and also [11] for a discussion). The algorithm SQEMA is based on the Ackermann lemma [1], which in its essence encodes the minimal valuation argument (cf. [6] for discussion about this point). Another interesting contribution to this line of research is the syntactic characterization of the class of the so called *inductive formulas* [20, 8], which properly extends the class of Sahlqvist formulas. Inductive formulas are shown to have first-order correspondent and be canonical. Intuitively, inductive formulas are defined in such a way that the minimal valuation argument, encoded in the form of the Ackermann lemma, is guaranteed to succeed.

The second approach to canonicity, the one referred to as *Jónsson-style*, originates in [21] and, independently, in [19], and has been pursued further in [17] (which builds on [21] and in which the similarities with [19] are recognized), and in [26] (which builds on [19] and, similarly, presents a *constructive* treatment of canonicity). The main features of this approach are its being purely algebraic, pursuing canonicity independently from correspondence, and above all, its crucially relying on the construction and theory of *canonical extensions*: indeed, there are two canonical ways to extend any operation (interpreting e.g. a given logical connective) of a given lattice expansion to operations on the canonical extension of the given lattice expansion. Canonicity results are then obtained via a study of the order-theoretic properties of the compositions of these extended operations.

Up to this point, the literature displays a division of labour of some sort between the two approaches. Namely, algebras are used for studying canonicity independently from correspondence, and Kripke frames and general frames are used for canonicity-via-correspondence.

In [9], this division of labour breaks down, and the algorithmic correspondence results of [8] are generalized to a setting of perfect distributive lattices with operators by means of the algorithm ALBA. In [10], a purely algebraic algorithmic correspondence result in a non-distributive setting is given, via a version of ALBA which is sound on non-distributive lattices. The results in [9] and [10] make clear that correspondence, and hence canonicity-via-correspondence, can also be developed on algebras. Moreover, together with [11], they make clear that algebraic correspondence is grounded on the same order-theoretic principles which guarantee the algebraic canonicity Jónsson-style, which forms the basis of *unified correspondence theory* [6]. This approach has also been generalized to a wide array of logics: monotone modal logic [14], modal mu-calculus [5, 4], hybrid logic [12], and regular modal logic [23].

Open issues. However, it is striking that, even if the Jónsson-style and the canonicity-via-correspondence approaches use the same order-theoretic principles and the same setting of perfect algebras, they still look radically different. So it is natural to try and clarify how they relate to one another.

In [28], an analysis was given of the Jónsson-style proof in [17] of the canonicity of Sahlqvist inequalities in distributive modal logic. This analysis was motivated by an attempt at extending the Jónsson-style method to the inductive/ALBA inequalities. As a result of this analysis, a close connection emerged between the key tool in the proof of [17]—which is referred to as "the **n**-trick" in [28]—and the Ackermann lemma, which, as mentioned above, is the key tool in the SQEMA-type algorithms. However, in [28], the problem of giving a Jónsson-style proof of canonicity for inductive inequality remained open, and it was observed that the proof method used in [17] to prove the canonicity of Sahlqvist formulas in the language of distributive modal logic cannot be straightforwardly extended to inductive formulas in the same language as defined in [9] (see Section 3.5 below for more details on this issue). So these remarks made the situation even more mysterious: from the point of view of correspondence, both Sahlqvist and inductive formulas are designed to guarantee that the minimal valuation argument, as encoded by the Ackermann lemma, succeeds; moreover, the Ackermann lemma and the **n**-trick are essentially one and the same thing. So why should the intimate similarity between Sahlqvist and inductive formulas break down when it comes to Jónsson-style canonicity?

Contributions. The present paper addresses these open issues: firstly, it clarifies the relationship between the two approaches to canonicity, and, as an application of the new insights, it extends the Jónsson-style canonicity proof to the inductive and ALBA inequalities.

Structure of the paper. In Section 2, the needed preliminaries are collected on canonical extensions and distributive modal logic. In Section 3, the Jónsson-style proof of canonicity of Sahlqvist DML-inequalities is reported on and discussed. In passing, we show that the proof in [17] can be simplified: specifically, the "brand new" Lemma 5.11 in [17] (cf. Lemma 3.12 below), is actually not needed. Section 4 contains the order-theoretic core of the canonicity result: the theory of canonical extensions of maps $\mathbb{A} \to \mathbb{B}^\delta$ to maps $\mathbb{A}^\delta \to \mathbb{B}^\delta$ is developed. In Section 5, the theory of generalized canonical extensions is applied to the terms of an expansion of the original language which originates in [9] motivated by the algorithmic correspondence theory. In Section 6, we prove that all DML-inequalities on which ALBA succeeds are canonical, using the Jónsson-style approach. In Section 7, we collect our findings about the relationship between the two strategies for achieving canonicity results.

2. Preliminaries

In the present section, we briefly review the preliminaries on DML and on canonical extensions. We refer the reader to [9], [13], [16], [17] and [28] for a more detailed discussion.

2.1. **Distributive modal logic.** Distributive modal logics (DMLs) are introduced in [17, Definition 2.1], and further studied in [9] and [28]. Given a set Prop of proposition variables, the *formulas* of the language $\mathcal{L}$ of distributive modal logic are defined as follows:

$$\alpha ::= p \mid \top \mid \bot \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \Diamond\alpha \mid \Box\alpha \mid \rhd\alpha \mid \lhd\alpha,$$

where $p \in \mathsf{Prop}$.
The intended meaning of $\Diamond, \Box, \rhd, \lhd$ is "it is possible that", "it is necessary that", "it is impossible that", "it is possibly not the case that", respectively. Since the resulting logic does not have the deduction detachment theorem, entailment cannot be captured by theoremhood, and hence it is encoded by sequents of the form $\alpha \Rightarrow \beta$, where α and β are formulas, and $\Rightarrow$ is a meta-level implication capturing entailment.

Definition 2.1 (**Distributive modal logic**)**.** A *distributive modal logic* (DML) is a set of $\mathcal{L}$-sequents containing the following sequents:

$$p \Rightarrow p \qquad \bot \Rightarrow p \qquad p \Rightarrow \top \qquad p \wedge (q \vee r) \Rightarrow (p \wedge q) \vee (p \wedge r)$$

$$p \Rightarrow p \vee q \qquad q \Rightarrow p \vee q \qquad p \wedge q \Rightarrow p \qquad p \wedge q \Rightarrow q$$

$$\Diamond(p \vee q) \Rightarrow \Diamond p \vee \Diamond q \qquad \Diamond\bot \Rightarrow \bot \qquad \Box p \wedge \Box q \Rightarrow \Box(p \wedge q) \qquad \top \Rightarrow \Box\top$$

$$\rhd p \wedge \rhd q \Rightarrow \rhd(p \vee q) \qquad \top \Rightarrow \rhd\bot \qquad \lhd(p \wedge q) \Rightarrow \lhd p \vee \lhd q \qquad \lhd\top \Rightarrow \bot$$

and closed under the following inference rules:

$$\frac{\alpha \Rightarrow \beta \quad \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma} \qquad \frac{\alpha \Rightarrow \beta}{\alpha(\gamma/x) \Rightarrow \beta(\gamma/x)} \qquad \frac{\alpha \Rightarrow \gamma \quad \beta \Rightarrow \gamma}{\alpha \vee \beta \Rightarrow \gamma} \qquad \frac{\gamma \Rightarrow \alpha \quad \gamma \Rightarrow \beta}{\gamma \Rightarrow \alpha \wedge \beta}$$

$$\frac{\alpha \Rightarrow \beta}{\Diamond\alpha \Rightarrow \Diamond\beta} \qquad \frac{\alpha \Rightarrow \beta}{\Box\alpha \Rightarrow \Box\beta} \qquad \frac{\alpha \Rightarrow \beta}{\rhd\beta \Rightarrow \rhd\alpha} \qquad \frac{\alpha \Rightarrow \beta}{\lhd\beta \Rightarrow \lhd\alpha}$$

where $p, q, r \in \mathsf{Prop}$ and α, β, γ are formulas.

The algebraic semantics of any distributive modal logic is given by a suitable class of bounded distributive lattice expansions. Since we are keeping little distinction between the syntax of the logic and its interpretation in algebras, formulas will be sometimes referred to as *terms*, and we will sometimes use the symbol $\mathcal{L}_{term}$ instead of $\mathcal{L}$. Likewise, since sequents are interpreted as inequalities in algebras, we will mostly use the notation $\alpha \leq \beta$ for sequents, and we let $\mathcal{L}_{\leq}$ be the set of $\mathcal{L}$-sequents. In what follows, we will also work with *quasi-inequalities* $(\alpha_1 \leq \beta_1 \& \ldots \& \alpha_n \leq \beta_n) \Rightarrow \alpha \leq \beta$, where & and $\Rightarrow$ denote the meta-level conjunction and implication, respectively. Let $\mathcal{L}_{quasi}$ be the set of $\mathcal{L}$-quasi-inequalities. We let DML denote the smallest distributive modal logic.

2.2. **Distributive modal algebras.** As is well known, given an algebra $\mathbb{A}$, a formula $\alpha(p_1, \ldots, p_n)$[1] induces an n-ary term function $\alpha^{\mathbb{A}} : \mathbb{A}^n \to \mathbb{A}$ on $\mathbb{A}$. For any assignment $h : \mathsf{Prop} \to \mathbb{A}$, the formula α is then interpreted as the element $[\![\alpha]\!]^{\mathbb{A}}_h = \alpha^{\mathbb{A}}(h(p_1), \ldots, h(p_n)) \in \mathbb{A}$ under h. Then, satisfaction with respect to assignments and validity of sequents $\alpha \Rightarrow \beta$ in $\mathbb{A}$ (notation: $(\mathbb{A}, h) \models \alpha \Rightarrow \beta$ and $\mathbb{A} \models \alpha \Rightarrow \beta$) are respectively defined in terms of the inequality $[\![\alpha]\!]^{\mathbb{A}}_h \leq [\![\beta]\!]^{\mathbb{A}}_h$ being true in $\mathbb{A}$, and of the inequality $\alpha^{\mathbb{A}} \leq \beta^{\mathbb{A}}$ being true in the function space $[\mathbb{A}^n \to \mathbb{A}]$ ordered pointwise. The sound and complete algebraic semantics of DML is given by distributive modal algebras (cf. [17, Definition 2.9]), defined as follows:

[1]As usual, the notation $\alpha(p_1, \ldots, p_n)$ indicates that the free proposition variables occurring in α form a subset of $\{p_1 \ldots p_n\}$.

Definition 2.2 (**Distributive modal algebra**). A *distributive modal algebra* (DMA) is an algebra $\mathbb{A} = (A, \vee, \wedge, \bot, \top, \Diamond, \Box, \rhd, \lhd)$, such that $(A, \vee, \wedge, \bot, \top)$ is a bounded distributive lattice (BDL), and the unary operations $\Diamond, \Box, \rhd, \lhd$ satisfy the following identities:

$$\Diamond(a \vee b) = \Diamond a \vee \Diamond b \quad \Diamond \bot = \bot \quad \Box(a \wedge b) = \Box a \wedge \Box b \quad \Box\top = \top$$

$$\rhd(a \vee b) = \rhd a \wedge \rhd b \quad \rhd \bot = \top \quad \lhd(a \wedge b) = \lhd a \vee \lhd b \quad \lhd\top = \bot$$

For any DMA $\mathbb{A}$, let $\mathbb{D}_{\mathbb{A}}$ be the BDL-reduct of $\mathbb{A}$. Hence, $\mathbb{A} = (\mathbb{D}_{\mathbb{A}}, \Diamond, \Box, \rhd, \lhd)$.

Satisfaction of quasi-inequalities $(\alpha_1 \leq \beta_1 \& \ldots \& \alpha_n \leq \beta_n) \Rightarrow \alpha \leq \beta$ in $\mathcal{L}_{quasi}$, in symbols:

$$(\mathbb{A}, h) \models (\alpha_1 \leq \beta_1 \& \ldots \& \alpha_n \leq \beta_n) \Rightarrow \alpha \leq \beta,$$

is defined as usual by requiring that whenever $[\![\alpha_i]\!]_h^{\mathbb{A}} \leq [\![\beta_i]\!]_h^{\mathbb{A}}$ for all $1 \leq i \leq n$ one has $[\![\alpha]\!]_h^{\mathbb{A}} \leq [\![\beta]\!]_h^{\mathbb{A}}$. Validity of quasi-inequalities is defined as usual as satisfaction for every assignment.

Since the signature of DMAs consists of both order-preserving and order-reversing operations, it will be convenient to introduce the following notation (cf. [17, Definition 2.10]).

For any BDL $\mathbb{A}$, let $\mathbb{A}^\partial$ denote its dual lattice, and let $\mathbb{A}^1 = \mathbb{A}$. For any $1 \leq n \in \mathbb{N}$, an *n-order type* ϵ is an element of $\{1, \partial\}^n$, and its i-th coordinate is denoted by ϵ_i. We omit n when it is clear from the context. Let ϵ^∂ denote the *dual order type* of ϵ, i.e. the order-type such that $\epsilon_i^\partial = 1$ (resp. ∂) if $\epsilon_i = \partial$ (resp. 1). For any n-order type ϵ, we let $\mathbb{A}^\epsilon$ be the product algebra $\mathbb{A}^{\epsilon_1} \times \ldots \times \mathbb{A}^{\epsilon_n}$.

2.3. **Canonical extensions.**

Definition 2.3 (**Canonical extension of BDLs**). (cf. [17, Definition 2.12]) The *canonical extension* of a BDL $\mathbb{A}$ is a complete BDL $\mathbb{A}^\delta$ containing $\mathbb{A}$ as a sublattice, and such that:

(*denseness*) every element of $\mathbb{A}^\delta$ is both a join of meets and a meet of joins of elements from $\mathbb{A}$;

(*compactness*) for all $S, T \subseteq \mathbb{A}$ with $\bigwedge S \leq \bigvee T$ in $\mathbb{A}^\delta$, there exist some finite subsets $F \subseteq S$ and $G \subseteq T$ such that $\bigwedge F \leq \bigvee G$.

It is well known that the canonical extension of a BDL $\mathbb{A}$ is unique up to an isomorphism fixing $\mathbb{A}$ (for a proof, see e.g. [18, Theorem 1]), and that the canonical extension of a BDL is a perfect BDL (cf. [17, Definition 2.14]):

Definition 2.4 (**Perfect BDL**). A *perfect* BDL $\mathbb{A}$ is a complete and completely distributive lattice which is both completely join generated by the set $J^\infty(\mathbb{A})$ of its completely join-irreducible elements and is completely meet generated by the set $M^\infty(\mathbb{A})$ of its completely meet-irreducible elements.

Theorem 2.5. *If $\mathbb{A}$ is a BDL, then $\mathbb{A}^\delta$ is a perfect BDL.*

An element $x \in \mathbb{A}^\delta$ is *closed* (resp. *open*), if it is the meet (resp. join) of some subset of $\mathbb{A}$. We let $K(\mathbb{A}^\delta)$ and $O(\mathbb{A}^\delta)$ respectively denote the sets of closed and open elements of $\mathbb{A}^\delta$. The denseness condition in the definition of canonical extension straightforwardly implies that $J^\infty(\mathbb{A}^\delta) \subseteq K(\mathbb{A}^\delta)$ and $M^\infty(\mathbb{A}^\delta) \subseteq O(\mathbb{A}^\delta)$.

The following properties can be readily checked to hold of the interaction between canonical extension, order duals, and products:

Lemma 2.6. (1) $(\mathbb{A}^\partial)^\delta \cong (\mathbb{A}^\delta)^\partial$;
(2) $(\mathbb{A}^n)^\delta \cong (\mathbb{A}^\delta)^n$;
(3) $(\mathbb{A}^\epsilon)^\delta \cong (\mathbb{A}^\delta)^\epsilon$;
(4) $K((\mathbb{A}^\partial)^\delta) = O(\mathbb{A}^\delta)^\partial$;

(5) $O((\mathbb{A}^\partial)^\delta) = K(\mathbb{A}^\delta)^\partial$;
(6) $K((\mathbb{A}^n)^\delta) = (K(\mathbb{A}^\delta))^n$;
(7) $O((\mathbb{A}^n)^\delta) = (O(\mathbb{A}^\delta))^n$.

2.3.1. *Canonical extensions of maps.* Let $\mathbb{A}, \mathbb{B}$ be BDLs. A map $f : \mathbb{A} \to \mathbb{B}$ can be extended to a map $\mathbb{A}^\delta \to \mathbb{B}^\delta$ in two canonical ways. Let f^σ and f^π respectively denote the σ and π-*extension* of f (cf. [17, Definition 2.15]) defined as follows:

Definition 2.7 (**σ- and π-extension**). For any $f : \mathbb{A} \to \mathbb{B}$ and all $u \in \mathbb{A}^\delta$, we define

$$f^\sigma(u) = \bigvee\{\bigwedge\{f(a) : a \in \mathbb{A} \text{ and } x \leq a \leq y\} : K(\mathbb{A}^\delta) \ni x \leq u \leq y \in O(\mathbb{A}^\delta)\};$$
$$f^\pi(u) = \bigwedge\{\bigvee\{f(a) : a \in \mathbb{A} \text{ and } x \leq a \leq y\} : K(\mathbb{A}^\delta) \ni x \leq u \leq y \in O(\mathbb{A}^\delta)\}.$$

Notice that the order of the codomain is all that matters to the definitions above: indeed, if $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A}^\partial \to \mathbb{B}$ are such that $f(a) = g(a)$ for all $a \in \mathbb{A}$, then $f^\sigma = g^\sigma$ and $f^\pi = g^\pi$. However, if $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{B}^\partial$ are such that $f(a) = g(a)$ for all $a \in \mathbb{A}$, then $f^\sigma = g^\pi$ and $f^\pi = g^\sigma$. Moreover, if for any map $f : \mathbb{A} \to \mathbb{B}$ we let $f^\partial : \mathbb{A}^\partial \to \mathbb{B}^\partial$ be such that $f^\partial(a) := f(a)$ for all $a \in \mathbb{A}$, we have that $(f^\partial)^\sigma = (f^\pi)^\partial$ and $(f^\partial)^\pi = (f^\sigma)^\partial$.
For order-preserving maps, the definition of canonical extensions can be simplified as follows:

Proposition 2.8. (cf. [17, Remark 2.17]) *If $f : \mathbb{A} \to \mathbb{B}$ is order-preserving, then for all $u \in \mathbb{A}^\delta$,*

$$f^\sigma(u) = \bigvee\{\bigwedge\{f(a) : x \leq a \in \mathbb{A}\} : u \geq x \in K(\mathbb{A}^\delta)\}$$
$$f^\pi(u) = \bigwedge\{\bigvee\{f(a) : y \geq a \in \mathbb{A}\} : u \leq y \in O(\mathbb{A}^\delta)\}.$$

2.3.2. *Continuity properties.* In the present subsection, we report on the definitions of some continuity properties relevant to the development of the paper. Throughout the present subsection, $\mathbb{A}_1, \ldots, \mathbb{A}_m$ and $\mathbb{B}$ denote arbitrary BDLs, and $\Pi_{i=1}^n \mathbb{A}_i$ denote $\mathbb{A}_1 \times \ldots \times \mathbb{A}_n$.

Definition 2.9 (**Upper and lower continuity**). For any map $f : \mathbb{A}^\delta \to \mathbb{B}^\delta$,

(1) f is *upper continuous* (UC) iff, for any $u \in \mathbb{A}^\delta$ and any $q \in J^\infty(\mathbb{B}^\delta)$, if $q \leq f(u)$ then there exist some $x \in K(\mathbb{A}^\delta)$ and $y \in O(\mathbb{A}^\delta)$ such that $x \leq u \leq y$ and $q \leq f(v)$ for all $v \in \mathbb{A}^\delta$ such that $x \leq v \leq y$;
(2) f is *lower continuous* (LC) iff, for any $u \in \mathbb{A}^\delta$ and any $n \in M^\infty(\mathbb{B}^\delta)$, if $n \geq f(u)$ then there exist some $x \in K(\mathbb{A}^\delta)$ and $y \in O(\mathbb{A}^\delta)$ such that $x \leq u \leq y$ and $n \geq f(v)$ for all $v \in \mathbb{A}^\delta$ such that $x \leq v \leq y$.

Theorem 2.10. (cf. [16, Theorem 2.15]) *For any map $f : \mathbb{A} \to \mathbb{B}$,*

(1) *$f^\sigma : \mathbb{A}^\delta \to \mathbb{B}^\delta$ is the largest UC extension of f to $\mathbb{A}^\delta$;*
(2) *$f^\pi : \mathbb{A}^\delta \to \mathbb{B}^\delta$ is the smallest LC extension of f to $\mathbb{A}^\delta$.*

The general definition of (UC) and (LC) will be only needed in Section 3.4 to discuss the "brand new lemma". Everywhere else, we will only make use of the following, simpler characterization which applies to order-preserving maps:

Proposition 2.11. *For any order-preserving map $f : \mathbb{A}^\delta \to \mathbb{B}^\delta$,*

($\forall J \exists K$) *f is UC iff, for any $u \in \mathbb{A}^\delta$ and any $q \in J^\infty(\mathbb{B}^\delta)$, if $q \leq f(u)$ then there exists some $x \in K(\mathbb{A}^\delta)$ such that $x \leq u$ and $q \leq f(x)$;*
($\forall M \exists O$) *f is LC iff, for any $u \in \mathbb{A}^\delta$ and any $n \in M^\infty(\mathbb{B}^\delta)$, if $n \geq f(u)$ then there exists some $y \in O(\mathbb{A}^\delta)$ such that $y \geq u$ and $n \geq f(y)$.*

Definition 2.12. For all order-preserving maps $f : \mathbb{A}^\delta \to \mathbb{B}^\delta$, and $g : \Pi_{i=1}^m \mathbb{A}_i^\delta \to \mathbb{B}^\delta$, (**Strong upper and lower continuity**)

($\forall K\exists K$) f is *strongly upper continuous* (SUC) iff, for any $u \in \mathbb{A}^\delta$ and any $q \in K(\mathbb{B}^\delta)$, if $q \leq f(u)$ then there exists some $x \in K(\mathbb{A}^\delta)$ such that $x \leq u$ and $q \leq f(x)$;

($\forall O\exists O$) f is *strongly lower continuous* (SLC) iff, for any $u \in \mathbb{A}^\delta$ and any $n \in O(\mathbb{B}^\delta)$, if $n \geq f(u)$ then there exists some $y \in O(\mathbb{A}^\delta)$ such that $y \geq u$ and $n \geq f(y)$.

(**Scott- and dual Scott-continuity**)

($\forall J\exists J$) f is *Scott continuous* if for all $u \in \mathbb{A}^\delta$ and all $q \in J^\infty(\mathbb{B}^\delta)$, if $q \leq f(u)$ then there exists some $x \in J^\infty(\mathbb{A}^\delta)$ such that $x \leq u$ and $q \leq f(x)$;

($\forall M\exists M$) f is *dually Scott continuous* if for all $u \in \mathbb{A}^\delta$ and all $n \in M^\infty(\mathbb{B}^\delta)$, if $n \geq f(u)$ then there exists some $y \in M^\infty(\mathbb{A}^\delta)$ such that $y \geq u$ and $n \geq f(y)$.

(**Weak Scott- and dual weak Scott-continuity**)

($\forall J\exists J^m$) g is *weakly Scott continuous* if for any $\overline{u} \in \Pi_{i=1}^m \mathbb{A}_i^\delta$, and any $q \in J^\infty(\mathbb{B}^\delta)$, if $q \leq f(\overline{u})$ then there exist some $\overline{x} \in \Pi_{i=1}^m J^\infty(\mathbb{A}_i^\delta)$ such that $\overline{x} \leq \overline{u}$ and $q \leq g(\overline{x})$;

($\forall M\exists M^m$) g is *dually weakly Scott continuous* if for any $\overline{u} \in \Pi_{i=1}^m \mathbb{A}_i^\delta$, and any $n \in M^\infty(\mathbb{B}^\delta)$, if $n \geq g(u_1, \ldots, u_m)$ then there exist some $\overline{y} \in \Pi_{i=1}^m M^\infty(\mathbb{A}_i^\delta)$ such that $\overline{y} \geq \overline{u}$ and $n \geq g(\overline{y})$.

Notice that these continuity properties are special cases of upper (resp. lower) continuity.

2.3.3. *Join- and meet-preservation properties.* In the present subsection, we collect the definitions of the preservation properties for joins and meets which are relevant to our treatment. Throughout the present subsection, $\mathbb{A}_1, \ldots, \mathbb{A}_m$ and $\mathbb{B}$ denote arbitrary BDLs.

Definition 2.13. Let $1 \leq i \leq n$. An n-ary map $f : \Pi_{i=1}^n \mathbb{A}_i \to \mathbb{B}$ is:[2]

- *additive* if f preserves non-empty finite joins in each coordinate.
- *multiplicative* if f preserves non-empty finite meets in each coordinate.
- *p-additive* if f preserves non-empty finite joins in $\Pi_{i=1}^n \mathbb{A}_i$.
- *p-multiplicative* if f preserves non-empty finite meets in $\Pi_{i=1}^n \mathbb{A}_i$.
- *normal* when for any $\overline{a} \in \Pi_{i=1}^n \mathbb{A}_i$, if $a_i = \bot$ for some $1 \leq i \leq n$, then $f(\overline{a}) = \bot$.
- *dually normal* when for any $\overline{a} \in \Pi_{i=1}^n \mathbb{A}_i$, if $a_i = \top$ for some $1 \leq i \leq n$, then $f(\overline{a}) = \top$.
- an *operator* if f is additive and normal.
- a *dual operator* if f is multiplicative and dually normal.
- *join-preserving* if f preserves arbitrary finite joins[3] in the product.
- *meet-preserving* if f preserves arbitrary finite meets[4] in the product.
- *completely additive* if f preserves non-empty arbitrary joins in each coordinate.
- *completely multiplicative* if f preserves non-empty arbitrary meets in each coordinate.
- *completely p-additive* if f preserves non-empty arbitrary joins in $\Pi_{i=1}^n \mathbb{A}_i$.
- *completely p-multiplicative* if f preserves non-empty arbitrary meets in $\Pi_{i=1}^n \mathbb{A}_i$.
- a *complete operator* if f preserves arbitrary joins in each coordinate.
- a *complete dual operator* if f preserves arbitrary meets in each coordinate.
- *completely join-preserving* if f preserves arbitrary joins in the product.
- *completely meet-preserving* if f preserves arbitrary meets in the product.

2.3.4. *Canonical extensions of DMAs.* Since each unary operation in a DMA is either meet-preserving, join-preserving, join-reversing, or meet-reversing, each of them is *smooth*, that is: $f^\sigma = f^\pi$ for each $f \in \{\Box, \Diamond, \lhd, \rhd\}$ (cf. [16, Lemma 2.25]). Therefore, the canonical extension of a DMA can be defined as follows:

[2] This terminology is taken from [3].

[3] Hence also $\bot = \bigvee \varnothing$ is preserved.

[4] Hence also $\top = \bigwedge \varnothing$ is preserved.

Definition 2.14 (**Canonical extension of DMAs**). (cf. [17, Definitions 2.19 and 2.20]) The canonical extension of any DMA $\mathbb{A} = (\mathbb{D}_\mathbb{A}, \Diamond, \Box, \rhd, \lhd)$ is

$$\mathbb{A}^\delta = (\mathbb{D}^\delta_\mathbb{A}, \Diamond^\sigma, \Box^\sigma, \rhd^\sigma, \lhd^\sigma) = (\mathbb{D}^\delta_\mathbb{A}, \Diamond^\pi, \Box^\pi, \rhd^\pi, \lhd^\pi).$$

We will sometimes abuse notation and write $(\mathbb{D}^\delta_\mathbb{A}, \Diamond, \Box, \rhd, \lhd)$ for $\mathbb{A}^\delta$.

Definition 2.15 (**Perfect DMA**). A DMA $\mathbb{A} = (\mathbb{D}_\mathbb{A}, \Diamond, \Box, \rhd, \lhd)$ is *perfect* if $\mathbb{D}_\mathbb{A}$ is a perfect BDL, and the operations of the modal expansion satisfy the following complete distribution properties: for any $X \subseteq \mathbb{D}_\mathbb{A}$,

$$\Diamond(\bigvee X) = \bigvee \Diamond(X) \quad \rhd(\bigvee X) = \bigwedge \rhd(X)$$
$$\Box(\bigwedge X) = \bigwedge \Box(X) \quad \lhd(\bigwedge X) = \bigvee \lhd(X).$$

Theorem 2.16. (cf. [17, Lemma 2.21]) *If $\mathbb{A}$ is a DMA, then $\mathbb{A}^\delta$ is a perfect DMA.*

Clearly, every $\alpha \in \mathcal{L}_{term}$ gives rise to its corresponding term function in any DMA, and hence also in the canonical extension $\mathbb{A}^\delta$ of any given DMA $\mathbb{A}$. In addition to this, the term function $\alpha^\mathbb{A}$ can also be extended to $\mathbb{A}^\delta$ via its σ- and π-extension. The following terminology (cf. [17, Definition 5.2]) helps in comparing these functions.

Definition 2.17 (**Stable, expanding and contracting extensions of maps**). Let $\mathbb{A}$, $\mathbb{B}$ be BDLs, let $f : \mathbb{A} \to \mathbb{B}$, and let $g : \mathbb{A}^\delta \to \mathbb{B}^\delta$ be an extension of f. For $\lambda \in \{\sigma, \pi\}$,

- g is λ-*stable* if $g = f^\lambda$;
- g is λ-*expanding* if $g \leq f^\lambda$;
- g is λ-*contracting* if $g \geq f^\lambda$.

A term $\alpha \in \mathcal{L}$ is λ-*stable* (resp. λ-*expanding*, λ-*contracting*) if so is its corresponding term function $\alpha^\mathbb{A}$ for any DMA $\mathbb{A}$.

Lemma 2.18 (cf. [17], Lemma 5.5). *Every uniform map $f : \mathbb{A}^\epsilon \to \mathbb{A}$ is both σ-contracting and π-expanding.*

2.4. **The distributive modal languages $\mathcal{L}^+$ and $\mathcal{L}^{++}$.** The algebraic and algorithmic correspondence theory (cf. [9]) makes crucial use of an expanded language $\mathcal{L}^+$, which is naturally interpreted on perfect DMAs. In the present section, we review the language $\mathcal{L}^+$, and introduce a further expansion $\mathcal{L}^{++}$ of it, both of which will be used in the following sections.

The expanded language $\mathcal{L}^+$ includes the connectives corresponding to the adjoint and residuals of all the operations in the DMA signature, as well as a denumerably infinite set of sorted variables NOM called *nominals*, ranging over the completely join-prime elements of perfect DMAs, and a denumerably infinite set of sorted variables $\mathsf{CO\text{-}NOM}$, called *co-nominals*, ranging over the completely meet-prime elements of perfect DMAs. The elements of NOM and $\mathsf{CO\text{-}NOM}$ will be respectively denoted by $\mathbf{i}, \mathbf{j}$, and $\mathbf{m}, \mathbf{n}$, possibly indexed. Let us introduce the language $\mathcal{L}^+$ formally:

$$\alpha ::= \bot \mid \top \mid p \mid \mathbf{i} \mid \mathbf{m} \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \Diamond\alpha \mid \Box\alpha \mid \rhd\alpha \mid \lhd\alpha \mid \alpha \to \beta \mid \alpha - \beta \mid \blacklozenge\alpha \mid \blacksquare\alpha \mid \blacktriangleleft\alpha \mid \blacktriangleright\alpha$$

where $p \in \mathsf{Prop}$, $\mathbf{i} \in \mathsf{NOM}$ and $\mathbf{m} \in \mathsf{CO\text{-}NOM}$. As is well known from general order theory (cf. [9, Section 2.5] for an expanded discussion), each operation of a perfect DMA has an adjoint or a residual, introduced in the table below. Each operation in the lower row is intended to be interpreted as the adjoint or residual of the interpretation of the corresponding operation in the upper row:

$\wedge$	$\vee$	$\Diamond$	$\Box$	$\lhd$	$\rhd$
$\to$	$-$	$\blacksquare$	$\blacklozenge$	$\blacktriangleleft$	$\blacktriangleright$

In particular, for every $u, v, w \in \mathbb{C}$,

$$u \wedge v \leq w \ \text{ iff } \ u \leq v \rightarrow w \quad \text{ and } \quad u - v \leq w \ \text{ iff } \ u \leq v \vee w,$$

and as for the remaining operations, for every $u, v \in \mathbb{C}$,

$$\begin{array}{ll} \Diamond u \leq v \ \text{ iff } \ u \leq \blacksquare v & \quad u \leq \Box v \ \text{ iff } \ \blacklozenge u \leq v \\ \lhd u \leq v \ \text{ iff } \ \blacktriangleleft v \leq u & \quad u \leq \rhd v \ \text{ iff } \ v \leq \blacktriangleright u. \end{array}$$

Summing up:

Complete operators	$\wedge : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$	$\Diamond : \mathbb{C} \rightarrow \mathbb{C}$	$\lhd : \mathbb{C}^\partial \rightarrow \mathbb{C}$
Right adjoints	$\rightarrow : \mathbb{C}^\partial \times \mathbb{C} \rightarrow \mathbb{C}$	$\blacksquare : \mathbb{C} \rightarrow \mathbb{C}$	$\blacktriangleleft : \mathbb{C} \rightarrow \mathbb{C}^\partial$
Complete dual operators	$\vee : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$	$\Box : \mathbb{C} \rightarrow \mathbb{C}$	$\rhd : \mathbb{C}^\partial \rightarrow \mathbb{C}$
Left adjoints	$- : \mathbb{C} \times \mathbb{C}^\partial \rightarrow \mathbb{C}$	$\blacklozenge : \mathbb{C} \rightarrow \mathbb{C}$	$\blacktriangleright : \mathbb{C} \rightarrow \mathbb{C}^\partial$

We let $\mathcal{L}^+_\leq$ and $\mathcal{L}^+_{quasi}$ respectively be the sets of sequents and quasi inequalities in the expanded language $\mathcal{L}^+$. A formula $\alpha \in \mathcal{L}^+$ is *pure* if no proposition variables occur in it.
One technical tool which is key to our study is the language expansion $\mathcal{L}^{++}$ of $\mathcal{L}^+$. The language $\mathcal{L}^{++}$ is expressive enough so that truth-preserving and reflecting translations can be established (in fact, more than one) from quasi-inequalities in $\mathcal{L}^+$ to inequalities in $\mathcal{L}^{++}$, and from inequalities in $\mathcal{L}^+$ to terms in $\mathcal{L}^{++}$.
The language $\mathcal{L}^{++}$ is obtained by expanding $\mathcal{L}^+$ with the binary connectives $\mathbf{n}$ and $\mathbf{l}$, the intended interpretation of which, on any perfect DMA $\mathbb{A}$, is respectively given by the binary operations $\mathbf{n} : \mathbb{A} \times \mathbb{A}^\partial \rightarrow \mathbb{A}$ and $\mathbf{l} : \mathbb{A}^\partial \times \mathbb{A} \rightarrow \mathbb{A}$ defined as follows: for all $x, y \in \mathbb{A}$,

$$\mathbf{n}(x, y) := \begin{cases} \bot & \text{if } x \leq_\mathbb{A} y \\ \top & \text{if } x \not\leq_\mathbb{A} y, \end{cases} \qquad \mathbf{l}(x, y) := \begin{cases} \bot & \text{if } x \not\leq_\mathbb{A} y \\ \top & \text{if } x \leq_\mathbb{A} y. \end{cases}$$

An assignment in the setting of $\mathcal{L}^+$ and $\mathcal{L}^{++}$ is a map $h : \mathsf{Prop} \cup \mathsf{NOM} \cup \mathsf{CO\text{-}NOM} \rightarrow \mathbb{A}^\delta$ such that $h(\mathbf{i}) \in J^\infty(\mathbb{A}^\delta)$ for any $\mathbf{i} \in \mathsf{NOM}$ and $h(\mathbf{m}) \in M^\infty(\mathbb{A}^\delta)$ for any $\mathbf{m} \in \mathsf{CO\text{-}NOM}$. Satisfaction and validity of terms, inequalities and quasi-inequalities in $\mathcal{L}^+$ and $\mathcal{L}^{++}$ are then defined as usual (cf. Section 2.2). The sets of inequalities and quasi-inequalities in $\mathcal{L}^+$ and $\mathcal{L}^{++}$ are denoted $\mathcal{L}^+_\leq$, $\mathcal{L}^+_{quasi}$, $\mathcal{L}^{++}_\leq$ and $\mathcal{L}^{++}_{quasi}$, respectively. The following proposition can be easily verified:

Proposition 2.19. *For any DMA $\mathbb{A}$, any assignment $h : \mathsf{Prop} \rightarrow \mathbb{A}$, any $\mathcal{L}$-inequality $\alpha \leq \beta$ and any $\mathcal{L}$-terms φ, ψ, α_i, β_i , γ_j, δ_j for $1 \leq i \leq n$ and $1 \leq j \leq m$*

(1) $(\mathbb{A}, h) \models \alpha \leq \beta$ *iff* $(\mathbb{A}, h) \models \mathbf{l}(\alpha, \beta)$;
(2) $(\mathbb{A}, h) \models \varphi \wedge \bigwedge_{i=1}^n \mathbf{l}(\alpha_i, \beta_i) \leq \psi \vee \bigvee_{j=1}^m \mathbf{l}(\gamma_j, \delta_j)$ *iff* $(\mathbb{A}, h) \models \varphi \wedge \bigwedge_{j=1}^m \mathbf{n}(\gamma_j, \delta_j) \leq \psi \vee \bigvee_{i=1}^n \mathbf{n}(\alpha_i, \beta_i)$.

For any $\mathcal{L}^+$-quasi-inequality $\Phi := (\&_{k=0}^{n-1} \alpha_k \leq \beta_k) \Rightarrow \alpha_n \leq \beta_n$, let us consider the following $\mathcal{L}^{++}$-inequalities:

$$\mathsf{T}_1(\Phi) := \bigwedge_{k=0}^{n-1} \mathbf{l}(\alpha_k, \beta_k) \leq \mathbf{l}(\alpha_n, \beta_n) \qquad \mathsf{T}_2(\Phi) := \mathbf{n}(\alpha_n, \beta_n) \leq \bigvee_{k=0}^{n-1} \mathbf{n}(\alpha_k, \beta_k).$$

The following proposition can be easily verified:

Proposition 2.20. *For every perfect DMA $\mathbb{A}$, any assignment $h : \mathsf{Prop} \rightarrow \mathbb{A}$, and any $\Phi \in \mathcal{L}^+_{quasi}$,*

$$(\mathbb{A}, h) \models \Phi \ \textit{ iff } \ (\mathbb{A}, h) \models \mathsf{T}_1(\Phi) \ \textit{ iff } \ (\mathbb{A}, h) \models \mathsf{T}_2(\Phi).$$

2.5. **Sahlqvist and inductive inequalities.** In the present subsection, we apply the unified correspondence versions of the definitions of Sahlqvist and inductive inequalities (cf. [9, Definition 3.1] and [6, Definition 36.5]) to the languages $\mathcal{L}$, $\mathcal{L}^+$ and $\mathcal{L}^{++}$.

Definition 2.21 (**Signed generation trees of $\mathcal{L}^{++}$-terms**)**.** The two *signed generation trees* associated with any $s \in \mathcal{L}^{++}$ are denoted $+s$ and $-s$ respectively and are obtained by assigning signs ($+$ and $-$) to the nodes of the generation tree of s, as follows:

- The root node of $+s$ (resp. $-s$) is the root node of the generation tree of s, signed with $+$ (resp. $-$).
- If a node is labelled with $\vee, \wedge, \Box, \Diamond, \blacksquare$, or $\blacklozenge$, the same sign is assigned to its child(ren) node(s).
- If a node is labelled with $\lhd$, $\rhd$, $\blacktriangleleft$, or $\blacktriangleright$, the opposite sign is assigned to its child node.
- If a node is labelled with $\rightarrow$ or $\mathbf{l}$, the opposite sign is assigned to its left child node, and the same sign is assigned to its right child node.
- If a node is labelled with $-$ or $\mathbf{n}$, the same sign is assigned to its left child node, and the opposite sign is assigned to its right child node.

A node in a signed generation tree is *positive* (resp. *negative*) if it is signed $+$ (resp. $-$).

We let $+s$ (resp. $-s$) denote the positive (resp. negative) signed generation tree of the term $s \in \mathcal{L}^{++}_{term}$. For $\star, * \in \{+, -\}$, the symbol $\star s \prec * \alpha$ (sometimes $s \prec \alpha$) indicates that $\star s$ is a signed subtree (a subterm) of $*\alpha$.

Definition 2.22. For any order-type $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, any term $s(p_1, \ldots, p_n) \in \mathcal{L}^{++}_{term}$ and $* \in \{+, -\}$,

- an *ϵ-critical node* in the signed generation tree $*s$ is a leaf node $+p_i$ if $\epsilon_i = 1$ or $-p_i$ if $\epsilon_i = \partial$. In this case, we will say that this occurrence of p_i in $*s$ *agrees with* ϵ;
- an *ϵ-critical branch* in the tree is a branch terminating in an ϵ-critical node;
- $*s$ *agrees with* ϵ (notation: $\epsilon(*s)$) if every occurrence of every proposition variable in $*s$ agrees with ϵ. We also say that $*s$ is *ϵ-uniform*;
- $*s$ is *uniform* if $*s$ is *ϵ-uniform* for some order-type ϵ.

The following definitions are given for $\mathcal{L}$ only:

Skeleton	PIA
Δ-adjoints	SRA
$+ \quad \vee \quad \wedge$	$+ \quad \Box \quad \rhd \quad \wedge$
$- \quad \wedge \quad \vee$	$- \quad \Diamond \quad \lhd \quad \vee$
SLR	SRR
$+ \quad \wedge \quad \Diamond \quad \lhd$	$+ \quad \vee$
$- \quad \vee \quad \Box \quad \rhd$	$- \quad \wedge$

TABLE 1. Classification of nodes

Definition 2.23. (cf. [6, Definition 36.4]) Nodes in generation trees of $\mathcal{L}$-terms are classified according to table 1. For $* \in \{+, -\}$, a branch in a signed generation tree $*s$ is:

- a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes, and P_2 consists (apart from variable nodes) only of Skeleton-nodes.

- an *excellent branch* if it is good, and moreover P_1 consists (apart from variable nodes) only of SRA-nodes.

Definition 2.24 (Sahlqvist inequalities). (cf. [6, Definition 36.5]) Given an order type ϵ, the signed generation tree of a term $s(p_1, \ldots p_n)$ is ϵ-*Sahlqvist* if every ϵ-critical branch is excellent. An inequality $s \leq t$ is ϵ-*Sahlqvist* if the trees $+s$ and $-t$ are both ϵ-Sahlqvist. An $\mathcal{L}$-inequality is *Sahlqvist* if it is ϵ-Sahlqvist for some ϵ.

Definition 2.25 (Inductive inequalities). (cf. [10, Section 2.2]) Given an order type ϵ and a strict partial order $<_\Omega$ on the variables $p_1, \ldots p_n$, the signed generation tree of a term $s(p_1, \ldots p_n)$ is (Ω, ϵ)-*inductive* if for all $1 \leq i \leq n$, every ϵ-critical branch with leaf labelled p_i is good, and moreover, for every SRR node $*(\alpha \circledast \beta)$ on the branch, it holds that for some $\gamma \in \{\alpha, \beta\}$

(1) $\epsilon^\partial(*\gamma)$, and
(2) $p_j <_\Omega p_i$ for every p_j occurring in γ.

We will refer to $<_\Omega$ as the *dependency order* on the variables. An inequality $s \leq t$ is (Ω, ϵ)-*inductive* if the trees $+s$ and $-t$ are both (Ω, ϵ)-inductive. An inequality $s \leq t$ is *inductive* if it is (Ω, ϵ)-inductive for some Ω and ϵ.

Remark 2.26. Definitions 2.24 and 2.25 are equivalent but different from those of [17] and [9], and rather follow [6]. As discussed in [6], this way of defining Sahlqvist and inductive inequalities, given purely in terms of the order-theoretic properties of the algebraic interpretations of the logical connectives, is designed so as to remain essentially unchanged when applied to different signatures and different logics (the differences between signatures being captured by the classifications of nodes appropriate to each setting). In Section 3.3, we will expand on how the order theoretic-properties of the 'Sahlqvist shape' guarantee the Jónsson-style proof of canonicity to go through.

Example 2.27. The inequality $\Box(p \vee \lhd q) \wedge \Box q \leq \Diamond(p \wedge q)$ in $\mathcal{L}_\leq$ is (Ω, ϵ)-inductive with $q \leq_\Omega p$, $\epsilon_p = 1$ and $\epsilon_q = 1$, but it is not ϵ-Sahlqvist for any ϵ.

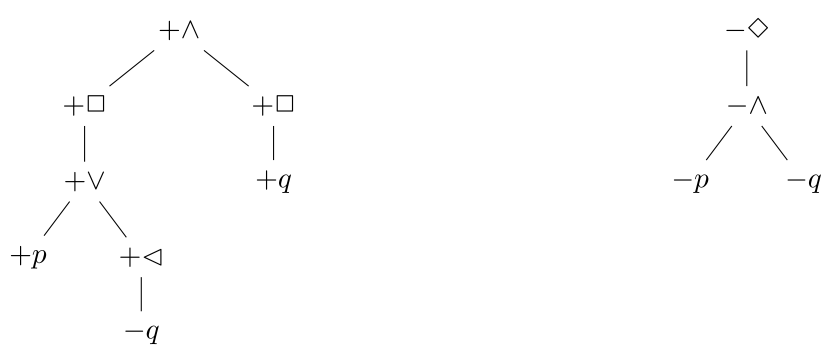

FIGURE 1. Signed generation tree for $\Box(p \vee \lhd q) \wedge \Box q \leq \Diamond(p \wedge q)$

3. Jónsson-style canonicity for Sahlqvist inequalities

In the present section, we review the Jónsson-style proof of canonicity for Sahlqvist DML-inequalities, as given in [17]. In Section 3.4, we show how this proof can be simplified, and then in Section 3.5, we briefly discuss the difficulties in generalizing this method to inductive inequalities, and propose some directions for a solution.

3.1. **The canonicity theorem.** In [17], it is shown (modulo some corrections detailed in [23]) that any Sahlqvist inequality is preserved under taking canonical extensions of DMAs:

Theorem 3.1 (cf. [17], Theorem 5.1). *For any DMA $\mathbb{A}$ and any Sahlqvist inequality $\alpha \leq \beta$,*

$$\mathbb{A} \models \alpha \leq \beta \quad \Rightarrow \quad \mathbb{A}^\delta \models \alpha \leq \beta. \tag{1}$$

Proof. The proof can be schematically illustrated by the following chain:

$$\begin{array}{lll} & \mathbb{A} \models \alpha \leq \beta & \\ \Longleftrightarrow & \alpha^{\mathbb{A}} \leq \beta^{\mathbb{A}} & \text{(by definition)} \\ \Longleftrightarrow & \alpha_1^{\mathbb{A}} \leq \beta_1^{\mathbb{A}} \vee \gamma^{\mathbb{A}} & \text{(Lemma 3.3)} \\ \Longleftrightarrow & \left(\alpha_1^{\mathbb{A}}\right)^\sigma \leq \left(\beta_1^{\mathbb{A}} \vee \gamma^{\mathbb{A}}\right)^\sigma & \text{(definition of } \sigma\text{-extension)} \\ \Longrightarrow & \left(\alpha_1^{\mathbb{A}}\right)^\sigma \leq \left(\beta_1^{\mathbb{A}}\right)^\pi \vee \left(\gamma^{\mathbb{A}}\right)^\sigma & \text{(Lemma 3.12)} \\ \Longrightarrow & \alpha_1^{\mathbb{A}^\delta} \leq \beta_1^{\mathbb{A}^\delta} \vee \gamma^{\mathbb{A}^\delta} & \text{(Lemma 2.18 and 3.11)} \\ \Longleftrightarrow & \alpha^{\mathbb{A}^\delta} \leq \beta^{\mathbb{A}^\delta} & \text{(Lemma 3.3)} \\ \Longleftrightarrow & \mathbb{A}^\delta \models \alpha \leq \beta. & \text{(by definition)} \end{array}$$

In the chain above, α_1, β_1 and γ are such that $+\alpha_1$ is ϵ'-uniform and Sahlqvist, $-\beta_1$ is ϵ'-uniform and Sahlqvist, and $+\gamma$ is ϵ'-uniform for some order-type ϵ' extending ϵ. □

There are two key steps to the proof of Theorem 3.1. The first step equivalently transforms the inequality $\alpha \leq \beta$ into some inequality $\alpha_1 \leq \beta_1 \vee \gamma$ s.t. $\alpha_1 \leq \beta_1$ is a uniform Sahlqvist inequality, γ is a uniform term, and certain additional conditions are satisfied, as detailed in Lemma 3.3. The second step consists in showing that if $\alpha_1 \leq \beta_1$ is uniform Sahlqvist, then α_1 is σ-expanding and β_1 is π-contracting (cf. [17, Lemma 5.10], emended as indicated in [23, Remark 5.4]), and that if γ is uniform, then γ is σ-contracting (cf. [17, Lemma 5.5]). This step is not the object of the refinement of the present paper. For a discussion about it, the reader is referred to the companion paper [23] in which this step is discussed, refined, and generalized.[5]
In the following subsection, we give a closer look to the first step, and among other things, we discuss an alternative proof which does not make use of [17, Lemma 5.11].
These two steps will be expanded on in Sections 3.2 and 3.3 respectively. In Section 3.4 we will provide a refinement of the proof above, which does not rely on Lemma 3.12 (i.e. [17, Lemma 5.11]).

3.2. **Minimal collapse algorithm, n-trick, and Ackermann lemma.** In the present subsection, we provide some details on how any ϵ-Sahlqvist inequality $\alpha \leq \beta$ is proved to be equivalent to some inequality $\alpha_1 \leq \beta_1 \vee \gamma$ such that $+\alpha_1$ is ϵ'-uniform and Sahlqvist, $-\beta_1$ is ϵ'-uniform and Sahlqvist, and γ is ϵ'-uniform for some order-type ϵ' extending ϵ. Towards this end, the connective $\mathbf{n}$, defined in subsection 2.5, will be used to "extract" maximally ϵ^∂-uniform subterms from $+\alpha$ and $-\beta$, and to form γ.

[5] In [23], this second step, as laid out in [17], is closely analyzed. As one consequence of this analysis, some mistakes in the proof of canonicity given in [17] are found and emended. Specifically, some steps in that proof rely on certain order-theoretic assumptions about the interpretations of the logical connectives which turn out to not be satisfied by all connectives involved; however, it is shown that Jónsson strategy goes through all the same under weaker assumptions, which are satisfied in the setting of [17].

Recall that, for any DMA $\mathbb{A}$, the interpretation of the connective $\mathbf{n}$ is the map $\mathbf{n}^{\mathbb{A}} : \mathbb{A}\times\mathbb{A}^{\partial} \to \mathbb{A}$ defined as follows:

$$\mathbf{n}^{\mathbb{A}}(a,b) := \begin{cases} \bot & \text{if } a \leq b \\ \top & \text{if } a \not\leq b. \end{cases}$$

It is easy to check that $\mathbf{n}^{\mathbb{A}} : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is join-preserving in the first coordinate and meet-reversing in the second coordinate.

Let $\mathcal{L}^{\mathbf{n}}$ denote the language $\mathcal{L}$ expanded with $\mathbf{n}$. The languages $\mathcal{L}^{\mathbf{n}}_{term}$, $\mathcal{L}^{\mathbf{n}}_{\leq}$ and $\mathcal{L}^{\mathbf{n}}_{quasi}$ can be defined similarly to what is done in Section 2.4, and stable, expanding and contracting $\mathcal{L}^{\mathbf{n}}$-terms can be defined similarly to Definition 2.17.

In what follows, for a given term α, a subterm $s \prec \alpha$, and a fresh variable z, we will abuse notation and let the symbol $\alpha(z/s)$ denote the term obtained by replacing the given *occurrence* of s in α with the fresh variable z.

The desired equivalent transformation from $\alpha \leq \beta$ to $\alpha_1 \leq \beta_1 \vee \gamma$ relies on the following lemma:

Lemma 3.2 (**n-trick**)**.** *Let $\alpha, \beta, s \in \mathcal{L}^{\mathbf{n}}_{term}$, and z be a new variable not occurring in α or β.*

(1) *if $+s \prec +\alpha$ and $+s \prec -\beta$, then $\mathbb{A} \models \alpha \leq \beta$ iff $\mathbb{A} \models \alpha(z/s) \leq \beta(z/s) \vee \mathbf{n}(z,s)$;*
(2) *if $-s \prec +\alpha$ and $-s \prec -\beta$, then $\mathbb{A} \models \alpha \leq \beta$ iff $\mathbb{A} \models \alpha(z/s) \leq \beta(z/s) \vee \mathbf{n}(s,z)$.*

The following lemma is obtained by exhaustive application of Lemma 3.2:

Lemma 3.3 (cf. Lemma 5.14 in [17])**.** *Let $\alpha \leq \beta$ be an $\mathcal{L}$-inequality, and let $+s_i \prec +\alpha$ for $1 \leq i \leq m$, $-s'_i \prec +\alpha$ for $1 \leq i \leq m'$, $+t_i \prec -\beta$ for $1 \leq i \leq l$ and $-t'_i \prec -\beta$ for $1 \leq i \leq l'$ be all the maximally ϵ^{∂}-uniform subterms*[6] *of $+\alpha$ and $-\beta$. The following are equivalent:*

(1) *$\alpha^{\mathbb{A}} \leq \beta^{\mathbb{A}}$;*
(2) *$\alpha_1^{\mathbb{A}} \leq \beta_1^{\mathbb{A}} \vee \gamma^{\mathbb{A}}$, where*
  (a) *$\alpha_1 = \alpha(\vec{z}/\vec{s}, \vec{z'}/\vec{s'})$,*
  (b) *$\beta_1 = \beta(\vec{w}/\vec{t}, \vec{w'}/\vec{t'})$,*
  (c) *$\gamma = \bigvee_{i=1}^{m} \mathbf{n}(z_i, s_i) \vee \bigvee_{i=1}^{m'} \mathbf{n}(s'_i, z'_i) \vee \bigvee_{i=1}^{l} \mathbf{n}(w_i, t_i) \vee \bigvee_{i=1}^{l'} \mathbf{n}(t'_i, w'_i)$.*

Moreover, if $\alpha \leq \beta$ is ϵ-Sahlqvist, then $+\alpha_1$ is ϵ'-uniform and Sahlqvist, $-\beta_1$ is ϵ'-uniform and Sahlqvist, and γ is ϵ'-uniform, where ϵ' is the order-type extending ϵ to the new variables $\vec{z}, \vec{w}, \vec{z'}, \vec{w'}$ by assigning all variables in $\vec{z}, \vec{w}$ to 1 and all variables in $\vec{z'}, \vec{w'}$ to ∂.

Before moving on, we expand on an observation, made in [28], that Lemma 3.3 bears a close resemblance with the Ackermann lemma [1]. Indeed, it is easy to see that the following universal[7] versions of the Ackermann lemma are equivalent to Lemma 3.3.

Lemma 3.4 (Universal Ackermann lemmas)**.** *Let $\mathbb{A}$ be a DMA, $\alpha \leq \beta$ be an inequality in $\mathcal{L}^{\mathbf{n}}_{\leq}$, and z be a new variable which does not occur in both α and β. Then:*

(1) *if $+s \prec +\alpha$ and $+s \prec -\beta$, then $\mathbb{A} \models \alpha \leq \beta$ iff $\mathbb{A} \models z \leq s \Rightarrow \alpha(z/s) \leq \beta(z/s)$;*
(2) *if $-s \prec +\alpha$ and $-s \prec -\beta$, then $\mathbb{A} \models \alpha \leq \beta$ iff $\mathbb{A} \models s \leq z \Rightarrow \alpha(z/s) \leq \beta(z/s)$.*

The equivalence between e.g. items 1. of Lemma 3.3 and Lemma 3.4 relies on the equivalence between the following conditions, for any DMA $\mathbb{A}$ and any assignment h on $\mathbb{A}$:

(1) $\mathbb{A}, h \models \alpha(z/s) \leq \beta(z/s) \vee \mathbf{n}(z,s)$;
(2) $\mathbb{A}, h \models z \leq s \Rightarrow \alpha(z/s) \leq \beta(z/s)$.

[6] For any terms s, α such that $\star s \prec *\alpha$, we say that $\star s$ is a maximally ϵ^{∂}-uniform subterm if $\epsilon^{\partial}(\star s)$ and any other term s' such that $s' \neq s$ and $s \prec s' \prec \alpha$ is not ϵ^{∂}-uniform.

[7] The statement of Lemma 3.4 is referred to as "universal" because when applied to a logical setting, it is about the validity of term inequalities on algebras. In Section B, we will treat the existential version, which, when applied to a logical setting, is about the satisfaction of term inequalities.

Indeed, if $z \nleq s$ is true under h, then the two conditions are trivially true. If the $z \leq s$ is true, then both conditions hold iff $\alpha(z/s) \leq \beta(z/s)$ is true under h.

3.3. **Contracting and expanding terms.** In the present subsection, we outline the analysis of the second step in the Jónsson-style canonicity. Our account is based on [23, Sections 4.2 and 5].

The core of Jónsson's method as applied in [17] is the proof that if $\alpha_1 \leq \beta_1$ is uniform Sahlqvist, then α_1 is σ-expanding and β_1 is π-contracting. We only consider the case of α_1 in the present subsection, the case of β_1 being order-dual.

Recall that the σ-extension of a monotone map is its greatest UC extension (cf. Fact 2.10). Hence, in order to show that a term t is σ-expanding, i.e. that $t^{\mathbb{A}^\delta} \leq (t^{\mathbb{A}})^\sigma$, it suffices to show that $t^{\mathbb{A}^\delta}$ is UC. The term function $t^{\mathbb{A}^\delta}$ is the composition of the σ-extensions of the interpretations of the logical connectives occurring in t.

Let us start by considering the composition of two monotone maps $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{C}$, in search for conditions which guarantee the composition $g^\sigma f^\sigma$ to be UC.

Recall that upper continuity, Scott-continuity, and strong upper-continuity are similar conditions, and specifically, their quantification patterns are of the following forms, respectively:

for all $q \in J^\infty(\mathbb{B}^\delta) \ldots$ there exists some $x \in K(\mathbb{A}^\delta) \ldots$ $(\forall J \exists K)$

for all $q \in J^\infty(\mathbb{B}^\delta) \ldots$ there exists some $x \in J^\infty(\mathbb{A}^\delta) \ldots$ $(\forall J \exists J)$

for all $q \in K(\mathbb{B}^\delta) \ldots$ there exists some $x \in K(\mathbb{A}^\delta) \ldots$ $(\forall K \exists K)$

This quantification pattern suggests that there are two immediately available sufficient conditions on f and g which guarantee $g^\sigma f^\sigma$ to be UC:

- either g^σ is Scott continuous $(\forall J \exists J)$ and f^σ is UC $(\forall J \exists K)$;
- or g^σ is UC $(\forall J \exists K)$ and f^σ is SUC $(\forall K \exists K)$.

Since σ-extensions of monotone functions are always UC, only half of each condition needs to be guaranteed: namely, in the first case g^σ needs to be Scott continuous, and in the second case f^σ needs to be SUC.

In fact, this strategy can be generalized by replacing Scott continuity with the following weaker notion of *weak m-Scott continuity* (see Definition 3.5 below). In what follows, we will find it useful to let the symbols $J^\infty_\bot(\mathbb{A}^\delta)$ and $M^\infty_\top(\mathbb{A}^\delta)$ abbreviate the sets $J^\infty(\mathbb{A}^\delta) \cup \{\bot\}$ and $M^\infty(\mathbb{A}^\delta) \cup \{\top\}$ respectively, for any DLE $\mathbb{A}$. For the sake of readability, Π will abbreviate $\Pi_{i=1}^m$.

Definition 3.5 (Definition 5.1, [23])**.** For any monotone map $f : \Pi\mathbb{A}_i^\delta \to \mathbb{B}^\delta$,

$(\forall J \exists J^m_\bot)$ f is *weakly m-Scott continuous* if for any $\overline{u} \in \Pi\mathbb{A}_i^\delta$ and any $q \in J^\infty(\mathbb{B}^\delta)$, if $q \leq f(\overline{u})$ then $q \leq f(\overline{x})$ for some $\overline{x} \in \Pi J^\infty_\bot(\mathbb{A}_i^\delta)$ s.t. $\overline{x} \leq \overline{u}$.

$(\forall M \exists M^m_\top)$ f is *dually weakly m-Scott continuous* if for any $\overline{u} \in \Pi\mathbb{A}_i^\delta$ and any $q \in M^\infty(\mathbb{B}^\delta)$, if $q \geq f(\overline{u})$ then $q \geq f(\overline{x})$ for some $\overline{x} \in \Pi M^\infty_\top(\mathbb{A}_i^\delta)$ s.t. $\overline{x} \geq \overline{u}$.

Remark 3.6. Note that both weak m-Scott continuity and dual weak m-Scott continuity are special cases of upper continuity (cf. Definition 2.9). These properties are used later in the proof of Theorem 4.12.

Corollaries 3.8 and 3.10 below implement this strategy. Indeed, the map g being additive is a sufficient condition for the first case to apply (Corollary 3.8), and f being p-multiplicative is a sufficient condition for the second case to apply (Corollary 3.10). Based on these two corollaries, [17, Lemma 5.10], with its proof emended as indicated in [23, Remark 5.4], shows that if $\alpha_1 \leq \beta_1$ is uniform Sahlqvist, then α_1 is σ-expanding and β_1 is π-contracting.

Hence, the syntactic shape of uniform Sahlqvist inequalities is conceptually motivated in terms of the order-theoretic behaviour of the interpretation of the logical connectives, and of the properties of their resulting composition.

Lemma 3.7 ([23], Lemma 5.2)**.** *For any monotone map $f : \Pi\mathbb{A}_i^\delta \to \mathbb{B}^\delta$, if f is completely additive, then f is weakly m-Scott continuous.*

Corollary 3.8 ([23], Corollary 5.3)**.** *For all monotone maps $f_i : \mathbb{A}_i \to \mathbb{B}_i$, $1 \leq i \leq m$ and $g : \Pi\mathbb{B}_i \to \mathbb{C}$, if g is additive, then $g^\sigma(f_1^\sigma, \ldots, f_m^\sigma) \leq (g(f_1, \ldots, f_m))^\sigma$.*

Lemma 3.9 ([3], Proposition 117.5)**.** *If $f : \mathbb{A} \to \mathbb{B}$ is p-multiplicative, then f^σ is SUC.*

Corollary 3.10 ([23], Corollary 6.12)**.** *If $f : \mathbb{A}^\delta \to \mathbb{B}^\delta$ is p-multiplicative and $g : \mathbb{B}^\delta \to \mathbb{C}^\delta$ is order-preserving, then $g^\sigma f^\sigma \leq (gf)^\sigma$.*

The following lemma holds for Sahlqvist terms, as discussed in [23]. Its proof is by induction on the shape of the terms, using the corollaries above.

Lemma 3.11. *Every ϵ-uniform positive (resp. negative) Sahlqvist term is σ-expanding (resp. π-contracting).*

3.4. **A proof refinement.** In the present subsection, we give an alternative proof of Theorem 3.1, in which the following lemma is not needed:

Lemma 3.12 (cf. Lemma 5.11 in [17])**.** *For all maps $f, g : \mathbb{A} \to \mathbb{B}$ between BDLs such that f is order-preserving and g is order-reversing, $(f \vee g)^\sigma \leq f^\sigma \vee g^\pi$.*

Alternative proof of Theorem 3.1.

$$
\begin{array}{lll}
 & \mathbb{A} \models \alpha \leq \beta & \\
\iff & \alpha^{\mathbb{A}} \leq \beta^{\mathbb{A}} & \text{(by definition)} \\
\iff & \alpha_1^{\mathbb{A}} \leq \beta_1^{\mathbb{A}} \vee \gamma^{\mathbb{A}} & \text{(Lemma 3.3)} \\
\iff & (\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}} \leq \gamma^{\mathbb{A}} & \text{(Lemma 3.13)} \\
\iff & \left((\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}}\right)^\sigma \leq \left(\gamma^{\mathbb{A}}\right)^\sigma & \text{(definition of the } \sigma\text{-extension)} \\
\implies & (\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}^\delta} \leq \left(\gamma^{\mathbb{A}}\right)^\sigma & \text{(Lemma 3.15)} \\
\implies & (\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}^\delta} \leq \gamma^{\mathbb{A}^\delta} & \text{(Lemma 2.18)} \\
\iff & \alpha_1^{\mathbb{A}^\delta} \leq \beta_1^{\mathbb{A}^\delta} \vee \gamma^{\mathbb{A}^\delta} & \text{(Lemma 3.13)} \\
\iff & \alpha^{\mathbb{A}^\delta} \leq \beta^{\mathbb{A}^\delta} & \text{(Lemma 3.3)} \\
\iff & \mathbb{A}^\delta \models \alpha \leq \beta. & \text{(by definition)}
\end{array}
$$

□

In the remainder of the present subsection, we prove the lemmas needed in the schematic proof above.

Lemma 3.13. *Let $\alpha, \beta, \gamma \in \mathcal{L}^{\mathbf{n}}_{term}$ s.t. γ is of the form $\bigvee_{i=1}^n \mathbf{n}(s_i, t_i)$. For any DMA $\mathbb{A}$,*

$$\alpha_1^{\mathbb{A}} \leq \beta_1^{\mathbb{A}} \vee \gamma^{\mathbb{A}} \quad \textit{iff} \quad (\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}} \leq \gamma^{\mathbb{A}}.$$

Proof. We need to show that for any assignment h,

$$[\![\alpha_1]\!]^{\mathbb{A}}_h \leq [\![\beta_1]\!]^{\mathbb{A}}_h \vee [\![\gamma]\!]^{\mathbb{A}}_h \quad \text{iff} \quad [\![\mathbf{n}(\alpha_1, \beta_1)]\!]^{\mathbb{A}}_h \leq [\![\gamma]\!]^{\mathbb{A}}_h.$$

Fix an assignment h. Notice preliminarily that γ being of the form $\bigvee_{i=1}^n \mathbf{n}(s_i, t_i)$ implies that either $[\![\gamma]\!]^{\mathbb{A}}_h = \top^{\mathbb{A}}$ or $[\![\gamma]\!]^{\mathbb{A}}_h = \bot^{\mathbb{A}}$.
The equivalence immediately holds if $[\![\gamma]\!]^{\mathbb{A}}_h = \top$. If $[\![\gamma]\!]^{\mathbb{A}}_h = \bot$, then the left-hand inequality is equivalent to $[\![\alpha_1]\!]^{\mathbb{A}}_h \leq [\![\beta_1]\!]^{\mathbb{A}}_h$, which is equivalent to $[\![\mathbf{n}(\alpha_1, \beta_1)]\!]^{\mathbb{A}}_h = \bot^{\mathbb{A}} \leq \bot^{\mathbb{A}} = [\![\gamma]\!]^{\mathbb{A}}_h$. □

To prove that $\mathbf{n}(\alpha_1, \beta_1)$ is σ-expanding, we will use the following:

Lemma 3.14 (cf. Lemma 5.15 in [17]). $(\mathbf{n}^{\mathbb{A}})^\sigma = \mathbf{n}^{\mathbb{A}^\delta}$, *i.e.* $\mathbf{n}$ *is* σ*-stable.*

Lemma 3.15. *If* $+\alpha_1$ *is* ϵ*-uniform and Sahlqvist and* $-\beta_1$ *is* ϵ^∂*-uniform and Sahlqvist, then* $\mathbf{n}(\alpha_1, \beta_1)$ *is* σ*-expanding.*

Proof. By assumption, $\alpha_1^{\mathbb{A}} : \mathbb{A}^\epsilon \to \mathbb{A}$ and $(\beta_1^{\mathbb{A}})^\partial : \mathbb{A}^\epsilon \to \mathbb{A}^\partial$ are monotone, hence so is $(\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}} = \mathbf{n}^{\mathbb{A}}(\alpha_1^{\mathbb{A}}, (\beta_1^{\mathbb{A}})^\partial)$ as a map $\mathbb{A}^\epsilon \to \mathbb{A}$. Also by the assumption and Lemma 3.11, $(\alpha_1^{\mathbb{A}})^\sigma \geq \alpha_1^{\mathbb{A}^\delta}$ and $(\beta_1^{\mathbb{A}})^\pi \leq \beta_1^{\mathbb{A}^\delta}$. Hence, the following chain holds, which proves that $\mathbf{n}(\alpha_1, \beta_1)$ is σ-expanding, as required.

$$\begin{array}{cll}
& ((\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}})^\sigma & \\
\geq & (\mathbf{n}^{\mathbb{A}})^\sigma(\alpha_1^{\mathbb{A}}, (\beta_1^{\mathbb{A}})^\partial)^\sigma & \text{(Corollary 3.8)} \\
= & \mathbf{n}^{\mathbb{A}^\delta}((\alpha_1^{\mathbb{A}})^\sigma, ((\beta_1^{\mathbb{A}})^\partial)^\sigma) & \text{(Lemma 3.14)} \\
= & \mathbf{n}^{\mathbb{A}^\delta}((\alpha_1^{\mathbb{A}})^\sigma, ((\beta_1^{\mathbb{A}})^\pi)^\partial) & \\
\geq & \mathbf{n}^{\mathbb{A}^\delta}(\alpha_1^{\mathbb{A}^\delta}, (\beta_1^{\mathbb{A}^\delta})^\partial) & (\mathbf{n}^{\mathbb{A}} : \mathbb{A} \times \mathbb{A}^\partial \to \mathbb{A} \text{ monotone, } (\alpha_1^{\mathbb{A}})^\sigma \geq \alpha_1^{\mathbb{A}^\delta} \text{ and } (\beta_1^{\mathbb{A}})^\pi \leq \beta_1^{\mathbb{A}^\delta}) \\
= & (\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}^\delta}. &
\end{array}$$

□

Remark 3.16. Having dispensed with Lemma 3.12 shows more clearly that the key to the Jónsson-style proof of the canonicity of any given inequality $\alpha \leq \beta$ is being able to equivalently rewrite it so as to separate its σ-contracting part from its σ-expanding part. It is a specific feature of the signature $\mathcal{L}^{\mathbf{n}}$ that all $\mathcal{L}^{\mathbf{n}}$-terms which are ϵ-uniform for some ϵ are σ-contracting. Hence, in a sense, this signature is not the right setting to a deeper understanding of the property of being σ-contracting. In Sections 3.6 and 4.2, we will show that σ-contracting terms are exactly the *closed Esakia* ones (cf. Definitions 3.19 and 4.5). As to the property of being σ-expanding, we saw that, after the σ-contracting part has been extracted from a Sahlqvist inequality, what is left is an inequality consisting of ϵ'*-uniform* Sahlqvist terms, the compositional structure of which guarantees them to be σ-expanding, as discussed in Section 3.3. Thus, in a sense, the extraction operation improves the compositional structure of (the ϵ-uniform part of) Sahlqvist terms, which can be syntactically characterized as uniform Sahlqvist. As we will see in the next subsection, the extraction procedure of Lemmas 3.2 and 3.3 demotes (the ϵ-uniform part of) inductive terms to terms the compositional structure of which cannot be argued to be σ-expanding in the same way in which the canonicity of Sahlqvist inequalities is proven.

3.5. **Why the n-trick alone fails on inductive inequalities, and ideas for a solution.** In the present section, we discuss why the attempt in [28] to prove the canonicity of inductive inequalities using the **n**-trick alone fails. For simplicity of discussion, we use the original proof strategy in [17].
Similarly to the case of Sahlqvist inequalities, we apply the same minimal collapse algorithm to transform a given ϵ-inductive inequality $\alpha \leq \beta$ into an inequality $\alpha_1 \leq \beta_1 \vee \gamma$, where α_1, γ are ϵ'-uniform and β_1 is ϵ'^∂-uniform for a suitable order-type ϵ' extending ϵ. However, as observed in [28], the terms α_1 and β_1 obtained after executing the minimal collapse algorithm might not be even inductive:

Example 3.17 (cf. Example 5.1.3 in [28])**.** The term $\alpha = \Box(\lhd x \vee y)$ is (Ω, ϵ)-left inductive for $\epsilon = (1, 1)$ and $x \leq_\Omega y$. After the minimal collapse algorithm, the resulting formula is $\alpha_1 = \Box(z \vee y)$, and the prescribed order-type ϵ' extending ϵ is $\epsilon' = (1, 1, 1)$. However, $+\alpha_1$ is not (Ω', ϵ')-inductive for any Ω'.

As discussed in the subsection above, even if $+\alpha_1$ is not inductive, we still hope to show that it is σ-expanding. However, with our existing tools, we cannot show that this is the case.
Indeed, as discussed in the previous section, Jónsson's strategy provides two possibilities to prove that the inequality $g^\sigma f^\sigma \leq (gf)^\sigma$ holds, namely either applying Corollary 3.8 or Corollary 3.10. In order to apply the first possibility, we need to require f^σ to satisfy UC ($\forall J \exists K$) and g^σ to satisfy weak m-Scott continuity ($\forall J \exists J^m_\perp$). In order to apply the second possibility, we need to require f^σ to satisfy SUC ($\forall K \exists K$) and g^σ to satisfy UC ($\forall J \exists K$). For any order-preserving maps f and g, the first case holds when g is additive (cf. Lemma 3.7), and the second case holds when f is p-multiplicative (cf. Lemma 3.9).
For $\alpha = \Box(z \vee y)$, we cannot employ the first possibility, since in general $\Box$ is not additive. We cannot employ the second possibility either, since $\vee$ is not in general SUC, as shown in the example below (which is very similar to [16, Example 2.26]).

Example 3.18. Let $\vee : \mathbb{A}^\delta \times \mathbb{A}^\delta \to \mathbb{A}^\delta$ where $\mathbb{A}^\delta$ is the canonical extension of some DMA $\mathbb{A}$ such that the underlying lattice of $\mathbb{A}^\delta$ is an infinite Boolean algebra, and $K(\mathbb{A}^\delta) \cup O(\mathbb{A}^\delta) \neq \mathbb{A}^\delta$. Let $(u, v) \in \mathbb{A}^\delta \times \mathbb{A}^\delta$ such that u is the complement of v and $u, v \notin K(\mathbb{A}^\delta) \cup O(\mathbb{A}^\delta)$. For $z = \top \in K(\mathbb{A}^\delta)$, we have $z \leq u \vee v$, but there is no $(x, y) \in K(\mathbb{A}^\delta \times \mathbb{A}^\delta)$ such that $z \leq x \vee y$ (that is, $x \vee y = \top$) and $(x, y) \leq (u, v)$.

Therefore, the tools of the **n**-trick and minimal collapse algorithm are not enough to equivalently transform inductive terms into a combination of contracting and expanding terms suitable to implement Jónsson's strategy.
Notice that the operations interpreting the $\mathcal{L}^{\mathbf{n}}$-connectives in any perfect DMA are either completely join- or meet-preserving or reversing in each coordinate, or they are completely join- or meet-preserving or reversing in the product where they are defined as binary operations. It is well known (cf. [13]) that in the context of complete lattices in which we are, these properties can be equivalently stated in terms of the existence of the appropriate adjoint and residuals of each connective. Hence, the language expansion $\mathcal{L}^+$, which we have already seen in Section 2.4, will hopefully provide us with a better array of tools with which more syntactic manipulations will be possible, aimed at reaching a shape in which one of the two possibilities mentioned above, namely Corollary 3.8 and Corollary 3.10, can be applied. Our basic idea consists in applying the syntactic machinery already employed in the proof of canonicity-via-correspondence to the implementation of Jónsson's strategy to inductive inequalities. This is in line with the spirit of the **n**-trick: indeed, the **n**-trick consists in expanding the language with a new term which makes it possible to express at the level of terms more things that could be expressed before. In Section 4, we will develop the order-theoretic theory which makes it possible to pursue our basic idea. Before moving on to it, in the following subsection, we show that σ-contracting maps are exactly those which are closed Esakia (cf. Definition 3.19).

3.6. Characterization of σ-contracting maps.

Definition 3.19. Let $f^\mathbb{A} : \mathbb{A} \to \mathbb{B}$ be a monotone function and $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$ be its extension.

(1) The map $f^{\mathbb{A}^\delta}$ is *closed Esakia* if $f^{\mathbb{A}^\delta}(\bigwedge\{c_i : i \in I\}) = \bigwedge\{f^{\mathbb{A}^\delta}(c_i) : i \in I\}$ for any non-empty downward-directed collection $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$;

(2) The map $f^{\mathbb{A}^\delta}$ is *open Esakia* if $f^{\mathbb{A}^\delta}(\bigvee\{o_i : i \in I\}) = \bigvee\{f^{\mathbb{A}^\delta}(o_i) : i \in I\}$ for any non-empty upward-directed collection $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$.

Proposition 3.20. *For any monotone function $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}$ and its extension $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$,*

(1) $(f^{\mathbb{A}})^\sigma \leq f^{\mathbb{A}^\delta}$ *iff* $f^{\mathbb{A}^\delta}$ *is closed Esakia.*
(2) $f^{\mathbb{A}^\delta} \leq (f^{\mathbb{A}})^\pi$ *iff* $f^{\mathbb{A}^\delta}$ *is open Esakia.*

Proof. We prove 1, statement 2 being an order-variant of 1.
($\Leftarrow$): Assume $f^{\mathbb{A}^\delta}$ is closed Esakia. To show that $(f^{\mathbb{A}})^\sigma \leq f^{\mathbb{A}^\delta}$, we have to show that for all $u \in \mathbb{A}^\delta$, $\bigvee\{\bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\} : u \geq x \in K(\mathbb{A}^\delta)\} \leq f^{\mathbb{A}^\delta}(u)$, i.e., for all $u \in \mathbb{A}^\delta$, all $x \in K(\mathbb{A}^\delta)$ such that $x \leq u$, $\bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\} \leq f^{\mathbb{A}^\delta}(u)$. By the monotonicity of $f^{\mathbb{A}^\delta}$ and the fact that $f^{\mathbb{A}}$ and $f^{\mathbb{A}^\delta}$ coincide on $\mathbb{A}$, it suffices to show that for all $x \in K(\mathbb{A}^\delta)$, $\bigwedge\{f^{\mathbb{A}^\delta}(a) : x \leq a \in \mathbb{A}\} \leq f^{\mathbb{A}^\delta}(x)$, which follows easily from $f^{\mathbb{A}^\delta}$ being closed Esakia.
($\Rightarrow$): Suppose that $(f^{\mathbb{A}})^\sigma \leq f^{\mathbb{A}^\delta}$, i.e., for all $u \in \mathbb{A}^\delta$, $\bigvee\{\bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\} : u \geq x \in K(\mathbb{A}^\delta)\} \leq f^{\mathbb{A}^\delta}(u)$. Then let us preliminarily show that $\bigwedge\{f^{\mathbb{A}^\delta}(a) : x \leq a \in \mathbb{A}\} = f^{\mathbb{A}^\delta}(x)$ for all $x \in K(\mathbb{A}^\delta)$:
For any $x \in K(\mathbb{A}^\delta)$, we have $\bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\} = \bigvee\{\bigwedge\{f^{\mathbb{A}}(a) : x' \leq a \in \mathbb{A}\} : x \geq x' \in K(\mathbb{A}^\delta)\} \leq f^{\mathbb{A}^\delta}(x)$. Since $f^{\mathbb{A}}$ and $f^{\mathbb{A}^\delta}$ coincide on $\mathbb{A}$, we have $\bigwedge\{f^{\mathbb{A}^\delta}(a) : x \leq a \in \mathbb{A}\} \leq f^{\mathbb{A}^\delta}(x)$. The converse inequality follows from the monotonicity of $f^{\mathbb{A}^\delta}$.
Let us prove that $f^{\mathbb{A}^\delta}$ is closed Esakia, i.e., for any non-empty downward-directed collection $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$, $f^{\mathbb{A}^\delta}(\bigwedge\{c_i : i \in I\}) = \bigwedge\{f^{\mathbb{A}^\delta}(c_i) : i \in I\}$:
The inequality $f^{\mathbb{A}^\delta}(\bigwedge\{c_i : i \in I\}) \leq \bigwedge\{f^{\mathbb{A}^\delta}(c_i) : i \in I\}$ straightforwardly follows from the monotonicity of $f^{\mathbb{A}^\delta}$. For the converse direction $\bigwedge\{f^{\mathbb{A}^\delta}(c_i) : i \in I\} \leq f^{\mathbb{A}^\delta}(\bigwedge\{c_i : i \in I\})$, let $x = \bigwedge\{c_i : i \in I\} \in K(\mathbb{A}^\delta)$. By the preliminary fact shown above, we have that $f^{\mathbb{A}^\delta}(x) = \bigwedge\{f^{\mathbb{A}^\delta}(a) : a \in \mathbb{A}$ and $x \leq a\}$. Hence, it is enough to show that $\bigwedge\{f^{\mathbb{A}^\delta}(c_i) : i \in I\} \leq \bigwedge\{f^{\mathbb{A}^\delta}(a) : a \in \mathbb{A}$ and $x \leq a\}$, i.e., we need to show that for each $a \in \mathbb{A}$, if $x \leq a$, then there exists an $i_0 \in I$ such that $f^{\mathbb{A}^\delta}(c_{i_0}) \leq f^{\mathbb{A}^\delta}(a)$. By compactness, from $\bigwedge\{c_i : i \in I\} = x \leq a$ we get that $x' = \bigwedge\{c_i : i \in I'\} \leq a$ for some finite non-empty $I' \subseteq I$. By the downward-directedness of $\{c_i : i \in I\}$, we get $c_{i_0} \leq x' \leq a$ for some $i_0 \in I$. Therefore, $f^{\mathbb{A}^\delta}(c_{i_0}) \leq f^{\mathbb{A}^\delta}(a)$. □

The proposition above gives a characterization of the order-theoretic property of being σ-contracting (resp. π-expanding) in terms of the topological property of being closed (resp. open) Esakia. From the proof of this proposition, one can see that the assumption that $f^{\mathbb{A}}$ maps elements in $\mathbb{A}$ to elements in $\mathbb{B}$ plays no role. Therefore the proposition above can be straightforwardly extended to maps $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^\delta$, which is the setting treated in the next section.

4. Generalized canonical extensions of maps

The main technical difficulty we face when trying to employ the expanded language manipulated by ALBA for the Jónsson strategy is that DMAs are not in general closed under the additional operations (i.e. under the adjoints and residuals of the operations interpreting the connectives of the original signature). This is the reason why a generalization of the standard theory of canonical extensions of maps is needed, accounting for maps $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^\delta$ such that the value of $f^{\mathbb{A}}$ is not restricted to clopen elements in $\mathbb{B}$. Providing this treatment is the aim of the present section.

Specifically, building on [16, Section 2.3], we consider primitive maps $f^{\mathbb{A}^\delta}: \mathbb{A}^\delta \to \mathbb{B}^\delta$, then consider their restrictions $f^{\mathbb{A}}: \mathbb{A} \to \mathbb{B}^\delta$ to $\mathbb{A}$, and then define the *generalized* λ-extensions $(f^{\mathbb{A}})^\lambda : \mathbb{A}^\delta \to \mathbb{B}^\delta$ (for $\lambda \in \{\sigma, \pi\}$) of these restrictions. Comparing these generalized extensions to the original maps $f^{\mathbb{A}^\delta}$ gives rise to associated definitions of λ-contracting or expanding maps. This theory can be applied to the adjoints and residuals of the operations interpreting the connectives of the original DML signature.
The intuition behind these generalized extensions of maps is based e.g. on the similarity with the way in which the value of a continuous real variable function on an irrational real number can be taken as a limit of its values on rational numbers.
In [16], canonical extensions are fully discussed only for functions $f^{\mathbb{A}}: \mathbb{A} \to \mathbb{B}$, and not for functions $f^{\mathbb{A}}: \mathbb{A} \to \mathbb{B}^\delta$. In the present section we fill this gap.

4.1. **Generalized canonical extensions of maps and their basic properties.** In what follows, we use superscripts to indicate the domain of the function. For instance, the symbol $f^{\mathbb{A}}$ indicates the restriction of $f^{\mathbb{A}^\delta}$ to $\mathbb{A}$ or to $\mathbb{A}^\epsilon$, depending on the context. Abusing notation, we extend this convention also to operation symbols. For instance, instead of denoting an operation on $\mathbb{A}$, the symbol $\blacklozenge^{\mathbb{A}}$ denotes a map from $\mathbb{A}$ to $\mathbb{A}^\delta$, which is the restriction of the map $\blacklozenge^{\mathbb{A}^\delta}: \mathbb{A}^\delta \to \mathbb{A}^\delta$ to $\mathbb{A}$. Throughout the present section, $\mathbb{A}, \mathbb{B}, \mathbb{C}$ denote BDLs.

Definition 4.1 (**Generalized canonical extensions of maps**). For any order-preserving map $f^{\mathbb{A}}: \mathbb{A} \to \mathbb{B}^\delta$ and any $u \in \mathbb{A}^\delta$, we let:

$$(f^{\mathbb{A}})^\sigma(u) = \bigvee\{\bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\} : u \geq x \in K(\mathbb{A}^\delta)\}$$

$$(f^{\mathbb{A}})^\pi(u) = \bigwedge\{\bigvee\{f^{\mathbb{A}}(a) : y \geq a \in \mathbb{A}\} : u \leq y \in O(\mathbb{A}^\delta)\}.$$

For example, for a given DMA $\mathbb{A}$, consider the left adjoint $\blacklozenge^{\mathbb{A}^\delta}: \mathbb{A}^\delta \to \mathbb{A}^\delta$ of $\Box^{\mathbb{A}^\delta}: \mathbb{A}^\delta \to \mathbb{A}^\delta$. Let $\blacklozenge^{\mathbb{A}}: \mathbb{A} \to \mathbb{A}^\delta$ be its restriction to $\mathbb{A}$. Let the generalized canonical extension of $\blacklozenge^{\mathbb{A}}$ be $(\blacklozenge^{\mathbb{A}})^\lambda$, where $\lambda \in \{\sigma, \pi\}$. We will show that $(\blacklozenge^{\mathbb{A}})^\sigma = (\blacklozenge^{\mathbb{A}})^\pi = \blacklozenge^{\mathbb{A}^\delta}$ (see Section 5.2).
Many properties of σ- and π-extensions still hold in the generalized setting. Here we state some of them without proof.

Proposition 4.2. *For any order-preserving map $f^{\mathbb{A}}: \mathbb{A} \to \mathbb{B}^\delta$,*

(1) *Both $(f^{\mathbb{A}})^\sigma$ and $(f^{\mathbb{A}})^\pi$ are order-preserving;*
(2) *$(f^{\mathbb{A}})^\sigma(x) = \bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\}$ for any $x \in K(\mathbb{A}^\delta)$;*
(3) *$(f^{\mathbb{A}})^\pi(y) = \bigvee\{f^{\mathbb{A}}(a) : y \geq a \in \mathbb{A}\}$ for any $y \in O(\mathbb{A}^\delta)$;*
(4) *$(f^{\mathbb{A}})^\sigma(a) = (f^{\mathbb{A}})^\pi(a) = f^{\mathbb{A}}(a)$ for any $a \in \mathbb{A}$;*
(5) *$(f^{\mathbb{A}})^\sigma(u) \leq (f^{\mathbb{A}})^\pi(u)$ for any $u \in \mathbb{A}^\delta$, and the equality holds if $u \in K(\mathbb{A}^\delta) \cup O(\mathbb{A}^\delta)$.*

Proof. Items 1-4 follow straightforwardly from the definition. The proof of item 5 is very similar to the proof of Theorem 3.1 in [15], and is omitted. □

The next proposition extends to the generalized setting a fundamental property of canonical extensions, and will play a crucial role in Section 4.3.

Proposition 4.3. *Given any order-preserving map $f^{\mathbb{A}}: \mathbb{A} \to \mathbb{B}^\delta$,*

(1) *$(f^{\mathbb{A}})^\sigma$ is the largest order-preserving UC extension of $f^{\mathbb{A}}$ to $\mathbb{A}^\delta$.*
(2) *$(f^{\mathbb{A}})^\pi$ is the smallest order-preserving LC extension of $f^{\mathbb{A}}$ to $\mathbb{A}^\delta$.*

Proof. We only prove item 1, the proof of item 2 being order dual. The upper continuity of $(f^{\mathbb{A}})^\sigma$ easily follows from the definition of generalized σ-extension. To show that $(f^{\mathbb{A}})^\sigma$ is the largest such map, let $g^{\mathbb{A}^\delta}: \mathbb{A}^\delta \to \mathbb{B}^\delta$ be an order-preserving map extending $f^{\mathbb{A}}$ satisfying UC, and let us show that $g^{\mathbb{A}^\delta} \leq (f^{\mathbb{A}})^\sigma$. Suppose for contradiction that there exists some $u \in \mathbb{A}^\delta$ such that $g^{\mathbb{A}^\delta}(u) \nleq (f^{\mathbb{A}})^\sigma(u)$. Then there exists some $i \in J^\infty(\mathbb{B}^\delta)$ such that $i \leq g^{\mathbb{A}^\delta}(u)$ and $i \nleq (f^{\mathbb{A}})^\sigma(u)$. Since $g^{\mathbb{A}^\delta}$ is UC, the first inequality implies that there exists some $x \in K(\mathbb{A}^\delta)$ such that $x \leq u$ and $i \leq g^{\mathbb{A}^\delta}(x)$. Then by the monotonicity of $g^{\mathbb{A}^\delta}$ and the fact that both $(f^{\mathbb{A}})^\sigma$ and $g^{\mathbb{A}^\delta}$ extend $f^{\mathbb{A}}$, the following chain of (in)equalities holds for all $a \in \mathbb{A}$ such that $x \leq a$:

$$i \leq g^{\mathbb{A}^\delta}(x) \leq g^{\mathbb{A}^\delta}(a) = f^{\mathbb{A}}(a).$$

Hence $i \leq \bigwedge\{f^{\mathbb{A}}(a) : \mathbb{A} \ni a \geq x\} = (f^{\mathbb{A}})^\sigma(x) \leq (f^{\mathbb{A}})^\sigma(u)$, contradicting $i \nleq (f^{\mathbb{A}})^\sigma(u)$. □

Below, we generalize the definition of stable, expanding, and contracting maps to this new setting:

Definition 4.4 (**Generalized stable, expanding and contracting maps**)**.** For any order-preserving map $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$, and for $\lambda \in \{\sigma, \pi\}$,

- $f^{\mathbb{A}^\delta}$ is λ-*stable*, if $f^{\mathbb{A}^\delta} = (f^{\mathbb{A}})^\lambda$;
- $f^{\mathbb{A}^\delta}$ is λ-*expanding*, if $f^{\mathbb{A}^\delta} \leq (f^{\mathbb{A}})^\lambda$;
- $f^{\mathbb{A}^\delta}$ is λ-*contracting*, if $f^{\mathbb{A}^\delta} \geq (f^{\mathbb{A}})^\lambda$).

4.2. **Generalized σ-contracting and π-expanding maps, and Esakia conditions.** As mentioned at the end of Section 3.6, Proposition 3.20 can be generalized to the present setting. In the present subsection, we characterize σ-contracting and π-expanding maps in terms of the generalized Esakia conditions (cf. Theorem 4.7) and also in terms of the intersection conditions (cf. Lemma 4.9).

Definition 4.5 (**Generalized Esakia conditions**)**.** For any order-preserving map $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$,

(1) $f^{\mathbb{A}^\delta}$ is *closed Esakia* if for any non-empty downward-directed collection $\mathcal{C} = \{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$, $f^{\mathbb{A}^\delta}(\bigwedge\{c_i : i \in I\}) = \bigwedge f^{\mathbb{A}^\delta}(\{c_i : i \in I\})$;
(2) $f^{\mathbb{A}^\delta}$ is *open Esakia* if for any non-empty upward-directed collection $\mathcal{O} = \{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$, $f^{\mathbb{A}^\delta}(\bigvee\{o_i : i \in I\}) = \bigvee f^{\mathbb{A}^\delta}(\{o_i : i \in I\})$.

Theorem 4.6. *For any order-preserving map $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$, both $(f^{\mathbb{A}})^\sigma$ and $(f^{\mathbb{A}})^\pi$ are both open and closed Esakia.*

Proof. We only prove that $(f^{\mathbb{A}})^\sigma$ and $(f^{\mathbb{A}})^\pi$ are open Esakia, the proof of the remaining part being order-dual.
As to $(f^{\mathbb{A}})^\pi$, for any non-empty upward-directed collection $\mathcal{O} = \{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$, we have that $(f^{\mathbb{A}})^\pi(\bigvee \mathcal{O}) = \bigvee\{f^{\mathbb{A}}(a) : \mathbb{A} \ni a \leq \bigvee \mathcal{O}\}$ and $\bigvee\{(f^{\mathbb{A}})^\pi(o) : o \in \mathcal{O}\} = \bigvee\{f^{\mathbb{A}}(a) : \mathbb{A} \ni a \leq o$ for some $o \in \mathcal{O}\}$, therefore it suffices to show that for all $a \in \mathbb{A}$, $a \leq \bigvee \mathcal{O}$ iff $a \leq o$ for some $o \in \mathcal{O}$. This equivalence is easy to check using the non-emptiness, compactness and upward-directedness of $\mathcal{O}$.
As to $(f^{\mathbb{A}})^\sigma$, for any non-empty upward-directed collection $\mathcal{O} \subseteq O(\mathbb{A}^\delta)$, the following chain of identities holds, which proves that $(f^{\mathbb{A}})^\sigma$ is open Esakia, as required.

$$\begin{array}{rcll} (f^{\mathbb{A}})^\sigma(\bigvee \mathcal{O}) & = & (f^{\mathbb{A}})^\pi(\bigvee \mathcal{O}) & \text{(by Proposition 4.2.5)} \\ & = & \bigvee\{(f^{\mathbb{A}})^\pi(o) : o \in \mathcal{O}\} & \text{(by } (f^{\mathbb{A}})^\pi \text{ is open Esakia)} \\ & = & \bigvee\{(f^{\mathbb{A}})^\sigma(o) : o \in \mathcal{O}\} & \text{(by Proposition 4.2.5).} \end{array}$$

□

Theorem 4.7. *For any order-preserving map $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$,*

(1) *$f^{\mathbb{A}^\delta}$ is σ-contracting iff $f^{\mathbb{A}^\delta}$ is closed Esakia;*

(2) $f^{\mathbb{A}^\delta}$ *is π-expanding iff* $f^{\mathbb{A}^\delta}$ *is open Esakia.*

Proof. Straightforward generalization the proof of Proposition 3.20. □

In fact, for n-ary order-preserving maps, we can also show that the Esakia conditions of Definition 4.5 are equivalent to the following much more manageable intersection conditions, which will be used in the discussion about the comparison between the two approaches to canonicity.

In what follows, we will represent a given map $f^{\mathbb{A}^\delta} : (\mathbb{A}^\delta)^\epsilon \to \mathbb{B}^\delta$ by grouping together the monotone and the antitone coordinates. Hence, we will write $f^{\mathbb{A}^\delta}$ as $f^{\mathbb{A}^\delta}(\vec{u}, \vec{v}) : (\mathbb{A}^\delta)^\epsilon \to \mathbb{B}^\delta$, and assume that the function $f^{\mathbb{A}^\delta}(\vec{u}, \vec{v})$ is monotone in its $\vec{u}$ coordinates and antitone in its $\vec{v}$ coordinates. Moreover, we use the variable w to select a given coordinate in $\vec{u}$ or $\vec{v}$, and, abusing notation, we write the function $f^{\mathbb{A}^\delta}(\vec{u}, \vec{v})$ as $f^{\mathbb{A}^\delta}(\vec{u}, \vec{v}, w)$. Also, symbols such as $\vec{c} \in K(\mathbb{A}^\delta)$ indicate that every element in the array $\vec{c}$ belongs to $K(\mathbb{A}^\delta)$.

Definition 4.8 (**Intersection conditions**)**.** For any order-preserving map $f^{\mathbb{A}^\delta}(\vec{u}, \vec{v})$: $(\mathbb{A}^\delta)^\epsilon \to \mathbb{B}^\delta$,

(1) the map $f^{\mathbb{A}^\delta}$ satisfies the *closed intersection condition* if the following holds for any coordinate w, for all $\vec{c} \in K(\mathbb{A}^\delta), \vec{o} \in O(\mathbb{A}^\delta)$, any non-empty downward-directed (resp. upward-directed) collection $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$ (resp. $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$):
- if the order type of f in the coordinate w is 1, then
$$f^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, \bigwedge\{c_i : i \in I\}) = \bigwedge\{f^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, c_i) : i \in I\};$$
- if the order type of f in the coordinate w is ∂, then
$$f^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, \bigvee\{o_i : i \in I\}) = \bigwedge\{f^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, o_i) : i \in I\}.$$

(2) the map $f^{\mathbb{A}^\delta}$ satisfies the *open intersection condition* if the following holds for any coordinate w, for all $\vec{c} \in K(\mathbb{A}^\delta), \vec{o} \in O(\mathbb{A}^\delta)$, any non-empty downward-directed (resp. upward-directed) collection $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$ (resp. $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$):
- if the order type of f in the coordinate w is 1, then
$$f^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, \bigvee\{o_i : i \in I\}) = \bigvee\{f^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, o_i) : i \in I\};$$
- if the order type of f in the coordinate w is ∂, then
$$f^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, \bigwedge\{c_i : i \in I\}) = \bigvee\{f^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, c_i) : i \in I\}.$$

Clearly, the intersection conditions above are the coordinatewise versions of the Esakia conditions. In the following lemma, we show that the Esakia conditions and their coordinatewise counterparts are equivalent.

We remind our notational convention that the function $f^{\mathbb{A}^\delta}(\vec{u}, \vec{v})$ is monotone in its $\vec{u}$ coordinates and antitone in its $\vec{v}$ coordinates.

Lemma 4.9. *For any order-preserving map* $f^{\mathbb{A}^\delta}(\vec{u}, \vec{v}) : (\mathbb{A}^\delta)^\epsilon \to \mathbb{B}^\delta$*,*

(1) $f^{\mathbb{A}^\delta}$ *is closed Esakia iff it satisfies the closed intersection condition;*
(2) $f^{\mathbb{A}^\delta}$ *is open Esakia iff it satisfies the open intersection condition;*

Proof. ($\Rightarrow$): It is easy to see that the open intersection condition is a special case of the open Esakia condition. Indeed, for any w in $\vec{u}$ or in $\vec{v}$, for any $\vec{o}, \vec{c}$, $\{o_i : i \in I\}$ and $\{c_i : i \in I\}$ as above, the set $\{(\vec{o}, \vec{c}, o_i) : i \in I\}$ is a non-empty upward-directed collection in $O((\mathbb{A}^\delta)^\epsilon)$ if w is in $\vec{u}$, and the set $\{(\vec{o}, \vec{c}, c_i) : i \in I\}$ is a non-empty upward-directed collection in $O((\mathbb{A}^\delta)^\epsilon)$ if w is in $\vec{v}$.

($\Leftarrow$): To avoid notational overload, we only discuss the special case in which both $\vec{u}$ and $\vec{v}$ are of length 2, the general case being not conceptually different, but much more complicated notationally. Let us denote $\mathbb{A}^\delta \times \mathbb{A}^\delta \times (\mathbb{A}^\delta)^\partial \times (\mathbb{A}^\delta)^\partial$ by $(\mathbb{A}^\delta)^\epsilon$. Then $f^{\mathbb{A}^\delta}(u_1, u_2, v_1, v_2) : (\mathbb{A}^\delta)^\epsilon \to \mathbb{B}^\delta$, and $O((\mathbb{A}^\delta)^\epsilon) = O(\mathbb{A}^\delta) \times O(\mathbb{A}^\delta) \times K(\mathbb{A}^\delta) \times K(\mathbb{A}^\delta)$.
We are going to show that, for any non-empty upward-directed collection $\{(o_i, o'_i, c_i, c'_i) : i \in I\} \subseteq O((\mathbb{A}^\delta)^\epsilon)$,

$$f^{\mathbb{A}^\delta}(o, o', c, c') = \bigvee\{f^{\mathbb{A}^\delta}(o_i, o'_i, c_i, c'_i) : i \in I\},$$

where $o = \bigvee\{o_i : i \in I\}$, $o' = \bigvee\{o'_i : i \in I\}$, $c = \bigwedge\{c_i : i \in I\}$, $c' = \bigwedge\{c'_i : i \in I\}$.
The right-to-left inequality follows by the assumption that $f^{\mathbb{A}^\delta} : (\mathbb{A}^\delta)^\epsilon \to \mathbb{B}^\delta$ is monotone. As to the converse inequality, by assumption, $f^{\mathbb{A}^\delta}$ satisfies the open intersection condition. Hence,

$$f^{\mathbb{A}^\delta}(o, o', c, c') = \bigvee\{f^{\mathbb{A}^\delta}(o_{i_1}, o'_{i_2}, c_{i_3}, c'_{i_4}) : i_1, i_2, i_3, i_4 \in I\}.$$

Therefore, it suffices to show that for any $i_1, i_2, i_3, i_4 \in I$ there exists some $i \in I$ such that

$$f^{\mathbb{A}^\delta}(o_{i_1}, o'_{i_2}, c_{i_3}, c'_{i_4}) \leq f^{\mathbb{A}^\delta}(o_i, o'_i, c_i, c'_i).$$

Fix $i_1, i_2, i_3, i_4 \in I$, and consider the finite subset $\{(o_{i_k}, o'_{i_k}, c_{i_k}, c'_{i_k}) : 1 \leq k \leq 4\}$. Clearly,

$$\{(o_{i_k}, o'_{i_k}, c_{i_k}, c'_{i_k}) : 1 \leq k \leq 4\} \subseteq \{(o_i, o'_i, c_i, c'_i) : i \in I\}.$$

By the upward-directedness of $\{(o_i, o'_i, c_i, c'_i) : i \in I\}$ as a sub-poset of $(\mathbb{A}^\delta)^\epsilon$, this finite subset has a common upper bound in $\{(o_i, o'_i, c_i, c'_i) : i \in I\}$. That is, there exists some $i \in I$ such that $(o_{i_k}, o'_{i_k}, c_{i_k}, c'_{i_k}) \leq_\epsilon (o_i, o'_i, c_i, c'_i)$ for $1 \leq k \leq 4$. This implies that $(o_{i_1}, o'_{i_2}, c_{i_3}, c'_{i_4}) \leq_\epsilon (o_i, o'_i, c_i, c'_i)$, and thus, by the ϵ-monotonicity of $f^{\mathbb{A}^\delta}$, we get $f^{\mathbb{A}^\delta}(o_{i_1}, o'_{i_2}, c_{i_3}, c'_{i_4}) \leq f^{\mathbb{A}^\delta}(o_i, o'_i, c_i, c'_i)$. □

4.3. **Sufficient conditions for generalized σ-expanding or π-contracting maps.** In the present subsection, we give conditions guaranteeing maps to be σ-expanding or π-contracting. In what follows, we only discuss the property of being σ-expanding, since the treatment for π-contracting maps can be obtained by order-duality.

4.3.1. *Upper and lower continuity.* In Section 4.1, we proved that the generalized σ-extension (resp. π-extension) is the greatest (resp. least) upper (resp. lower) continuous extension of a given map (cf. Proposition 4.3). The following is an immediate consequence of this fact.

Corollary 4.10. *For any order-preserving map $g^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$,*

(1) *if $g^{\mathbb{A}^\delta}$ is UC, then $g^{\mathbb{A}^\delta} \leq (g^{\mathbb{A}})^\sigma$, i.e. g is σ-expanding.*
(2) *if $g^{\mathbb{A}^\delta}$ is LC, then $g^{\mathbb{A}^\delta} \geq (g^{\mathbb{A}})^\pi$, i.e. g is π-contracting.*

4.3.2. *Weak m-Scott and dual weak m-Scott continuity.* In the present subsection we study the order-theoretic properties of generalized extensions of *additive* maps (cf. Definition 2.13), i.e., order-preserving maps $f^{\mathbb{A}} : \mathbb{A}_1 \times \ldots \times \mathbb{A}_m \to \mathbb{B}^\delta$ which preserve finite non-empty joins in each coordinate. Recall that *completely additive* maps are order-preserving maps $f^{\mathbb{A}^\delta} : \mathbb{A}_1^\delta \times \ldots \times \mathbb{A}_m^\delta \to \mathbb{B}^\delta$ which preserve arbitrary non-empty joins in each coordinate. *Multiplicative* and *completely multiplicative* maps are defined order-dually.

Lemma 4.11. *For any order-preserving map $f^{\mathbb{A}} : \mathbb{A}_1 \times \ldots \times \mathbb{A}_m \to \mathbb{B}^\delta$,*

(1) *if $f^{\mathbb{A}}$ is additive, then $(f^{\mathbb{A}})^\sigma : \mathbb{A}_1^\delta \times \ldots \times \mathbb{A}_m^\delta \to \mathbb{B}^\delta$ is completely additive;*
(2) *if $f^{\mathbb{A}}$ is multiplicative, then $(f^{\mathbb{A}})^\pi : \mathbb{A}_1^\delta \times \ldots \times \mathbb{A}_m^\delta \to \mathbb{B}^\delta$ is completely multiplicative.*

Proof. Similar to the proof of Lemma 3.7 in [23]. □

The following theorem relies on Definition 3.5:

Theorem 4.12. *For any order-preserving map $f^{\mathbb{A}^\delta} : \mathbb{A}_1^\delta \times \ldots \times \mathbb{A}_m^\delta \to \mathbb{B}^\delta$,*

(1) *if $f^{\mathbb{A}^\delta}$ is completely additive, then $f^{\mathbb{A}^\delta}$ is weakly m-Scott continuous, hence σ-expanding;*
(2) *if $f^{\mathbb{A}^\delta}$ is completely multiplicative, then $f^{\mathbb{A}^\delta}$ is dually weakly m-Scott continuous, hence π-contracting.*

Proof. Here we only prove item 1, since item 2 can be obtained order-dually. The proof that $f^{\mathbb{A}^\delta}$ is weakly m-Scott continuous is given in [23, Lemma 5.2]. As to showing that $f^{\mathbb{A}^\delta}$ is σ-expanding, recall that weak m-Scott continuity implies UC (cf. Remark 3.6), and by Corollary 4.10, UC implies being σ-expanding. □

The following lemma generalizes Corollary 5.7 in [17]. In what follows, we let $f_i^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}_i^\delta$ be order-preserving for any $1 \leq i \leq m$. We let $\mathbb{B}^\delta := \mathbb{B}_1^\delta \times \ldots \times \mathbb{B}_m^\delta$ and $g^{\mathbb{B}^\delta} : \mathbb{B}_1^\delta \times \ldots \times \mathbb{B}_m^\delta \to \mathbb{C}^\delta$ be order-preserving, and $(g \circ (f_1, \ldots, f_m))^{\mathbb{A}}$ be the restriction of $g^{\mathbb{B}^\delta}(f_1^{\mathbb{A}^\delta}, \ldots, f_m^{\mathbb{A}^\delta})$ to $\mathbb{A}$.

Lemma 4.13. *For all $f_i^{\mathbb{A}^\delta}$, $g^{\mathbb{B}^\delta}$, and $(g \circ (f_1, \ldots, f_m))^{\mathbb{A}}$ as above,*

(1) *if $g^{\mathbb{B}^\delta}$ is completely additive, then $g^{\mathbb{B}^\delta}((f_1^{\mathbb{A}})^\sigma, \ldots, (f_m^{\mathbb{A}})^\sigma) \leq ((g \circ (f_1, \ldots, f_m))^{\mathbb{A}})^\sigma$;*
(2) *if $g^{\mathbb{B}^\delta}$ is completely multiplicative, then $g^{\mathbb{B}^\delta}((f_1^{\mathbb{A}})^\pi, \ldots, (f_m^{\mathbb{A}})^\pi) \geq ((g \circ (f_1, \ldots, f_m))^{\mathbb{A}})^\pi$.*

Proof. Cf. the proof of Corollary 5.3 in [23]. □

4.3.3. *Strong upper and lower continuity.* In [17], it is shown that if $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}$ is meet-preserving, then $(f^{\mathbb{A}})^\sigma$ is SUC. This result generalizes as follows:

Theorem 4.14. (1) *Let $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$ be completely p-multiplicative and open Esakia (cf. Definition 4.5), and such that $f^{\mathbb{A}^\delta}(a) \in O(\mathbb{B}^\delta)$ for all $a \in \mathbb{A}$. Then $f^{\mathbb{A}^\delta}$ is SUC, and hence is also σ-expanding.*
(2) *Let $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$ be completely p-additive and closed Esakia (cf. Definition 4.5), and such that $f^{\mathbb{A}^\delta}(a) \in K(\mathbb{B}^\delta)$ for all $a \in \mathbb{A}$. Then $f^{\mathbb{A}^\delta}$ is SLC, and hence is also π-contracting.*

Proof. We only prove item 1, since item 2 can be obtained order-dually. Let $u \in \mathbb{A}^\delta$ and $q \in K(\mathbb{B}^\delta)$, let us assume that $q \leq f^{\mathbb{A}^\delta}(u)$, and let us show that $q \leq f^{\mathbb{A}^\delta}(k)$ for some $k \in K(\mathbb{A}^\delta)$ such that $k \leq u$. Recall that by denseness, $u = \bigwedge\{\bigvee\{a : y \geq a \in \mathbb{A}\} : u \leq y \in O(\mathbb{A}^\delta)\}$, and the latter set is non-empty, given that $\top$ belongs to it.

	$q \leq f^{\mathbb{A}^\delta}(u)$	
$\Leftrightarrow$	$q \leq f^{\mathbb{A}^\delta}(\bigwedge\{\bigvee\{a : y \geq a \in \mathbb{A}\} : u \leq y \in O(\mathbb{A}^\delta)\})$	(denseness)
$\Leftrightarrow$	$q \leq \bigwedge\{f^{\mathbb{A}^\delta}(\bigvee\{a : y \geq a \in \mathbb{A}\}) : u \leq y \in O(\mathbb{A}^\delta)\}$	($f^{\mathbb{A}^\delta}$ c. p-multiplicative)
$\Leftrightarrow$	$q \leq f^{\mathbb{A}^\delta}(\bigvee\{a : y \geq a \in \mathbb{A}\})$ for any $y \in O(\mathbb{A}^\delta)$ s.t. $u \leq y$	
$\Leftrightarrow$	$q \leq \bigvee\{f^{\mathbb{A}^\delta}(a) : y \geq a \in \mathbb{A}\}$ for any $y \in O(\mathbb{A}^\delta)$ s.t. $u \leq y$	($f^{\mathbb{A}^\delta}$ open Esakia)
$\Rightarrow$	$q \leq \bigvee\{f^{\mathbb{A}^\delta}(a) : a \in \mathcal{A}_y\}$ for any $y \in O(\mathbb{A}^\delta)$ s.t. $u \leq y$	(compactness, $f^{\mathbb{A}^\delta}(a) \in O(\mathbb{B}^\delta)$, and $\mathcal{A}_y \subseteq_{fin} \{a \in \mathbb{A} : a \leq y\}$)
$\Rightarrow$	$q \leq f^{\mathbb{A}^\delta}(a_y)$ for any $y \in O(\mathbb{A}^\delta)$ s.t. $u \leq y$	($a_y := \bigvee \mathcal{A}_y \in \mathbb{A}$, and $f^{\mathbb{A}^\delta}$ monotone)
$\Leftrightarrow$	$q \leq \bigwedge\{f^{\mathbb{A}^\delta}(a_y) : u \leq y \in O(\mathbb{A}^\delta)\}$	
$\Leftrightarrow$	$q \leq f^{\mathbb{A}^\delta}(\bigwedge\{a_y : u \leq y \in O(\mathbb{A}^\delta)\})$.	($f^{\mathbb{A}^\delta}$ c. p-multiplicative)

The last equivalence above holds since the set $\{a_y : u \leq y \in O(\mathbb{A}^\delta)\}$ is non-empty, given that $\top$ belongs to the set of its indices. Let $k := \bigwedge\{a_y : u \leq y \in O(\mathbb{A}^\delta)\}$. Clearly, $k \in K(\mathbb{A}^\delta)$. Moreover, for any $y \in O(\mathbb{A}^\delta)$ such that $u \leq y$,

$$a_y = \bigvee \mathcal{A}_y \leq \bigvee\{a : y \geq a \in \mathbb{A}\} = y,$$

which implies that $k := \bigwedge\{a_y : O(\mathbb{A}^\delta) \ni y \geq u\} \leq u$, as required.
As to showing that $f^{\mathbb{A}^\delta}$ is σ-expanding, notice that SUC implies UC, and by Corollary 4.10, UC implies being σ-expanding. □

Corollary 5.9 in [17] is generalized in the lemma below. In what follows, $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$ and $g^{\mathbb{B}^\delta} : \mathbb{B}^\delta \to \mathbb{C}^\delta$ are order-preserving maps, and $(g \circ f)^{\mathbb{A}}$ denotes the restriction of their composition $g^{\mathbb{B}^\delta} \circ f^{\mathbb{A}^\delta}$ to $\mathbb{A}$.

Lemma 4.15. *For any $f^{\mathbb{A}^\delta}$, $g^{\mathbb{B}^\delta}$, and $(g \circ f)^{\mathbb{A}}$ as above,*

1. *if $f^{\mathbb{A}^\delta}$ is completely p-multiplicative, open Esakia, and moreover $f^{\mathbb{A}^\delta}(a) \in O(\mathbb{B}^\delta)$ for all $a \in \mathbb{A}$, and $g^{\mathbb{B}^\delta}$ is open Esakia, then $(g^{\mathbb{B}})^\sigma \circ f^{\mathbb{A}^\delta} \leq ((g \circ f)^{\mathbb{A}})^\sigma$;*
2. *if $f^{\mathbb{A}^\delta}$ is completely p-additive, closed Esakia, and moreover $f^{\mathbb{A}^\delta}(a) \in K(\mathbb{B}^\delta)$ for all $a \in \mathbb{A}$, and $g^{\mathbb{B}^\delta}$ is closed Esakia, then $(g^{\mathbb{B}})^\pi \circ f^{\mathbb{A}^\delta} \geq ((g \circ f)^{\mathbb{A}})^\pi$.*

Proof. We only prove item 1, since item 2 can be obtained order-dually.
By Proposition 4.3, $((g \circ f)^{\mathbb{A}})^\sigma$ is the largest order-preserving UC extension of $(g \circ f)^{\mathbb{A}}$ to $\mathbb{A}^\delta$. Hence, to prove the statement, it is enough to show that $(g^{\mathbb{B}})^\sigma \circ f^{\mathbb{A}^\delta}$ is UC and that $(g^{\mathbb{B}})^\sigma \circ f^{\mathbb{A}^\delta}$ coincides with $g^{\mathbb{B}^\delta} \circ f^{\mathbb{A}^\delta}$ on $\mathbb{A}$.
By Proposition 4.3, $(g^{\mathbb{B}})^\sigma$ is UC. By Theorem 4.14, $f^{\mathbb{A}^\delta}$ is SUC. This proves that $(g^{\mathbb{B}})^\sigma \circ f^{\mathbb{A}^\delta}$ is UC.
Since $f^{\mathbb{A}^\delta}(a) \in O(\mathbb{B}^\delta)$ for all $a \in \mathbb{A}$, to show that $(g^{\mathbb{B}})^\sigma \circ f^{\mathbb{A}^\delta}(a) = g^{\mathbb{B}^\delta} \circ f^{\mathbb{A}^\delta}(a)$ for all $a \in \mathbb{A}$, it suffices to show that $(g^{\mathbb{B}})^\sigma(o) = g^{\mathbb{B}^\delta}(o)$ for all $o \in O(\mathbb{B}^\delta)$:

$$\begin{array}{rcll}
(g^{\mathbb{B}})^\sigma(o) & = & \bigvee\{(g^{\mathbb{B}})^\sigma(a) : a \in \mathbb{A} \text{ and } a \leq o\} & \text{(Theorem 4.6)}\\
& = & \bigvee\{g^{\mathbb{B}^\delta}(a) : a \in \mathbb{A} \text{ and } a \leq o\} & \\
& = & g^{\mathbb{B}^\delta}(\bigvee\{a : a \in \mathbb{A} \text{ and } a \leq o\}) & (g^{\mathbb{B}^\delta} \text{ open Esakia})\\
& = & g^{\mathbb{B}^\delta}(o). & (o = \bigvee\{a : a \in \mathbb{A} \text{ and } a \leq o\})
\end{array}$$

□

In the next section, we are going to apply the theory developed so far to the additional operations of the expanded language $\mathcal{L}^{++}$.

5. Generalized canonical extensions of $\mathcal{L}^{++}$-terms

In the present section, we adapt the definition of generalized canonical extensions to term functions of the expanded language $\mathcal{L}^{++}$, and show that $\mathcal{L}^{++}$-terms of certain syntactic shapes are contracting or expanding, which will be used in the canonicity proof for ALBA DML-inequalities in Section 6.

5.1. **Generalized canonical extensions for term functions in $\mathcal{L}^{++}$.** The language $\mathcal{L}^{++}$ involves not only new logical symbols, but also new variables, namely nominals and co-nominals (see Section 2.4) ranging in their special subdomains. Therefore we need to adapt the definitions of term functions, canonical extensions and contracting and expanding terms to this setting. For convenience of work, below we introduce two different symbolic conventions for terms functions: one in which nominals and co-nominals are already assigned as constants, the second one in which they are taken as arguments.

Definition 5.1 (**Term functions in $\mathcal{L}^{++}$**). Let $t(\vec{p}, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ be an $\mathcal{L}^{++}$-term such that the proposition variables, nominals and co-nominals actually occurring in t belong to $\vec{p}, \vec{\mathbf{j}}, \vec{\mathbf{m}}$, respectively. Let $\mathbb{A}$ be a DMA. For all arrays $\vec{x}, \vec{y}$ of suitable lengths such that $x \in J^{\infty}(\mathbb{A}^{\delta})$ for each coordinate x in $\vec{x}$, and $y \in M^{\infty}(\mathbb{A}^{\delta})$ for each coordinate y in $\vec{y}$, the symbol $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ denotes the term function with the following domain and codomain:

$$(\mathbb{A}^{\delta})^n \to \mathbb{A}^{\delta}$$

such that the variables in $\vec{\mathbf{j}}, \vec{\mathbf{m}}$ are orderly interpreted as elements in $\vec{x}, \vec{y}$, respectively. In what follows, we abuse notation and write e.g. $\vec{x} \in J^{\infty}(\mathbb{A}^{\delta})$, meaning that each coordinate of $\vec{x}$ is an element of $J^{\infty}(\mathbb{A}^{\delta})$. As usual, $t^{\mathbb{A}^{\delta}}$ denotes the term function with the following domain and codomain:

$$(\mathbb{A}^{\delta})^n \times (J^{\infty}(\mathbb{A}^{\delta}))^k \times (M^{\infty}(\mathbb{A}^{\delta}))^l \to \mathbb{A}^{\delta}.$$

Finally, the symbols $t^{\mathbb{A}}_{\vec{x},\vec{y}}$ and $t^{\mathbb{A}}$ respectively denote the restrictions of the term functions $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ and $t^{\mathbb{A}^{\delta}}$ to $\mathbb{A}^n$. In other words,

$$t^{\mathbb{A}}_{\vec{x},\vec{y}} : \mathbb{A}^n \to \mathbb{A}^{\delta} \quad \text{and} \quad t^{\mathbb{A}} : \mathbb{A}^n \times (J^{\infty}(\mathbb{A}^{\delta}))^k \times (M^{\infty}(\mathbb{A}^{\delta}))^l \to \mathbb{A}^{\delta}.$$

The theory developed in Section 4 can be now applied to $\mathcal{L}^{++}$-term functions where nominals and co-nominals have been fixed. The following definition is the application of Definition 4.1 to $t^{\mathbb{A}}_{\vec{x},\vec{y}}$.

Definition 5.2 (**Generalized canonical extensions for term functions in $\mathcal{L}^{++}$**). Let $t(\vec{p}, \vec{q}, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ be a uniform $\mathcal{L}^{++}$-term such that all the nominals and co-nominals actually occurring in t are in $\vec{\mathbf{j}}, \vec{\mathbf{m}}$, and let $\vec{p}, \vec{q}$ stand for the arrays of monotone and antitone coordinates, respectively. For any DMA $\mathbb{A}$ and all $\vec{u}, \vec{v} \in \mathbb{A}^{\delta}$, $\vec{x} \in J^{\infty}(\mathbb{A}^{\delta})$, $\vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$, let

$$(t^{\mathbb{A}}_{\vec{x},\vec{y}})^{\sigma} : (\mathbb{A}^{\delta})^{n_1} \times ((\mathbb{A}^{\delta})^{\partial})^{n_2} \to \mathbb{A}^{\delta} \quad \text{and} \quad (t^{\mathbb{A}}_{\vec{x},\vec{y}})^{\pi} : (\mathbb{A}^{\delta})^{n_1} \times ((\mathbb{A}^{\delta})^{\partial})^{n_2} \to \mathbb{A}^{\delta}$$

be respectively defined by the assignments $(\vec{u}, \vec{v}) \mapsto (t^{\mathbb{A}})^{\sigma}(\vec{u}, \vec{v}, \vec{x}, \vec{y})$, and $(\vec{u}, \vec{v}) \mapsto (t^{\mathbb{A}})^{\pi}(\vec{u}, \vec{v}, \vec{x}, \vec{y})$, where:

$$(t^{\mathbb{A}})^{\sigma}(\vec{u}, \vec{v}, \vec{x}, \vec{y}) = \bigvee\{\bigwedge\{t^{\mathbb{A}}(\vec{a}, \vec{b}, \vec{x}, \vec{y}) : \vec{c} \leq \vec{a} \in \mathbb{A}, \vec{o} \geq \vec{b} \in \mathbb{A}\} : \vec{u} \geq \vec{c} \in K(\mathbb{A}^{\delta}), \vec{v} \leq \vec{o} \in O(\mathbb{A}^{\delta})\}$$

$$(t^{\mathbb{A}})^{\pi}(\vec{u}, \vec{v}, \vec{x}, \vec{y}) = \bigwedge\{\bigvee\{t^{\mathbb{A}}(\vec{a}, \vec{b}, \vec{x}, \vec{y}) : \vec{o} \geq \vec{a} \in \mathbb{A}, \vec{c} \leq \vec{b} \in \mathbb{A}\} : \vec{u} \leq \vec{o} \in O(\mathbb{A}^{\delta}), \vec{v} \geq \vec{c} \in K(\mathbb{A}^{\delta})\}.$$

Next, we define contracting and expanding $\mathcal{L}^{++}$-terms corresponding to the definitions above:

Definition 5.3 (**Generalized contracting and expanding $\mathcal{L}^{++}$-terms**). For any uniform $\mathcal{L}^{++}$-term $t(\vec{p}, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ (cf. Definition 2.22) such that all nominals and co-nominals occurring in t are in $\vec{\mathbf{j}}, \vec{\mathbf{m}}$, and for $\lambda \in \{\sigma, \pi\}$,

- t is λ-*stable* if $t^{\mathbb{A}^{\delta}} = (t^{\mathbb{A}})^{\lambda}$ for all DMAs $\mathbb{A}$.
- t is λ-*expanding* if $t^{\mathbb{A}^{\delta}} \leq (t^{\mathbb{A}})^{\lambda}$ for all DMAs $\mathbb{A}$.
- t is λ-*contracting* if $t^{\mathbb{A}^{\delta}} \geq (t^{\mathbb{A}})^{\lambda}$ for all DMAs $\mathbb{A}$.

Clearly, t is λ-stable (resp. λ-expanding, λ-contracting) iff $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}} = (t^{\mathbb{A}}_{\vec{x},\vec{y}})^{\lambda}$ (resp. $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}} \leq (t^{\mathbb{A}}_{\vec{x},\vec{y}})^{\lambda}$, $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}} \geq (t^{\mathbb{A}}_{\vec{x},\vec{y}})^{\lambda}$) for all $\vec{x} \in J^{\infty}(\mathbb{A}^{\delta})$, $\vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$.

The additional connectives of $\mathcal{L}^{++}$ turn out to be very well behaved, as we are going to discuss in the next subsection.

5.2. **σ- and π-stability of connectives in the expanded signature.** In the present subsection, we study the stability of the additional connectives $\blacksquare, \blacklozenge, \blacktriangleright, \blacktriangleleft, -, \rightarrow, \mathbf{n}, \mathbf{l}$. In particular, we will make use of the fact that they are closed and open Esakia (see below and also in Appendix B). Throughout the present subsection, we omit superscripts of term functions when they are clear from the context, since their domain is typically $\mathbb{A}^\delta$.
In the lemma below, we list all the preservation properties which follow from the fact, well known from general order theory (see [13]), that each additional operator is either right or left residual or adjoint.

Lemma 5.4. *For any $\mathcal{A} \subseteq \mathbb{A}^\delta$ and any $u \in \mathbb{A}^\delta$,*

(1) $\blacksquare \bigwedge \mathcal{A} = \bigwedge\{\blacksquare v : v \in \mathcal{A}\}$;
(2) $\blacklozenge \bigvee \mathcal{A} = \bigvee\{\blacklozenge v : v \in \mathcal{A}\}$;
(3) $\blacktriangleright \bigvee \mathcal{A} = \bigwedge\{\blacktriangleright v : v \in \mathcal{A}\}$;
(4) $\blacktriangleleft \bigwedge \mathcal{A} = \bigvee\{\blacktriangleleft v : v \in \mathcal{A}\}$;
(5) $u - \bigwedge \mathcal{A} = \bigvee\{u - v : v \in \mathcal{A}\}$, $\bigvee \mathcal{A} - u = \bigvee\{v - u : v \in \mathcal{A}\}$;
(6) $u \rightarrow \bigwedge \mathcal{A} = \bigwedge\{u \rightarrow v : v \in \mathcal{A}\}$, $\bigvee \mathcal{A} \rightarrow u = \bigwedge\{v \rightarrow u : v \in \mathcal{A}\}$;
(7) $\mathbf{n}(u, \bigwedge \mathcal{A}) = \bigvee\{\mathbf{n}(u, v) : v \in \mathcal{A}\}$, $\mathbf{n}(\bigvee \mathcal{A}, u) = \bigvee\{\mathbf{n}(v, u) : v \in \mathcal{A}\}$;
(8) $\mathbf{l}(u, \bigwedge \mathcal{A}) = \bigwedge\{\mathbf{l}(u, v) : v \in \mathcal{A}\}$, $\mathbf{l}(\bigvee \mathcal{A}, u) = \bigwedge\{\mathbf{l}(v, u) : v \in \mathcal{A}\}$.

Identities (1),(3), (6) and (8) imply in particular that $\blacksquare, \blacktriangleright, \rightarrow, \mathbf{l}$ are closed Esakia, and the remaining identities imply that $\blacklozenge, \blacktriangleleft, -, \mathbf{n}$ are open Esakia (see Definition 4.5).

The next proposition makes use of Lemma B.10 in the Appendix.

Proposition 5.5. *The connectives $\blacksquare, \blacklozenge, \blacktriangleright, \blacktriangleleft, -, \rightarrow, \mathbf{n}, \mathbf{l}$ are both σ-contracting and π-expanding.*

Proof. By Theorem 4.7, it is enough to show that all the connectives mentioned above satisfy both the open and the closed Esakia condition (see Definition 4.5). By the conclusion of Lemma 5.4, it remains to be shown that $\blacksquare, \blacktriangleright, \rightarrow, \mathbf{l}$ are open Esakia, and $\blacklozenge, \blacktriangleleft, -, \mathbf{n}$ are closed Esakia. Lemma B.10 equivalently says that $\blacksquare, \blacktriangleright, \rightarrow$ are open Esakia, and that $\blacklozenge, \blacktriangleleft, -$ are closed Esakia. Actually, Lemma B.10 states that the binary connectives $-$ and $\rightarrow$ enjoy the intersection condition. However, by Lemma 4.9, this condition is equivalent to the Esakia condition we require. Finally, Lemma 5.6 below takes care of the remaining intersection conditions for $\mathbf{n}$ and $\mathbf{l}$, which again by Lemma 4.9 imply the required Esakia conditions. ☐

Lemma 5.6. *Let $\mathcal{C} = \{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$ be non-empty downward-directed, $\mathcal{O} = \{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$ be non-empty upward-directed. Then*

(1) $\mathbf{n}(\bigwedge \mathcal{C}, \bigvee \mathcal{O}) = \bigwedge\{\mathbf{n}(c, o) : c \in \mathcal{C}, o \in \mathcal{O}\}$;
(2) $\mathbf{l}(\bigwedge \mathcal{C}, \bigvee \mathcal{O}) = \bigvee\{\mathbf{l}(c, o) : c \in \mathcal{C}, o \in \mathcal{O}\}$.

Proof. We only prove item 2, since item 1 is an order-variant of item 2. The right-to-left inequality straightforwardly follows from the $(\partial, 1)$-monotonicity of $\mathbf{l}$. To prove the converse inequality, since the value of $\mathbf{l}$ is either $\top$ or $\bot$, we only consider the case in which $\mathbf{l}(\bigwedge \mathcal{C}, \bigvee \mathcal{O}) = \top$, and show that there exist some $c \in \mathcal{C}$ and $o \in \mathcal{O}$ such that $\mathbf{l}(c, o) = \top$. From $\mathbf{l}(\bigwedge \mathcal{C}, \bigvee \mathcal{O}) = \top$ it follows that $\bigwedge \mathcal{C} \leq \bigvee \mathcal{O}$. Hence, by compactness, $\bigwedge \mathcal{C}_0 \leq \bigvee \mathcal{O}_0$ for some non-empty finite $\mathcal{C}_0 \subseteq \mathcal{C}$ and $\mathcal{O}_0 \subseteq \mathcal{O}$. By the directedness of $\mathcal{C}$ and $\mathcal{O}$, there exist some $c \in \mathcal{C}$ and $o \in \mathcal{O}$ such that $c \leq \bigwedge \mathcal{C}_0 \leq \bigvee \mathcal{O}_0 \leq o$, therefore $\mathbf{l}(c, o) = \top$. ☐

We are now in a position to prove the following:

Proposition 5.7. *The connectives $\blacksquare, \blacklozenge, \blacktriangleright, \blacktriangleleft$ are both σ-stable and π-stable.*

Proof. By Theorem 4.14.1, in order to prove that $\blacksquare : \mathbb{A}^\delta \rightarrow \mathbb{A}^\delta$ and $\blacktriangleright : (\mathbb{A}^\delta)^\partial \rightarrow \mathbb{A}^\delta$ are σ-expanding, it is enough to show that they are completely p-multiplicative, open Esakia,

and such that their value on clopen input is always open. These conditions are all satisfied (cf. Lemmas 5.4, B.10, and B.7, respectively). Likewise, in order to prove that $\blacklozenge : \mathbb{A}^\delta \to \mathbb{A}^\delta$ and $\blacktriangleleft : (\mathbb{A}^\delta)^\partial \to \mathbb{A}^\delta$ are π-contracting, we appeal to Theorem 4.14.2 and Lemmas 5.4, B.10, and B.7. Together with Proposition 5.5, this proves that $\blacksquare$ and $\blacktriangleright$ are σ-stable, and that $\blacklozenge$ and $\blacktriangleleft$ are π-stable.
As to the remaining parts of the statement, let us prove that $\blacksquare$ is π-stable, the others being order-variants of it. In the following proof, to make the domain of the functions clear, we use superscripts. Indeed, for any $u \in \mathbb{A}^\delta$, since $\blacksquare^{\mathbb{A}^\delta}$ is completely meet-preserving and satisfies the open Esakia condition, the following chain of identities holds:

$$\begin{aligned}
\blacksquare^{\mathbb{A}^\delta} u &= \blacksquare^{\mathbb{A}^\delta}(\bigwedge\{\bigvee\{a : o \geq a \in \mathbb{A}\} : u \leq o \in O(\mathbb{A}^\delta)\}) \\
&= \bigwedge\{\blacksquare^{\mathbb{A}^\delta}(\bigvee\{a : o \geq a \in \mathbb{A}\}) : u \leq o \in O(\mathbb{A}^\delta)\} \\
&= \bigwedge\{\bigvee\{\blacksquare^{\mathbb{A}^\delta} a : o \geq a \in \mathbb{A}\} : u \leq o \in O(\mathbb{A}^\delta)\} \\
&= \bigwedge\{\bigvee\{\blacksquare^{\mathbb{A}} a : o \geq a \in \mathbb{A}\} : u \leq o \in O(\mathbb{A}^\delta)\} \\
&= (\blacksquare^{\mathbb{A}})^\pi(u).
\end{aligned}$$

□

Finally, we state the stability properties enjoyed by the remaining binary connectives.

Proposition 5.8. *The connectives* $\mathbf{n}$ *and* $-$ *are* σ*-stable, and the connectives* $\mathbf{l}$ *and* $\rightarrow$ *are* π*-stable.*

Proof. By Proposition 5.5, it remains to be shown that $\mathbf{n}$ and$-$ are σ-expanding and $\mathbf{l}$ and $\rightarrow$ are π-contracting. By Theorem 4.12.1, to show that $\mathbf{n}, - : \mathbb{A}^\delta \times (\mathbb{A}^\delta)^\partial \to \mathbb{A}^\delta$ are σ-expanding, it is enough to show that they are completely additive. In fact more is true, given that it follows from Lemma 5.4 that they are complete operators. Likewise, in order to prove that $\mathbf{l}, \rightarrow : (\mathbb{A}^\delta)^\partial \times \mathbb{A}^\delta \to \mathbb{A}^\delta$ are π-contracting, we appeal to Theorem 4.12.2, Proposition 5.5, and Lemma 5.4. □

5.3. **Generalized σ-contracting and π-expanding $\mathcal{L}^{++}$-terms.** The aim of the present subsection is to show (cf. Theorem 5.11) that all uniform and syntactically closed (resp. open) $\mathcal{L}^{++}$-terms are σ-contracting (resp. π-expanding). This result will be applied to the ϵ^∂-part in the Jónsson-style canonicity strategy (cf. Section 3).

Let us remind the reader of our notational convention: in what follows, we consider uniform $\mathcal{L}^{++}$-terms $\varphi(\vec{p}, \vec{q}, r, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ which are positive in their $\vec{p}$ coordinates and negative in their $\vec{q}$ coordinates. To fix a variable r either in $\vec{p}$ or in $\vec{q}$, we will write $\varphi(\vec{p}, \vec{q}, r, \vec{\mathbf{j}}, \vec{\mathbf{m}})$.

Lemma 5.9 (Intersection lemma)**.** *Let* $\varphi(\vec{p}, \vec{q}, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ *and* $\psi(\vec{p}, \vec{q}, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ *be* $\mathcal{L}^{++}$*-terms as above, which are syntactically closed and open respectively, and such that all the nominals and co-nominals occurring in* φ *or in* ψ *are in* $\vec{\mathbf{j}}, \vec{\mathbf{m}}$*. Let* $\vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, $\vec{c} \in K(\mathbb{A}^\delta), \vec{o} \in O(\mathbb{A}^\delta)$ *of suitable lengths, and let* $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$ *be non-empty and downward-directed, and* $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$ *be non-empty and upward-directed.*

1. *If* $\varphi(\vec{p}, \vec{q}, r, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ *is positive in* r*, then* $\varphi^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, \bigwedge\{c_i : i \in I\}, \vec{x}, \vec{y}) = \bigwedge_{i\in I} \varphi^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, c_i, \vec{x}, \vec{y})$.
2. *If* $\psi(\vec{p}, \vec{q}, r, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ *is negative in* r*, then* $\psi^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, \bigwedge\{c_i : i \in I\}, \vec{x}, \vec{y}) = \bigvee_{i\in I} \psi^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, c_i, \vec{x}, \vec{y})$.
3. *If* $\varphi(\vec{p}, \vec{q}, r, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ *is negative in* r*, then* $\varphi^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, \bigvee\{o_i : i \in I\}, \vec{x}, \vec{y}) = \bigwedge_{i\in I} \varphi^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, o_i, \vec{x}, \vec{y})$.
4. *If* $\psi(\vec{p}, \vec{q}, r, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ *is positive in* r*, then* $\psi^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, \bigvee\{o_i : i \in I\}, \vec{x}, \vec{y}) = \bigvee_{i\in I} \psi^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, o_i, \vec{x}, \vec{y})$.

Proof. The proof goes by simultaneous induction on φ and ψ, and is similar to the proof of Lemma B.12. □

Corollary 5.10. *Let $\varphi(\vec{p}, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ and $\psi(\vec{p}, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ be uniform $\mathcal{L}^{++}$-terms, syntactically closed and open respectively, where $\vec{p}, \vec{\mathbf{j}}, \vec{\mathbf{m}}$ are all proposition variables, nominals and co-nominals occurring in φ or ψ. Then, for all $\vec{x} \in J^{\infty}(\mathbb{A}^{\delta})$ and $\vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$,*

(1) $\varphi^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}(\vec{p})$ *is closed Esakia;*
(2) $\psi^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}(\vec{p})$ *is open Esakia.*

Proof. By Lemma 4.9, it is enough to show that the intersection conditions hold, which have been verified in Lemma 5.9. □

Theorem 5.11. *Every uniform and syntactically closed $\mathcal{L}^{++}$-term is σ-contracting, and every uniform and syntactically open $\mathcal{L}^{++}$-term is π-expanding.*

Proof. By Corollary 5.10, for any uniform and syntactically closed term $\varphi(\vec{p}, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ and any DMA $\mathbb{A}$, $\vec{x} \in J^{\infty}(\mathbb{A}^{\delta}), \vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$, the term function $\varphi^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}(\vec{p})$ is closed Esakia. Therefore by Theorem 4.7, $\varphi^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}(\vec{p})$ is σ-contracting, hence by Definition 5.3, φ is σ-contracting. The second part of the statement is proven similarly. □

5.4. **Generalized σ-expanding and π-contracting $\mathcal{L}^{++}$-terms.** In the present subsection, we define para-inductive $\mathcal{L}^{++}$-terms (cf. Definition 5.14), and show that left (resp. right) para-inductive terms are σ-expanding (resp. π-contracting) (Theorem 5.17).
The following preliminary lemma is proved, analogously to Lemma B.11, by simultaneous induction on the shape of φ and ψ.

Lemma 5.12. *Let $\varphi(\vec{p}, \vec{q}, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ and $\psi(\vec{p}, \vec{q}, \vec{\mathbf{j}}, \vec{\mathbf{m}})$ be uniform $\mathcal{L}^{++}$-terms, which are positive in $\vec{p}$ and negative in $\vec{q}$. Let φ and ψ be syntactically closed and open respectively. Then, for all $\vec{x} \in J^{\infty}(\mathbb{A}^{\delta}), \vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$, $\vec{c} \in K(\mathbb{A}^{\delta}), \vec{o} \in O(\mathbb{A}^{\delta})$,*

(1) $\varphi^{\mathbb{A}^{\delta}}(\vec{c}, \vec{o}, \vec{x}, \vec{y}) \in K(\mathbb{A}^{\delta})$*;*
(2) $\psi^{\mathbb{A}^{\delta}}(\vec{o}, \vec{c}, \vec{x}, \vec{y}) \in O(\mathbb{A}^{\delta})$*.*

By Corollary 5.10, for any uniform and syntactically open term t, the term function $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ satisfies the open Esakia condition for any suitable assignment of nominals and co-nominals. Hence the following:

Corollary 5.13. *For any uniform and syntactically open $\mathcal{L}^{++}$-term t,*

(1) *for any DMA $\mathbb{A}$, $\vec{x} \in J^{\infty}(\mathbb{A}^{\delta})$ and $\vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$, if $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ is completely p-multiplicative, then $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ is SUC;*
(2) *if t is p-multiplicative (cf. Definition 2.13) in its proposition variables, then is σ-expanding.*

Proof. The assumptions imply, by Corollary 5.10, that $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ is open Esakia; by Lemma 5.12, for all $\vec{a}, \vec{a}' \in \mathbb{A}$, $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}(\vec{a}, \vec{a}') \in O(\mathbb{A}^{\delta})$; since $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ is completely p-multiplicative, by Theorem 4.14, $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ is SUC. The second part follows easily from the first part and Theorem 4.14. □

The corollary above is all we need to apply in the proof of canonicity for the ALBA inequalities in Section 6. In fact, the results in Section 4.3, providing sufficient conditions for terms to be σ-expanding and π-contracting, can be used as a guideline to defining para-inductive terms in such a way that left (resp. right) para-inductive terms in $\mathcal{L}^{++}$ are σ-expanding (resp. π-contracting).
In Section 4, we showed that in order for a function $f^{\mathbb{A}^{\delta}}$ to be σ-expanding (resp. π-contracting), it suffices to show that $f^{\mathbb{A}^{\delta}}$ satisfies UC (resp. LC). These results make it possible to reproduce the Jónsson-style core strategy which was discussed in Section 3.3 in

Polarity	Outer Skeleton							
$+$	$\wedge$	$\vee$	$\Diamond$	$\lhd$	$\blacklozenge$	$\blacktriangleleft$	$-$	$\mathbf{n}$
$-$	$\vee$	$\wedge$	$\Box$	$\rhd$	$\blacksquare$	$\blacktriangleright$	$\rightarrow$	$\mathbf{l}$
	Inner Structure							
	Adjoints					Residuals		
$+$	$\wedge$	$\Box$	$\rhd$	$\blacksquare$	$\blacktriangleright$	$\rightarrow$	$\mathbf{l}$	$\vee$
$-$	$\vee$	$\Diamond$	$\lhd$	$\blacklozenge$	$\blacktriangleleft$	$-$	$\mathbf{n}$	$\wedge$

TABLE 2. Classification of nodes for para-inductive terms

the context of generalized canonical extensions. Here we take UC as an example: as discussed in Section 3.3, UC is of the form $(\forall J \exists K)$, and hence this property can be guaranteed in two ways: either by requiring the inner function to satisfy UC $(\forall J \exists K)$ and the outer function to satisfy weak m-Scott continuity $(\forall J \exists J^m_\bot)$ (cf. Definition 3.5), or by requiring the inner function to satisfy SUC $(\forall K \exists K)$ and the outer function to satisfy UC $(\forall J \exists K)$. In addition, the differences between this generalized setting and the standard one require that we put extra constraints on terms in order to make them satisfy the assumptions of Lemma 4.13 and Lemma 4.15. These extra constraints are given in the following definition in the form of Sahlqvist-style syntactic conditions on $\mathcal{L}^{++}$-terms. The following definition is given in the style of [6].

Definition 5.14 (Para-inductive $\mathcal{L}^{++}$-terms). For any order type ϵ on $p_1, \ldots, p_n$, the signed generation tree of an $\mathcal{L}^{++}$-term $+s(p_1, \ldots, p_n)$ (resp. $-s(p_1, \ldots, p_n)$) is *ϵ-left para-inductive*, abbreviated as ϵ-LPI (resp. *ϵ-right para-inductive*, abbreviated as ϵ-RPI), if:

- it agrees with ϵ;
- every branch with a proposition variable p_i as its leaf is the concatenation of two paths P_1 and P_2, possibly of length 0, such that P_1 is a path from the leaf consisting (apart from p_i) only of Inner Structure nodes as given in Table 2, and P_2 consists (apart from variable nodes) only of Outer Skeleton nodes.
- Every Residual node in P_1 has one of the following shapes:

Polarity						
$+$	$\alpha \rightarrow \beta$	$\gamma \rightarrow \delta$	$\mathbf{l}(\alpha, \beta)$	$\mathbf{l}(\gamma, \delta)$	$\delta \vee \gamma$	$\gamma \vee \delta$
$-$	$\gamma - \delta$	$\alpha - \beta$	$\mathbf{n}(\gamma, \delta)$	$\mathbf{n}(\alpha, \beta)$	$\alpha \wedge \beta$	$\beta \wedge \alpha$

  where α is a syntactically closed pure term (see Definition B.3), δ is a syntactically open pure term, β and γ are the subformulas through which P_1 goes.
- If an immediate subformula $*\varphi$ of a binary adjunction node in P_1 (that is, either $+\wedge$ or $-\vee$) is pure, then:
  (1) either $\varphi = \delta$ is syntactically open if $*$ is $+$, or
  (2) $\varphi = \alpha$ is syntactically closed if $*$ is $-$.

In what follows, para-inductive terms will be abbreviated as *PI-terms*.

Remark 5.15. The following facts are worth noticing about the definition above:

(1) The definition above ensures that if t is a left-LPI term and $\mathbb{A}$ is a DMA, then, regarding $t^{\mathbb{A}^\delta}$ as a composition of order-preserving maps with respect to an appropriate order-type, $t^{\mathbb{A}^\delta}$ is composed of two parts: the interpretation of the outer skeleton, which is the composition of completely additive maps, and the interpretation of the inner structure, which is the composition of maps which are completely p-multiplicative and such that each subterm in the inner structure is an open (resp.

closed) term if it is labelled + (resp. −). Proposition 5.16 below describes the inner structure more in detail.

(2) Pure terms are trivially both left and right ϵ-LPI for any ϵ, since none of their branches end with a proposition variable.

(3) All uniform Sahlqvist terms in the basic DML signature are para-inductive terms.

(4) Para-inductive terms are reminiscent of inductive formulas, in a sense which can be made more precise as follows. Consider a non-pure ϵ-PI-term t in which only symbols in $\mathcal{L}$ occur on branches ending with proposition variables p_i. Suppose also that $\mathbf{n}$ and $\mathbf{l}$ do not occur in t. If we replace each δ with a fresh proposition variable $p_{+\delta}$ and each α with a fresh proposition variable $p_{-\alpha}$, extend ϵ to the order-type ϵ' on the union of the set of fresh variables and the set of old ones by assigning $\epsilon'(p_{+\delta}) := \partial$ and $\epsilon'(p_{-\alpha}) := 1$, and define a dependency order Ω in which all the old variables are maximal elements and all the new variables are minimal elements, then the term we obtain is (Ω, ϵ')-inductive. Since the minimal valuation for a variable of order type 1 (resp. ∂) is a syntactically closed (resp. open) term, the syntactic shape of PI-terms is intended to describe the shape of inductive terms after that all proposition variables which are non-maximal w.r.t. the dependency order have been eliminated via Ackermann-substitution.

Proposition 5.16. *If $*s$ is a non-pure PI-term with no occurrences of Outer Skeleton nodes, then every subterm $\star t$ of $*s$ is syntactically open (resp. closed) if $\star$ is + (resp. −).*

Proof. Simple but very lengthy, hence is omitted. □

Theorem 5.17. *Every ϵ-LPI (resp. ϵ-RPI) term is σ-expanding (resp. π-contracting).*

Proof. By simultaneous induction on the shape of LPI- and RPI-terms. We only discuss the case of ϵ-LPI terms, the treatment of ϵ-RPI terms being order-dual. The base of the induction is trivial. It is also easy to check that pure terms are both σ-stable and π-stable. Hence, without loss of generality, we only consider non-pure terms. Let α be an ϵ-LPI term, and let f be its main connective. The discussion breaks into cases.
If f is a node in the outer skeleton, then it is among $\wedge$, $\vee$, $\diamondsuit$, $\lhd$, $\blacklozenge$, $\blacktriangleleft$, $-$, $\mathbf{n}$. We only discuss the case of $\alpha = \beta - \gamma$ and $\alpha = \beta \vee \gamma$, the other cases being easier.
For the case of $\alpha = \beta - \gamma$, it can be readily checked that $+\beta$ is ϵ-LPI and $-\gamma$ is ϵ-RPI; indeed, if not, then their flaws would make $+\alpha$ non-LPI. Hence, by induction hypothesis, β is σ-expanding and γ is π-contracting, i.e.

$$\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \leq (\beta^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma \text{ and } (\gamma^{\mathbb{A}}_{\vec{x},\vec{y}})^\pi \leq \gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, \qquad \text{i.e. } (\gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}})^\partial \leq ((\gamma^{\mathbb{A}}_{\vec{x},\vec{y}})^\partial)^\sigma$$

for any DMA $\mathbb{A}$, any $\vec{x} \in J^\infty(\mathbb{A}^\delta)$, any $\vec{y} \in M^\infty(\mathbb{A}^\delta)$, and where $\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}$ and $\gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}$ are regarded as follows:

$$\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} : (\mathbb{A}^\delta)^\epsilon \to \mathbb{A}^\delta \text{ and } \gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} : ((\mathbb{A}^\delta)^\epsilon)^\partial \to \mathbb{A}^\delta.$$

Recall that $-^{\mathbb{A}^\delta} : \mathbb{A}^\delta \times (\mathbb{A}^\delta)^\partial \to \mathbb{A}^\delta$ is a complete operator and $\alpha^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} = -^{\mathbb{A}^\delta}(\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, (\gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}})^\partial)$ is a composition of the following order-preserving maps:

$$(\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, (\gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}})^\partial) : (\mathbb{A}^\delta)^\epsilon \to \mathbb{A}^\delta \times (\mathbb{A}^\delta)^\partial \qquad -^{\mathbb{A}^\delta} : \mathbb{A}^\delta \times (\mathbb{A}^\delta)^\partial \to \mathbb{A}^\delta.$$

Hence, by Lemma 4.13,

$$\begin{aligned} (\beta - \gamma)^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} &= -^{\mathbb{A}^\delta}(\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, (\gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}})^\partial) \\ &\leq -^{\mathbb{A}^\delta}((\beta^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma, ((\gamma^{\mathbb{A}}_{\vec{x},\vec{y}})^\partial)^\sigma) && \text{(induction hypothesis)} \\ &= -^{\mathbb{A}^\delta}(\beta^{\mathbb{A}}_{\vec{x},\vec{y}}, (\gamma^{\mathbb{A}}_{\vec{x},\vec{y}})^\partial)^\sigma \\ &\leq ((\beta - \gamma)^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma && \text{(Lemma 4.13).} \end{aligned}$$

For the case of $\alpha = \beta \vee \gamma$, it can be readily checked that both $+\beta$ and $+\gamma$ are ϵ-LPI; indeed, if not, then their flaws would make $+\alpha$ non-LPI. Hence, by induction hypothesis, both β and γ are σ-expanding, i.e.

$$\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \leq (\beta^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma \quad \text{and} \quad \gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \leq (\gamma^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma,$$

for any DMA $\mathbb{A}$, any $\vec{x} \in J^\infty(\mathbb{A}^\delta)$, any $\vec{y} \in M^\infty(\mathbb{A}^\delta)$, and where $\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}$ and $\gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}$ are regarded as follows:

$$\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, \gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} : (\mathbb{A}^\delta)^\epsilon \to \mathbb{A}^\delta.$$

Recall that $\vee^{\mathbb{A}^\delta} : \mathbb{A}^\delta \times \mathbb{A}^\delta \to \mathbb{A}^\delta$ is completely additive and $\alpha^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} = \vee^{\mathbb{A}^\delta}(\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, \gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}})$ is a composition of the following order-preserving maps:

$$(\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, \gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}) : (\mathbb{A}^\delta)^\epsilon \to \mathbb{A}^\delta \times \mathbb{A}^\delta \qquad \vee^{\mathbb{A}^\delta} : \mathbb{A}^\delta \times \mathbb{A}^\delta \to \mathbb{A}^\delta.$$

Hence, by Lemma 4.13,

$$\begin{array}{rcll}
(\beta \vee \gamma)^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} & = & \vee^{\mathbb{A}^\delta}(\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, \gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}) & \\
& \leq & \vee^{\mathbb{A}^\delta}((\beta^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma, (\gamma^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma) & \text{(induction hypothesis)} \\
& = & \vee^{\mathbb{A}^\delta}(\beta^{\mathbb{A}}_{\vec{x},\vec{y}}, \gamma^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma & \\
& \leq & ((\beta \vee \gamma)^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma & \text{(Lemma 4.13).}
\end{array}$$

If f is a node in the inner structure, then by Proposition 5.16, α is syntactically open, and by appropriately reversing the polarity of coordinates and functions, $\alpha^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}$ is a composition of maps which are completely p-multiplicative w.r.t. some order-type, therefore by Corollary 5.13, α is σ-expanding. ☐

6. Jónsson-style canonicity for ALBA-inequalities

In the present section, we prove the canonicity of ALBA inequalities in the language of distributive modal logic using the method of generalized canonical extensions of maps (cf. Section 4), as stated in the theorem below.

Theorem 6.1. *The DML-inequalities on which ALBA succeeds are canonical.*

Proof. Our proof strategy consists in making use of the syntactic ALBA-manipulations (cf. Appendix A) in order to implement Jónsson-style canonicity. The structure of the proof follows the U-shaped argument illustrated on page 2, except for the bottom line, where the reduction is not performed up to the calculation of the first-order correspondent of the input inequality, and instead the proof proceeds Jónsson-style.
Let $\varphi \leq \psi$ be an inequality on which ALBA succeeds. Let us run ALBA on this inequality up to the point in the reduction-elimination stage (at which point we will be dealing one or more systems each of which consisting of finitely many inequalities, see discussion in Appendix A) in which only one proposition variable remains in each system of inequalities, and these systems are all in Ackermann shape (cf. [9, Definition 10.7]). Since each system of inequality is treated independently of the others, we can assume for simplicity that each system is in the following *right-Ackermann form* (denoted $\mathrm{RAF}(\varphi_i \leq \psi_i)$), the case of systems in left-Ackermann shape being symmetric:

$$\begin{cases} \alpha_1 \leq p \\ \vdots \\ \alpha_m \leq p \\ \beta_1(p) \leq \gamma_1(p) \\ \vdots \\ \beta_n(p) \leq \gamma_n(p) \\ \delta_1 \leq \theta_1 \\ \vdots \\ \delta_k \leq \theta_k \end{cases}$$

where the αs, δs and θs are pure, the $\beta_j(p)$s are positive in p, and the $\gamma_j(p)$s are negative in p. In such a situation, the fact that p is the last remaining propositional variable implies that $\mathsf{Prop}(\beta_j(p)) \cup \mathsf{Prop}(\gamma_j(p)) = \{p\}$, and moreover, [9, Lemma 9.5] guarantees that the αs and $\beta_j(p)$s are syntactically closed, and the $\gamma_j(p)$s are syntactically open.
Our goal is to show that the Jónsson-style strategy can be applied to prove the equivalence between items (2) and (3) in the following list:

(1) $\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi$;
(2) $\mathbb{A}^\delta \models_{\mathbb{A}} \&\mathrm{RAF}(\varphi_i \leq \psi_i) \Rightarrow i_0 \leq m_0$ for all $\varphi_i \leq \psi_i$;
(3) $\mathbb{A}^\delta \models \&\mathrm{RAF}(\varphi_i \leq \psi_i) \Rightarrow i_0 \leq m_0$ for all $\varphi_i \leq \psi_i$;
(4) $\mathbb{A}^\delta \models \varphi \leq \psi$.

The equivalence between (3) and (4) follows from the soundness of ALBA, and the equivalence between (1) and (2) similarly follows from the soundness of ALBA when the valuations are admissible[8].
To show that (2) implies (3), we adapt the chain of entailments discussed in the proof of Theorem 3.1. Below, we use the following abbreviations:

$$\begin{array}{lcl} \mathrm{A}(p_i) & = & \alpha_{i,1} \vee \ldots \vee \alpha_{i,m} \leq p_i, \\ \mathrm{BG}(p_i) & = & \beta_{i,1}(p_i) \leq \gamma_{i,1}(p_i) \& \ldots \& \beta_{i,n}(p_i) \leq \gamma_{i,n}(p_i), \\ \mathrm{DT} & = & \delta_{i,1} \leq \theta_{i,1} \& \ldots \& \delta_{i,k} \leq \theta_{i,k}, \\ \mathbf{l}(\alpha, p_i) & = & \mathbf{l}(\alpha_{i,1} \vee \ldots \vee \alpha_{i,m}, p_i), \\ \mathrm{PURE} & = & \mathbf{n}(i_0, m_0) \wedge \bigwedge_{1 \leq j \leq k} \mathbf{l}(\delta_{i,j}, \theta_{i,j}), \\ \gamma(p_i) & = & \bigvee_{1 \leq j \leq n} \mathbf{n}(\beta_{i,j}(p_i), \gamma_{i,j}(p_i)). \end{array}$$

In what follows, the arrays $\vec{x}$ and $\vec{y}$ are taken in $J^\infty(\mathbb{A}^\delta)$ and $M^\infty(\mathbb{A}^\delta)$, respectively.

[8]Notice that the fact that we are using the ALBA manipulations does not mean that this proof is a canonicity-via-correspondence one. More details on this are given in Remark 6.4 and Section 7.

$$
\begin{aligned}
& \mathbb{A}^\delta \models_{\mathbb{A}} \,\&\, \text{RAF } (\varphi_i \leq \psi_i) \Rightarrow i_0 \leq m_0 && \\
\iff\; & \mathbb{A}^\delta \models_{\mathbb{A}} (\text{A}(p_i) \;\&\; \text{ BG } (p_i) \;\&\; \text{ DT}) \Rightarrow i_0 \leq m_0 && \text{(stipulations above)} \\
\iff\; & \mathbb{A}^\delta \models_{\mathbb{A}} \mathbf{l}(\alpha, p_i) \wedge \bigwedge_{1\leq j\leq n} \mathbf{l}(\beta_{i,j}(p_i), \gamma_{i,j}(p_i)) \wedge \bigwedge_{1\leq j\leq k} \mathbf{l}(\delta_{i,j}, \theta_{i,j}) \leq \mathbf{l}(i_0, m_0) && \text{(Proposition 2.19)} \\
\iff\; & \mathbb{A}^\delta \models_{\mathbb{A}} \mathbf{l}(\alpha, p_i) \wedge \mathbf{n}(i_0, m_0) \wedge \bigwedge_{1\leq j\leq k} \mathbf{l}(\delta_{i,j}, \theta_{i,j}) \leq \bigvee_{1\leq j\leq n} \mathbf{n}(\beta_{i,j}(p_i), \gamma_{i,j}(p_i)) && (*) \\
\iff\; & (\mathbf{l}(\alpha, p_i) \wedge \text{ PURE})^{\mathbb{A}}_{\vec{x},\vec{y}} \leq (\gamma(p_i))^{\mathbb{A}}_{\vec{x},\vec{y}} \text{ for all } \vec{x}, \vec{y} && \text{(validity on } \mathbb{A}) \\
\iff\; & ((\mathbf{l}(\alpha, p_i) \wedge \text{ PURE})^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma \leq ((\gamma(p_i))^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma \text{ for all } \vec{x}, \vec{y} && \text{(Def. } \sigma\text{-extension)} \\
\implies\; & ((\mathbf{l}(\alpha, p_i) \wedge \text{ PURE})^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma \leq (\gamma(p_i))^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \text{ for all } \vec{x}, \vec{y} && \text{(Lemma 6.2)} \\
\implies\; & (\mathbf{l}(\alpha, p_i) \wedge \text{ PURE})^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \leq (\gamma(p_i))^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \text{ for all } \vec{x}, \vec{y} && \text{(Lemma 6.3)} \\
\iff\; & \mathbb{A}^\delta \models \mathbf{l}(\alpha, p_i) \wedge \text{ PURE} \leq \gamma(p_i) && \text{(validity on } \mathbb{A}^\delta) \\
\iff\; & \mathbb{A}^\delta \models \,\&\, \text{RAF } (\varphi_i \leq \psi_i) \Rightarrow i_0 \leq m_0 && \text{(Proposition 2.19).}
\end{aligned}
$$

The equivalence marked with $(*)$ can be shown by straightforward algebraic manipulations by using the definitions of $\mathbf{n}$ and $\mathbf{l}$. The claims needed to complete the proof are discussed below. □

Lemma 6.2. *$\gamma(p_i)$ is σ-contracting.*

Proof. Recall that $\gamma(p_i) = \bigvee_{1\leq j\leq n} \mathbf{n}(\beta_{i,j}(p_i), \gamma_{i,j}(p_i))$ where the βs are syntactically closed and positive in p_i, and the γs are syntactically open and negative in p_i. Hence $\gamma(p_i)$ is syntactically closed and positive in p_i. Then the statement follows from Theorem 5.11. □

Lemma 6.3. *$\mathbf{l}(\alpha, p_i) \wedge PURE$ is σ-expanding.*

Proof. By Theorem 5.17, it is enough to show that $\mathbf{l}(\alpha, p_i) \wedge$PURE is ϵ-LPI with $\epsilon(p_i) = 1$. Recall that pure terms are ϵ-LPI for any order-type (cf. Remark 5.15.2). Hence the only branch we need to check is the one ending in p_i. The segment P_2 only consists of the node $+\wedge$, and P_1 only consists of the node $+\mathbf{l}(\alpha, p_i)$, where $\alpha := \alpha_{i,1} \vee \ldots \vee \alpha_{i,m}$ is syntactically closed (cf. [9, Lemma 9.5]) and pure, as required by Definition 5.14. □

Remark 6.4.
- The tool of ALBA has been used in the present section for a different purpose than in [9]. Indeed, there it was used to calculate a first-order correspondent of formulas or inequalities; here, ALBA is used to manipulate the initial inequality into a shape to which the Jónsson strategy can be applied.
- Since ALBA succeeds on every inductive DML-inequality (cf. [9]), the result above accounts for the Jónsson-style canonicity of inductive inequalities.
- Some aspects of the proof might still look to be a bit mysterious. One such aspect is the fact that ALBA runs up to the point immediately before the last application of the Ackermann rule. We will discuss this aspect in Section 7.1.

7. Comparing the two approaches to canonicity

The main motivation for the result above is methodological. Indeed, by adapting Jónsson's argument so as to obtain a new proof of the canonicity of ALBA-inequalities, alternative to that of [9], we aimed at gaining a better understanding of the relationship between these two proof strategies for canonicity. Some ingredients of the two approaches can be now recognized as "two faces of the same coin". In the present section, we collect our findings in this respect.

7.1. **General strategy.**

- As discussed below Lemma 3.3, the so-called **n**-trick is used in the Jónsson-style proof of canonicity of Sahlqvist inequalities to transform Sahlqvist inequalities of the form $\alpha \leq \beta$ into inequalities of the form $\alpha_1 \leq \beta_1 \vee \gamma$, satisfying additional properties as detailed there. The latter shape allows for a neat separation between what we can call the **n**-*part* γ and the *Sahlqvist part* α_1, β_1; it is then shown that the Sahlqvist part α_1 (resp. β_1) is σ-expanding (resp. π-contracting), and the **n**-part γ, which is uniform, is σ-contracting.
- In the ALBA-style canonicity-via-correspondence, all the essentially algebraic and order-topological manipulation steps aim at achieving pure quasi-inequalities. Immediately before the last application of the Ackermann rule, the inequalities in a system can be classified into two types: the first type, which we call the *minimal valuation part*, contains inequalities which are used to compute the minimal valuation; the second type, which we will call *receiving part*, contains the inequalities where the minimal valuation is going to be substituted into.

The most notable difference between the two approaches is that the Jónsson-style transformations preserve the integrity of the initial inequality and proceed from the bottom of the generation tree by taking out subterms and moving them to the γ-part, whereas the ALBA-style approach systematically decomposes the initial inequality into systems of inequalities, and proceeds from the top of the generation tree. However, the two approaches are similar in that they both aim at achieving a state in which the ingredients are neatly separated into two types: in the Jónsson-style approach, the two types are the Sahlqvist part and the **n**-part, and in the ALBA-style approach, they are the minimal valuation part and the receiving part.

The core of the ALBA-aided Jónsson-style canonicity is the recognition that, modulo some manipulations, the minimal valuation part is σ-expanding, and the receiving part is σ-contracting[9]. Namely, the minimal valuation part and the receiving part, generated by ALBA, respectively enjoy the same properties enjoyed by the Sahlqvist part and the **n**-part respectively, and which are crucial in the Jónsson-style canonicity. In the following subsections, we will expand on this. Finally, the ad hoc aspect of the proof, namely the fact that ALBA runs up to the step immediately before the last application of the Ackermann rule, is in fact motivated by the need to highlight this recognition in a simple way; however, this is not in general the only step in which this recognition is possible.

7.2. **The Sahlqvist part and the minimal valuation part.** In this subsection, we expand on the roles of the Sahlqvist part in the Jónsson-style canonicity, and of the minimal valuation part in the ALBA-style canonicity. We first consider the case of Sahlqvist inequalities, then briefly discuss the situation in the case of inductive inequalities. Without loss of generality, we consider left-Sahlqvist and left-inductive terms.

7.2.1. *The Sahlqvist case.* Let us sum up the observations so far, relative to the treatment of Sahlqvist inequalities in the two approaches:

- In the Jónsson-style canonicity, the *compositional* structure of uniform left Sahlqvist terms crucially guarantees that they are σ-expanding. Indeed, uniform left-Sahlqvist

[9] Of course, in order to be able to formally express these facts, we had to introduce the extra connectives **n** and **l**.

terms consist of a composition of additive maps on the outside (i.e. as their Outside Skeleton) (cf. Table 1), the σ-extensions of which are weakly m-Scott continuous ($\forall J \exists J^m_\perp$), and of p-multiplicative maps on the inside (i.e. as their Inner Structure)[10], the σ-extensions of which are SUC ($\forall K \exists K$). This guarantees that the resulting composition is UC ($\forall J \exists K$), which guarantees their being σ-expanding.

- In the ALBA-style canonicity-via-correspondence, the uniform left-Sahlqvist terms are *decomposed* to "display" the critical occurrences of proposition variables, which are the ones to be minimally valued. After the first approximation, we get inequalities e.g. of the form $i \leq \alpha$, where α is of certain restricted left-Sahlqvist form. The algorithm proceeds deconstructing this inequality, and this deconstruction is made possible by the structure of α (without loss of generality we only consider critical occurrences of proposition variables of order type 1).

  Indeed, we recall that the structure of α consists of connectives in the outer skeleton (cf. Table 1) which are interpreted in $\mathbb{A}^\delta$ as maps which are completely additive in each coordinate, and of connectives in the inner structure which are interpreted in $\mathbb{A}^\delta$ as maps which are right adjoints. Hence, the inequality $i \leq \alpha$ is decomposed in two stages:

  (1) The first stage transforms the inequality $i \leq f(\beta_1, \ldots, \beta_n)$ into inequalities $i \leq f(i_1, \ldots, i_n)$, $i_1 \leq \beta_1$, …, $i_n \leq \beta_n$, which keep nominals (interpreted as completely join-irreducibles) in the left-hand side of the inequalities. These transformations are sound thanks to some properties of the outer skeleton connectives in the DML-setting which do not match the properties required by the Jónsson-style argument, namely that outer skeleton connectives are not just additive, but they are either complete operators (and hence completely join-preserving in each coordinate), or if not, they are the right adjoint to the diagonal map Δ. Also all occurrences of the left adjoint of Δ, (that is $\vee$) in the outer skeleton have received a special treatment: namely it has been already eliminated in the preprocessing (cf. Appendix A).

  (2) The second stage transforms inequalities $\beta \leq g(\alpha)$ such that g is completely meet-preserving[11], into inequalities $g'(\beta) \leq \alpha$, where $g' \dashv g$. These transformations are sound thanks to the properties of the inner structure, namely that completely meet-preserving maps are right adjoints. Again, this is another "discrepancy" between the two canonicity strategies, but which is resolved in the setting of regular modal logics [23].

  Finally, after having performed the transformations in the second stage exhaustively (recall that we are only considering critical occurrences of proposition variables of order type 1), all the critical occurrences of p have been dragged out and we are reduced to a set of inequalities of the form $\beta \leq p$, which the minimal valuation part of a set of inequalities in right-Ackermann form consists of.

This account is intended to stress the fact that one and the same syntactic structure, namely that of Sahlqvist terms, is used in two different ways to achieve canonicity. Namely, it is used in a compositional way in the Jónsson strategy, and in a decompositional way for the ALBA strategy. The order-theoretic properties relevant to Jónsson-style canonicity, and guaranteed by the Sahlqvist shape, are: complete additivity for the outer skeleton, and complete p-multiplicativity for the inner part (in fact, all the connectives in the inner part are completely meet-preserving), whereas the order-theoretic properties relevant to ALBA,

[10]Actually, the connectives in the Inner Part of Sahlqvist inequalities have even better properties: indeed, they are meet-preserving.

[11]Note that here g need to be completely meet-preserving, not just completely multiplicative.

and guaranteed by the Sahlqvist shape, are: being a complete operator or a Δ-adjoint for the outer skeleton, and being a right adjoint for the inner part. The fact that the order-theoretic properties required by the canonicity-via-correspondence in the DML-setting seem to be stronger than the corresponding requirements in the Jónsson-style canonicity is in fact not essential. In the setting of regular modal logics [23], the order-theoretic properties required by the two canonicity arguments will perfectly match.

7.2.2. *The inductive case.* In [28], it has been observed that the Jónsson-style canonicity proof cannot be straightforwardly extended to inductive inequalities. In Section 3.5, we have seen that the minimal collapse algorithm, which is based on the plain **n**-trick, is not enough in the setting of inductive inequalities, because it transforms inductive inequalities into inequalities which are not guaranteed to enjoy the required order-theoretic properties for the Jónsson argument. However, from this fact we can only conclude that this kind of manipulation is too rough, since the order-theoretic properties guaranteed by the inductive shape are enough for the order-topological ALBA-style analysis. For instance, by Lemma 9.5 in [9], we know that the ALBA manipulations preserve the syntactic closedness and openness of terms.
Therefore, in order to achieve Jónsson-style canonicity for inductive/ALBA inequalities, we supplemented the **n**-trick with additional syntactic manipulations coming from ALBA. Hence, we have combined the two approaches, and used ALBA to transform the inductive inequalities into some suitable form where the Jónsson-style strategy for canonicity can be applied. Analogously to the original proof in [17], this suitable form achieves a separation between a part corresponding to the minimal valuation part, which in its turn plays the role of what we called the "Sahlqvist part" in [17], and a part corresponding to the receiving part, which in its turn plays the role of what we called the "**n**-part" in [17]. In particular, it is shown in Lemma 6.3 that the term corresponding to the minimal valuation part is LPI. In Remark 5.15, we mention that para-inductive terms can be recognized as inductive terms with minimal valuations substituted into. The fact that para-inductive terms have good enough compositional structure to guarantee that they are σ-expanding or π-contracting makes it possible for the Jónsson-style strategy to go through.

7.3. **The n-part and the receiving part.** The similarity between the **n**-part and the receiving part is even more straightforward:

- In the Jónsson-style canonicity, the **n**-part is σ-contracting. However, in [17], the proof goes through only on the basis of the fact that it is uniform. This is due to the fact that the setting of that proof only treats the original signature. When moving to the expanded signature, the hidden mechanism of σ-contracting and π-expanding terms starts to emerge: namely, their being equivalent to the Esakia conditions, or equivalently, the intersection conditions (cf. Section 4.2).
- In the ALBA-style canonicity, at the stage before the last application of the Ackermann rule, the left-hand side of the receiving part is syntactically closed and uniform in p, its right-hand side is syntactically open and uniform in p. In Lemma 5.10 and the discussion in Section 4.2, we highlighted the connections between these properties and σ-contracting or π-expanding maps. In the ALBA-aided Jónsson-style canonicity, these two aspects, which provide another two-faces-of-same-coin instance, come together. Namely, the receiving part is shown to be σ-contracting.

References

[1] W. Ackermann. Untersuchung uber das Eliminationsproblem der Mathematischen Logic. *Mathematische Annalen*, 110:390-413, 1935.

[2] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2001.
[3] P. Blackburn, J. van Benthem, and F. Wolter, editors. *Handbook of Modal Logic*, volume 3 of *Studies in Logic and Practical Reasoning*. Elsevier Science, Dec. 2006.
[4] W. Conradie and A. Craig. Canonicity Results for mu-Calculi: An Algorithmic Approach. *Submitted*.
[5] W. Conradie, Y. Fomatati, A. Palmigiano, and S. Sourabh. Algorithmic correspondence for intuitionistic modal mu-calculus. *Theoretical Computer Science*, 564:30–62, 2015.
[6] W. Conradie, S. Ghilardi, and A. Palmigiano. Unified Correspondence. In A. Baltag and S. Smets, editors, *Johan van Benthem on Logic and Information Dynamics*, volume 5 of *Outstanding Contributions to Logic*, pages 933–975. Springer International Publishing, 2014.
[7] W. Conradie, V. Goranko, and D. Vakarelov. Algorithmic Correspondence and Completeness in Modal Logic. I. The Core Algorithm SQEMA. *Logical Methods in Computer Science*, 2006.
[8] W. Conradie, V. Goranko, and D. Vakarelov. Algorithmic Correspondence and Completeness in Modal Logic V: Recursive Extensions of SQEMA. *Journal of Applied Logic*, 8(4): 319-333, 2010.
[9] W. Conradie and A. Palmigiano. Algorithmic Correspondence and Canonicity for Distributive Modal Logic. *Annals of Pure and Applied Logic*, 163(3):338 – 376, 2012.
[10] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for non-distributive logics. *Journal of Logic and Computation*, forthcoming.
[11] W. Conradie, A. Palmigiano, and S. Sourabh. Algorithmic Modal Correspondence: Sahlqvist and Beyond. *Submitted*.
[12] W. Conradie and C. Robinson. On Sahlqvist theory for hybrid logic. *Journal of Logic and Computation*, forthcoming.
[13] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2002.
[14] S. Frittella, A. Palmigiano, and L. Santocanale. Dual Characterizations for Finite Lattices via Correspondence Theory for Monotone Modal Logic. *Journal of Logic and Computation*, forthcoming.
[15] M. Gehrke and B. Jónsson. Monotone Bounded Distributive Lattice Expansions. *Mathematica Japonica*, 52(2):197–213, 2000.
[16] M. Gehrke and B. Jónsson. Bounded Distributive Lattice Expansions. *Mathematica Scandinavica*, 94(94):13–45, 2004.
[17] M. Gehrke, H. Nagahashi, and Y. Venema. A Sahlqvist Theorem for Distributive Modal Logic. *Annals of Pure and Applied Logic*, 131(1-3):65–102, 2005.
[18] M. Gehrke and J. Vosmaer. A View of Canonical Extension. In N. Bezhanishvili, S. Löbner, K. Schwabe, and L. Spada, editors, *TbiLLC*, volume 6618 of *Lecture Notes in Computer Science*, pages 77–100. Springer, 2009.
[19] S. Ghilardi and G. Meloni. Constructive Canonicity in Non-Classical Logics. *Annals of Pure and Applied Logic*, 86(1):1–32, 1997.
[20] V. Goranko and D. Vakarelov. Elementary Canonical Formulae: Extending Sahlqvist's Theorem. *Annals of Pure and Applied Logic*, 141(1-2):180–217, 2006.
[21] B. Jónsson. On the Canonicity of Sahlqvist Identities. *Studia Logica*, 53:473–491, 1994.
[22] B. Jónsson and A. Tarski. Boolean Algebras with Operators. *American Journal of Mathematics*, 74:127–162, 1952.
[23] A. Palmigiano, S. Sourabh, and Z. Zhao. Sahlvist Theory for Impossible Worlds. *Journal of Logic and Computation*, forthcoming.
[24] H. Sahlqvist. Completeness and Correspondence in the First and Second Order Semantics for Modal Logic. In S. Kanger, editor, *Studies in Logic and the Foundations of Mathematics*, volume 82, pages 110–143. North-Holland, Amsterdam, 1975.
[25] G. Sambin and V. Vaccaro. A New Proof of Sahlqvist's Theorem on Modal Definability and Completeness. *Journal of Symbolic Logic*, 54(3):992–999, 1989.
[26] T. Suzuki. Canonicity Results of Substructural and Lattice-Based Logics. *The Review of Symbolic Logic*, 4(01):1–42, 2011.
[27] J. van Benthem. *Modal Logic and Classical Logic*. Indices. Monographs in Philosophical Logic and Formal Linguistics. Bibliopolis, 1983.
[28] S. van Gool. Methods for Canonicity. *Master's Thesis, Univesity of Amsterdam, The Netherlands*, 2009.

Appendix A. The algorithm ALBA

In what follows, we illustrate how ALBA works, while at the same time we introduce its rules. The proof of the soundness and invertibility of the general rules for the DML setting

is discussed in [9, 6]. We refer the reader to these papers, and we do not elaborate further on this topic.
ALBA manipulates input $\mathcal{L}$-inequalities $\varphi \leq \psi$ and proceeds in three stages:
First stage: preprocessing and first approximation. ALBA preprocesses the input inequality $\varphi \leq \psi$ by performing the following steps exhaustively in the signed generation trees $+\varphi$ and $-\psi$:
In the generation tree of $+\varphi$ and $-\psi$,

(1) Apply the distribution rules:
   (a) Push down, towards variables, occurrences of $+\Diamond, +\wedge$ and $-\rhd$, by distributing them over nodes labelled with $+\vee$ whenever these are not in the scope of (other) PIA nodes, and
   (b) Push down, towards variables, occurrences of $-\Box, -\vee$ and $+\lhd$, by distributing them over nodes labelled with $-\wedge$ whenever these are not in the scope of (other) PIA nodes.

(2) Apply the splitting rules:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

(3) Apply the monotone and antitone variable-elimination rules:

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\bot) \leq \beta(\bot)} \qquad \frac{\beta(p) \leq \alpha(p)}{\beta(\top) \leq \alpha(\top)}$$

for $\beta(p)$ positive in p and $\alpha(p)$ negative in p.

Let $\mathsf{Preprocess}(\varphi \leq \psi)$ be the finite set $\{\varphi_i \leq \psi_i \mid 1 \leq i \leq n\}$ of inequalities obtained after the exhaustive application of the previous rules. We proceed separately on each of them, and hence, in what follows, we focus only on one element $\varphi_i \leq \psi_i$ in $\mathsf{Preprocess}(\varphi \leq \psi)$, and we drop the subscript. Next, the following *first approximation rule* is applied *only once* to every inequality in $\mathsf{Preprocess}(\varphi \leq \psi)$:

$$\frac{\varphi_i \leq \psi_i}{\mathbf{i}_0 \leq \varphi_i \quad \psi_i \leq \mathbf{m}_0}$$

Here, $\mathbf{i}_0$ and $\mathbf{m}_0$ are a nominal and a co-nominal respectively. The first-approximation step gives rise to systems of inequalities $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$ for each inequality in $\mathsf{Preprocess}(\varphi \leq \psi)$. Each such system is called an *initial system*, and is now passed on to the reduction-elimination cycle.
Second stage: reduction-elimination cycle. The goal of the reduction-elimination cycle is to eliminate all propositional variables from the systems which it receives from the preprocessing phase. The elimination of each variable is effected by an application of one of the Ackermann rules given below. In order to apply an Ackermann rule, the system must have a specific shape. The adjunction, residuation, approximation, and splitting rules are used to transform systems into this shape. The rules of the reduction-elimination cycle, viz. the adjunction, residuation, approximation, splitting, and Ackermann rules, will be collectively called the *reduction* rules.
Residuation rules.

$$\frac{\alpha \wedge \beta \leq \gamma}{\alpha \leq \beta \rightarrow \gamma} \qquad \frac{\alpha \leq \beta \vee \gamma}{\alpha - \beta \leq \gamma} \qquad \frac{\Diamond\alpha \leq \beta}{\alpha \leq \blacksquare\beta} \qquad \frac{\alpha \leq \Box\beta}{\Diamondblack\alpha \leq \beta} \qquad \frac{\lhd\alpha \leq \beta}{\blacktriangleleft \beta \leq \alpha} \qquad \frac{\alpha \leq \rhd\beta}{\beta \leq \blacktriangleright \alpha}$$

Approximation rules.

$$\frac{\mathbf{i} \leq \Diamond\alpha}{\mathbf{j} \leq \alpha \quad \mathbf{i} \leq \Diamond\mathbf{j}} \qquad \frac{\Box\alpha \leq \mathbf{m}}{\alpha \leq \mathbf{n} \quad \Box\mathbf{n} \leq \mathbf{m}} \qquad \frac{\mathbf{i} \leq \lhd\alpha}{\alpha \leq \mathbf{m} \quad \mathbf{i} \leq \lhd\mathbf{m}} \qquad \frac{\rhd\alpha \leq \mathbf{m}}{\mathbf{i} \leq \alpha \quad \rhd\mathbf{i} \leq \mathbf{m}}$$

The nominals and co-nominals introduced by the approximation rules must be *fresh*, i.e. must not already occur in the system before applying the rule.
Ackermann rules. These rules are the core of ALBA, since their application eliminates proposition variables. As mentioned earlier, all the preceding steps are aimed at equivalently rewriting the input system into one or more systems, each of which of a shape in which the Ackermann rules can be applied. An important feature of Ackermann rules is that they are executed on the whole set of inequalities in which a given variable occurs, and not on a single inequality.

$$\frac{\exists p\left[\&_{i=1}^{n}\{\alpha_i \leq p\} \ \& \ \&_{j=1}^{m}\{\beta_j(p) \leq \gamma_j(p)\}\right]}{\&_{j=1}^{m}\{\beta_j(\bigvee_{i=1}^{n}\alpha_i) \leq \gamma_j(\bigvee_{i=1}^{n}\alpha_i)\}}(RAR)$$

where p does not occur in $\alpha_1, \ldots, \alpha_n$, the formulas $\beta_1(p), \ldots, \beta_m(p)$ are positive in p, and $\gamma_1(p), \ldots, \gamma_m(p)$ are negative in p. Here below is the left-Ackermann rule:

$$\frac{\exists p\left[\&_{i=1}^{n}\{p \leq \alpha_i\} \ \& \ \&_{j=1}^{m}\{\beta_j(p) \leq \gamma_j(p)\}\right]}{\&_{j=1}^{m}\{\beta_j(\bigwedge_{i=1}^{n}\alpha_i) \leq \gamma_j(\bigwedge_{i=1}^{n}\alpha_i)\}}(LAR)$$

where p does not occur in $\alpha_1, \ldots, \alpha_n$, the formulas $\beta_1(p), \ldots, \beta_m(p)$ are negative in p, and $\gamma_1(p), \ldots, \gamma_m(p)$ are positive in p.
Third stage: output. If there was some system in the second stage from which not all occurring propositional variables could be eliminated through the application of the reduction rules, then ALBA reports failure and terminates. Else, each system $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$ obtained from $\mathsf{Preprocess}(\varphi \leq \psi)$ has been reduced to a system, denoted $\mathsf{Reduce}(\varphi_i \leq \psi_i)$, containing no propositional variables. Let $\mathsf{ALBA}(\varphi \leq \psi)$ be the set of quasi-inequalities

$$[\& \, \mathsf{Reduce}(\varphi_i \leq \psi_i)] \Rightarrow \mathbf{i}_0 \leq \mathbf{m}_0$$

for each $\varphi_i \leq \psi_i \in \mathsf{Preprocess}(\varphi \leq \psi)$.
Notice that all members of $\mathsf{ALBA}(\varphi \leq \psi)$ are free of propositional variables. Hence, translating them as discussed in [9, Section 2.5] produces sentences in the first-order correspondence language of DML-Kripke frames. ALBA returns $\mathsf{ALBA}(\varphi \leq \psi)$ and terminates. An inequality $\varphi \leq \psi$ on which ALBA succeeds will be called an ALBA-*inequality*.

Appendix B. Algebraic Ackermann lemmas

In the present section we report on the point-free, purely algebraic proofs of the Ackermann lemmas, both the basic version and the topological version. These proofs are very similar to [10].

Lemma B.1 (Right-handed Ackermann lemma). *Let α be such that $p \notin \mathsf{Prop}(\alpha)$, $\beta_1(p), \ldots, \beta_n(p)$ be positive in p and $\gamma_1(p), \ldots, \gamma_n(p)$ negative in p, and let $\vec{q}, \vec{\mathbf{j}}, \vec{\mathbf{m}}$ be all the proposition variables, nominals, co-nominals, respectively, occurring in $\alpha, \beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p)$ other than p. Then for all $\vec{a} \in \mathbb{A}^\delta, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, the following are equivalent:*

(1) $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}))$ *for* $1 \leq i \leq n$*;*
(2) *There exists $a_0 \in \mathbb{A}^\delta$ such that $\alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}) \leq a_0$ and $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0)$ for $1 \leq i \leq n$.*

Lemma B.2 (Left-handed Ackermann lemma)**.** *Let α be such that $p \notin \mathsf{Prop}(\alpha)$, $\beta_1(p), \ldots, \beta_n(p)$ be negative in p and $\gamma_1(p), \ldots, \gamma_n(p)$ positive in p, and let $\vec{q}, \vec{\mathbf{j}}, \vec{\mathbf{m}}$ be all the proposition variables, nominals, co-nominals, respectively, occurring in $\alpha, \beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p)$ other than p. Then for all $\vec{a} \in \mathbb{A}^\delta, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, the following are equivalent:*

(1) $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}))$ *for* $1 \leq i \leq n$*;*

(2) *There exists* $a_0 \in \mathbb{A}^\delta$ *such that* $a_0 \leq \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})$ *and* $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0)$ *for* $1 \leq i \leq n$*.*

Proof. The two lemmas are order variants of each other, so we only prove the right-handed version here. The direction from top to bottom is shown by taking a_0 to be $\alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})$. The converse direction easily follows from the monotonicity of $\beta_i^{\mathbb{A}^\delta}(p)$ and the antitonicity of $\gamma_i^{\mathbb{A}^\delta}(p)$ in p. $\square$

However, the proof above cannot be straightforwardly adapted to the case in which the assignments are *admissible*, i.e. if we stipulate that the assignments map all proposition variables and a_0 into $\mathbb{A}$. Indeed, instantiating $a_0 := \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})$ might not result in an admissible assignment, since α is a term in the expanded language, therefore it might contain nominals, co-nominals and expanded operators under which the subalgebra $\mathbb{A}$ is in general not closed.
Therefore we prove a new version of the Ackermann lemmas (Lemma B.4 and Lemma B.5), which will depend on the syntactic shape of the formulas such that compactness can be applied so as to find a suitable admissible element $a_0 \in \mathbb{A}$.
In order to prove the new topological version of Ackermann lemmas, we first introduce the following definitions:

Definition B.3 (Syntactically closed and open formulas)**.** (1) A formula in $\mathcal{L}^+$ is *syntactically closed* if all occurrences of nominals, $\blacktriangleleft, \blacklozenge, -$ are positive, and all occurrences of co-nominals, $\blacktriangleright, \blacksquare, \rightarrow$ are negative;

(2) A formula in $\mathcal{L}^+$ is *syntactically open* if all occurrences of co-nominals, $\blacktriangleleft, \blacklozenge, -$ are negative, and all occurrences of nominals, $\blacktriangleright, \blacksquare, \rightarrow$ are positive.

The intuition behind these definitions is that the value of a syntactically open (resp. closed) formula under an admissible assignment is always an open (resp. closed) element in $\mathbb{A}^\delta$, i.e., in $O(\mathbb{A}^\delta)$(resp. $K(\mathbb{A}^\delta)$), therefore we can apply compactness to obtain an admissible a_0. In the section on canonical extension of the expanded language, the syntactic closedness and openness will play an important role in proving the Esakia's lemmas.

Lemma B.4 (Right-handed topological Ackermann lemma)**.** *Let α be syntactically closed, $p \notin \mathsf{Prop}(\alpha)$, let $\beta_1(p), \ldots, \beta_n(p)$ be syntactically closed and positive in p, let $\gamma_1(p), \ldots, \gamma_n(p)$ be syntactically open and negative in p, and let $\vec{q}, \vec{\mathbf{j}}, \vec{\mathbf{m}}$ be all the proposition variables, nominals, co-nominals, respectively, occurring in $\alpha, \beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p)$ other than p. Then for all $\vec{a} \in \mathbb{A}, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, the following are equivalent:*

(1) $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}))$ *for* $1 \leq i \leq n$*;*

(2) *There exists* $a_0 \in \mathbb{A}$ *such that* $\alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}) \leq a_0$ *and* $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0)$ *for* $1 \leq i \leq n$*.*

Lemma B.5 (Left-handed topological Ackermann lemma)**.** *Let α be syntactically open, $p \notin \mathsf{Prop}(\alpha)$, let $\beta_1(p), \ldots, \beta_n(p)$ be syntactically closed and negative in p, let $\gamma_1(p), \ldots, \gamma_n(p)$ be syntactically open and positive in p, and let $\vec{q}, \vec{\mathbf{j}}, \vec{\mathbf{m}}$ be all the proposition variables, nominals,* *co-nominals, respectively, occurring in $\alpha, \beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p)$ other than p. Then the following are equivalent for all $\vec{a} \in \mathbb{A}, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$:*

(1) $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}))$ *for* $1 \leq i \leq n$;

(2) *There exists* $a_0 \in \mathbb{A}$ *such that* $a_0 \leq \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})$ *and* $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0)$ *for* $1 \leq i \leq n$.

The proof of the topological Ackermann lemmas relies on some properties of the modalities in the expanded language, which will be proved in what follows. We Notice that the dependence order of the following lemmas is different from the dependence order of analogous lemmas in the frame-theoretic setting of [9].

The following statement immediately follows from the definition of the closed and open elements of $\mathbb{A}^\delta$, as well as the fact that $\Box^{\mathbb{A}^\delta} = (\Box^{\mathbb{A}})^\sigma, \Diamond^{\mathbb{A}^\delta} = (\Diamond^{\mathbb{A}})^\pi, \rhd^{\mathbb{A}^\delta} = (\rhd^{\mathbb{A}})^\sigma, \lhd^{\mathbb{A}^\delta} = (\lhd^{\mathbb{A}})^\pi$.

Lemma B.6. *For all $c \in K(\mathbb{A}^\delta)$ and $o \in O(\mathbb{A}^\delta)$,*

(1) $\Box c \in K(\mathbb{A}^\delta)$;
(2) $\Diamond o \in O(\mathbb{A}^\delta)$;
(3) $\rhd o \in K(\mathbb{A}^\delta)$;
(4) $\lhd c \in O(\mathbb{A}^\delta)$.

Lemma B.7. *For all $c \in K(\mathbb{A}^\delta)$ and $o \in O(\mathbb{A}^\delta)$,*

(1) $\blacklozenge c \in K(\mathbb{A}^\delta)$;
(2) $\blacksquare o \in O(\mathbb{A}^\delta)$;
(3) $\blacktriangleleft o \in K(\mathbb{A}^\delta)$;
(4) $\blacktriangleright c \in O(\mathbb{A}^\delta)$;
(5) $c - o \in K(\mathbb{A}^\delta)$;
(6) $c \to o \in O(\mathbb{A}^\delta)$.

Proof. 1. By denseness, $\blacklozenge c = \bigwedge\{o \in O(\mathbb{A}^\delta) : \blacklozenge c \leq o\}$. Let $Y = \{o \in O(\mathbb{A}^\delta) : \blacklozenge c \leq o\}$ and $X = \{a \in \mathbb{A} : \blacklozenge c \leq a\}$. To show that $\blacklozenge c \in K(\mathbb{A}^\delta)$, it is enough to show that $\bigwedge X = \bigwedge Y$. Since clopens are opens, $X \subseteq Y$, so $\bigwedge Y \leq \bigwedge X$. In order to show that $\bigwedge X \leq \bigwedge Y$, it suffices to show that for every $o \in Y$ there exists some $a \in X$ such that $a \leq o$. Let $o \in Y$, i.e., $\blacklozenge c \leq o$. By adjunction, $c \leq \Box o$. Since $c \in K(\mathbb{A}^\delta)$, and $\Box o = \Box^\pi o = \bigvee\{\Box a : a \in \mathbb{A}$ and $a \leq o\}$, and $\Box a \in \mathbb{A} \subseteq O(\mathbb{A}^\delta)$, we may apply compactness and get that $c \leq \Box a_1 \vee \cdots \vee \Box a_n$ for some $a_1, \ldots, a_n \in \mathbb{A}$ s.t. $a_1, \ldots, a_n \leq o$. Let $a = a_1 \vee \cdots \vee a_n \leq o$. By the monotonicity of $\Box$ we have $c \leq \Box a_1 \vee \cdots \vee \Box a_n \leq \Box a$, and hence $\blacklozenge c \leq a$. 2. 3. and 4. are order-variants of 1.

5. By denseness, $c - o = \bigwedge\{o' \in O(\mathbb{A}^\delta) : c - o \leq o'\}$. Let $Y = \{o' \in O(\mathbb{A}^\delta) : c - o \leq o'\}$ and $X = \{a \in \mathbb{A} : c - o \leq a\}$. To show that $c - o \in K(\mathbb{A}^\delta)$, it is enough to show that $\bigwedge X = \bigwedge Y$. Since clopens are opens, $X \subseteq Y$, so $\bigwedge Y \leq \bigwedge X$. In order to show that $\bigwedge X \leq \bigwedge Y$, it suffices to show that for every $o' \in Y$ there exists some $a \in X$ such that $a \leq o'$. If $o' \in Y$, then $c - o \leq o'$. Since $c \in K(\mathbb{A}^\delta)$ and $o, o' \in O(\mathbb{A}^\delta)$, it follows that $c = \bigwedge \mathcal{A}_1, o = \bigvee \mathcal{A}_2$ and $o' = \bigvee \mathcal{A}_3$ for some $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \subseteq \mathbb{A}$. Hence, by adjunction, $\bigwedge \mathcal{A}_1 \leq \bigvee \mathcal{A}_2 \vee \bigvee \mathcal{A}_3$, and so by compactness, there exists some finite $\mathcal{A}_3' \subseteq \mathcal{A}_3$ such that $\bigwedge \mathcal{A}_1 \leq \bigvee \mathcal{A}_2 \vee \bigvee \mathcal{A}_3'$. By adjunction again we have $\bigwedge \mathcal{A}_1 - \bigvee \mathcal{A}_2 \leq \bigvee \mathcal{A}_3'$, i.e., $c - o \leq \bigvee \mathcal{A}_3'$. Take $a = \bigvee \mathcal{A}_3'$. Then $a \in \mathbb{A}, c - o \leq a$, so $a \in X$ and also $a = \bigvee \mathcal{A}_3' \leq \bigvee \mathcal{A}_3 = o'$. 6. is an order-variant of 5. $\Box$

Lemma B.8. *Let $\mathcal{C} = \{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$ be downward-directed, $\mathcal{O} = \{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$ be upward-directed, then*

(1) $\Box \bigvee \mathcal{O} = \bigvee\{\Box o : o \in \mathcal{O}\}$;
(2) $\Diamond \bigwedge \mathcal{C} = \bigwedge\{\Diamond c : c \in \mathcal{C}\}$;

(3) $\rhd \bigwedge \mathcal{C} = \bigvee\{\rhd c : c \in \mathcal{C}\}$;
(4) $\lhd \bigvee \mathcal{O} = \bigwedge\{\lhd o : o \in \mathcal{O}\}$.

Proof. 1. By denseness, $\Box \bigvee \mathcal{O} = \bigvee\{c \in K(\mathbb{A}^\delta) : c \leq \Box \bigvee \mathcal{O}\}$. Let $Y = \{c \in K(\mathbb{A}^\delta) : c \leq \Box \bigvee \mathcal{O}\}$ and $X = \{\Box o : o \in \mathcal{O}\}$. Then it suffices to show that $\bigvee Y = \bigvee X$. That $\bigvee X \leq \Box \bigvee \mathcal{O} = \bigvee Y$ immediately follows from the monotonicity of $\Box$. For the converse direction, it suffices to show that for every $c \in Y$ there exists some $o \in \mathcal{O}$ such that $c \leq \Box o$. Fix $c \in Y$, i.e., assume $c \leq \Box \bigvee \mathcal{O}$. By adjunction, this implies $\blacklozenge c \leq \bigvee \mathcal{O}$. Since $\blacklozenge c \in K(\mathbb{A}^\delta)$ (cf. Lemma B.7) , by compactness there exists some finite $\mathcal{O}_0 \subseteq \mathcal{O}$ such that $\blacklozenge c \leq \bigvee \mathcal{O}_0$. By the upward-directedness of $\mathcal{O}$, there exists some $o \in \mathcal{O}$ such that $\bigvee \mathcal{O}_0 \leq o$. Therefore $\blacklozenge c \leq o$, and again by adjunction we have $c \leq \Box o$. Items 2. 3. and 4. are order-variants of 1. □

The following corollary is an immediate consequence of the previous lemma, given that any singleton is both upward-directed and downward-directed:

Corollary B.9. *For all $c \in K(\mathbb{A}^\delta)$ and $o \in O(\mathbb{A}^\delta)$,*
(1) $\Box o \in O(\mathbb{A}^\delta)$;
(2) $\Diamond c \in K(\mathbb{A}^\delta)$;
(3) $\rhd c \in O(\mathbb{A}^\delta)$;
(4) $\lhd o \in K(\mathbb{A}^\delta)$.

Notice that in the proof of Lemma B.8, the fact that the operations $\Box^{\mathbb{A}^\delta}, \Diamond^{\mathbb{A}^\delta}, \rhd^{\mathbb{A}^\delta}, \lhd^{\mathbb{A}^\delta}$ are σ- or π-extensions of the corresponding operations in $\mathbb{A}$ plays no role, hence the same proof applies also to $\blacksquare, \blacklozenge, \blacktriangleright, \blacktriangleleft$, respectively, and yields the following lemma:

Lemma B.10. *Let $\mathcal{C} = \{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$ be downward-directed, $\mathcal{O} = \{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$ be upward-directed, then*
(1) $\blacksquare \bigvee \mathcal{O} = \bigvee\{\blacksquare o : o \in \mathcal{O}\}$;
(2) $\blacklozenge \bigwedge \mathcal{C} = \bigwedge\{\blacklozenge c : c \in \mathcal{C}\}$;
(3) $\blacktriangleright \bigwedge \mathcal{C} = \bigvee\{\blacktriangleright c : c \in \mathcal{C}\}$;
(4) $\blacktriangleleft \bigvee \mathcal{O} = \bigwedge\{\blacktriangleleft o : o \in \mathcal{O}\}$;
(5) $\bigwedge \mathcal{C} - \bigvee \mathcal{O} = \bigwedge\{c - o : c \in \mathcal{C}, o \in \mathcal{O}\}$;
(6) $\bigwedge \mathcal{C} \to \bigvee \mathcal{O} = \bigvee\{c \to o : c \in \mathcal{C}, o \in \mathcal{O}\}$.

Proof. 5. By denseness, $\bigwedge \mathcal{C} - \bigvee \mathcal{O} = \bigwedge\{o' \in O(\mathbb{A}^\delta) : \bigwedge \mathcal{C} - \bigvee \mathcal{O} \leq o'\}$. Let $Y = \{o' \in O(\mathbb{A}^\delta) : \bigwedge \mathcal{C} - \bigvee \mathcal{O} \leq o'\}$ and $X = \{c - o : c \in \mathcal{C}, o \in \mathcal{O}\}$. To show that $\bigwedge \mathcal{C} - \bigvee \mathcal{O} = \bigwedge\{c - o : c \in \mathcal{C}, o \in \mathcal{O}\}$, it is enough to show that $\bigwedge X = \bigwedge Y$. Since $-$ is monotone in the first coordinate and antitone in the second coordinate, we have $\bigwedge Y \leq c - o$ for all $c - o \in X$, so $\bigwedge Y \leq \bigwedge X$. In order to show that $\bigwedge X \leq \bigwedge Y$, it suffices to show that for every $o' \in Y$ there exists some $c - o \in X$ such that $c - o \leq o'$. For any $o' \in Y$, it holds that $\bigwedge \mathcal{C} - \bigvee \mathcal{O} \leq o'$, hence by adjunction, $\bigwedge \mathcal{C} \leq \bigvee \mathcal{O} \vee o'$. Since $\bigwedge \mathcal{C} \in K(\mathbb{A}^\delta)$ and $\bigvee \mathcal{O}, o' \in O(\mathbb{A}^\delta)$, by compactness there exist some finite $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{O}' \subseteq \mathcal{O}$ such that $\bigwedge \mathcal{C}' \leq \bigvee \mathcal{O}' \vee o'$. By the downward-directedness of $\mathcal{C}$ and upward-directedness of $\mathcal{O}$, there exist some $c \in \mathcal{C}$ and $o \in \mathcal{O}$ such that $c \leq \bigwedge \mathcal{C}'$ and $\bigvee \mathcal{O}' \leq o$, so $c \leq \bigwedge \mathcal{C}' \leq \bigvee \mathcal{O}' \vee o' \leq o \vee o'$, so by adjunction again we have $c - o \leq o'$, and also $c - o \in X$. 6. is order-variant of 5. □

Lemma B.11. *Let $\varphi(p)$ be syntactically closed and $\psi(p)$ syntactically open, and let $\vec{p}, \vec{\mathbf{j}}, \vec{\mathbf{m}}$ be all the proposition variables, nominals, co-nominals, respectively, occuring in $\varphi(p)$ and $\psi(p)$ other than p, $\vec{a} \in \mathbb{A}, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, $c \in K(\mathbb{A}^\delta), o \in O(\mathbb{A}^\delta)$, then*
(1) *If $\varphi(p)$ is positive in p, then $\varphi^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, c) \in K(\mathbb{A}^\delta)$;*
(2) *If $\psi(p)$ is negative in p, then $\psi^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, c) \in O(\mathbb{A}^\delta)$;*

(3) *If* $\varphi(p)$ *is negative in* p, *then* $\varphi^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},o) \in K(\mathbb{A}^\delta)$;
(4) *If* $\psi(p)$ *is positive in* p, *then* $\psi^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},o) \in O(\mathbb{A}^\delta)$.

Proof. By simultaneous induction on φ and ψ, using Lemma B.6, B.7 and B.9. □

Lemma B.12. *Let* $\varphi(p)$ *be syntactically closed,* $\psi(p)$ *be syntactically open, and let* $\vec{p},\vec{\mathbf{j}},\vec{\mathbf{m}}$ *be all the proposition variables, nominals, co-nominals, respectively, occuring in* $\varphi(p)$ *and* $\psi(p)$ *other than* p, $\vec{a} \in \mathbb{A}, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$ *be downward-directed,* $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$ *be upward-directed, then*

(1) *If* $\varphi(p)$ *is positive in* p, *then* $\varphi^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},c_i) : i \in I\}$;
(2) *If* $\psi(p)$ *is negative in* p, *then* $\psi^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},\bigwedge\{c_i : i \in I\}) = \bigvee\{\psi^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},c_i) : i \in I\}$;
(3) *If* $\varphi(p)$ *is negative in* p, *then* $\varphi^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},\bigvee\{o_i : i \in I\}) = \bigwedge\{\varphi^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},o_i) : i \in I\}$;
(4) *If* $\psi(p)$ *is positive in* p, *then* $\psi^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},\bigvee\{o_i : i \in I\}) = \bigvee\{\psi^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},o_i) : i \in I\}$.

Proof. By simultaneous induction on φ and ψ. It is easy to see that φ cannot be $\mathbf{m}$ and the outmost connective of φ cannot be $\blacksquare, \blacktriangleright, \rightarrow$, and similarly, ψ cannot be $\mathbf{i}$ and the outmost connective of ψ cannot be $\blacklozenge, \blacktriangleleft, -$. In the proof we omit $\vec{a},\vec{x},\vec{y}$ and the superscript $\mathbb{A}^\delta$, and just write $\varphi(a)$ and $\psi(a)$ where p is assigned a.

(1) The basic cases in which $\varphi = \bot, \top, p, q, \mathbf{i}$ and $\psi = \bot, \top, p, q, \mathbf{m}$ are straightforward.
(2) As to the case in which $\varphi(p) = \varphi_1(p) \wedge \varphi_2(p)$, if $\varphi(p)$ is positive in p, then $\varphi_1(p)$ and $\varphi_2(p)$ are syntactically closed and positive in p. Hence, by induction hypothesis, $\varphi_1(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\}$ and $\varphi_2(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_2(c_i) : i \in I\}$, so $\varphi(\bigwedge\{c_i : i \in I\}) = \varphi_1(\bigwedge\{c_i : i \in I\}) \wedge \varphi_2(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\} \wedge \bigwedge\{\varphi_2(c_i) : i \in I\} = \bigwedge\{\varphi_1(c_i) \wedge \varphi_2(c_i) : i \in I\} = \bigwedge\{\varphi(c_i) : i \in I\}$. The case in which $\varphi(p)$ is negative in p is treated similarly.
    The case in which $\psi(p) = \psi_1(p) \vee \psi_2(p)$ is similar to the previous case.
(3) As to the case in which $\varphi(p) = \varphi_1(p) \vee \varphi_2(p)$, if $\varphi(p)$ is positive in p, then $\varphi_1(p)$ and $\varphi_2(p)$ are syntactically closed and positive in p. Hence, by induction hypothesis, $\varphi_1(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\}$ and $\varphi_2(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_2(c_i) : i \in I\}$, so
$$\begin{array}{rcl}\varphi(\bigwedge\{c_i : i \in I\}) & = & \varphi_1(\bigwedge\{c_i : i \in I\}) \vee \varphi_2(\bigwedge\{c_i : i \in I\}) \\ & = & \bigwedge\{\varphi_1(c_i) : i \in I\} \vee \bigwedge\{\varphi_2(c_i) : i \in I\} \\ & = & \bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_{i'}) : i, i' \in I\}.\end{array}$$
Since $\bigwedge\{\varphi(c_i) : i \in I\} = \bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_i) : i \in I\}$, it suffices to show that
$$\bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_{i'}) : i, i' \in I\} = \bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_i) : i \in I\}.$$
The left-to-right inequality is easy. As to the converse one, it suffices to show that for all $i, i' \in I$ there exists some $i'' \in I$ such that $\varphi_1(c_{i''}) \vee \varphi_2(c_{i''}) \leq \varphi_1(c_i) \vee \varphi_2(c_{i'})$. Fix $i, i' \in I$. By the downward-directedness of $\{c_i : i \in I\}$ there exists some $i'' \in I$ such that $c_{i''} \leq c_i$ and $c_{i''} \leq c_{i'}$. By the monotonicity of $\varphi_1(p)$ and $\varphi_2(p)$ this implies that $\varphi_1(c_{i''}) \leq \varphi_1(c_i)$ and $\varphi_2(c_{i''}) \leq \varphi_2(c_{i'})$, and hence $\varphi_1(c_{i''}) \vee \varphi_2(c_{i''}) \leq \varphi_1(c_i) \vee \varphi_2(c_{i'})$. The case in which $\varphi(p)$ is negative in p is treated similarly. The case in which $\psi(p) = \psi_1(p) \wedge \psi_2(p)$ is similar to $\varphi(p) = \varphi_1(p) \vee \varphi_2(p)$ case.
(4) As to the case in which $\varphi(p) = \Box\varphi_1(p)$, if $\varphi(p)$ is positive in p, then $\varphi_1(p)$ is syntactically closed and positive in p. Hence, by induction hypothesis, $\varphi_1(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\}$. Therefore by the complete distributivity of $\Box$ over meets, $\varphi(\bigwedge\{c_i : i \in I\}) = \Box\varphi_1(\bigwedge\{c_i : i \in I\}) = \Box\bigwedge\{\varphi_1(c_i) : i \in I\} = \bigwedge\{\Box\varphi_1(c_i) : i \in I\} = \bigwedge\{\varphi(c_i) : i \in I\}$. The case in which $\varphi(p)$ is negative in p is treated similarly.
    The cases in which $\varphi(p) = \rhd\varphi_1(p)$, $\psi(p) = \Diamond\psi_1(p), \varphi(p) = \lhd\psi_1(p)$ are similar to the previous case.

(5) As to the case in which $\varphi(p) = \Diamond\varphi_1(p)$, if $\varphi(p)$ is positive in p, then $\varphi_1(p)$ is syntactically closed and positive in p. Hence, by induction hypothesis, $\varphi_1(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\}$. Let $\mathcal{C} = \{\varphi_1(c_i) : i \in I\}$. Lemma B.11 implies that $\mathcal{C} \subseteq K(\mathbb{A}^\delta)$. Let us show that $\mathcal{C}$ is downward-directed. Fix $\varphi_1(c_i), \varphi_1(c_{i'}) \in \mathcal{C}$. By the downward-directedness of $\{c_i : i \in I\}$, there exists some $i'' \in I$ such that $c_{i''} \leq c_i \wedge c_{i'}$, and since $\varphi_1(p)$ is positive in p, this implies that $\varphi_1(c_{i''}) \leq \varphi_1(c_i \wedge c_{i'}) \leq \varphi_1(c_i)$ and similarly $\varphi_1(c_{i''}) \leq \varphi_1(c_{i'})$, which completes the proof of the claim above. Hence, Lemma B.8 implies that $\varphi(\bigwedge\{c_i : i \in I\}) = \Diamond\varphi_1(\bigwedge\{c_i : i \in I\}) = \Diamond\bigwedge\{\varphi_1(c_i) : i \in I\} = \bigwedge\{\Diamond\varphi_1(c_i) : i \in I\} = \bigwedge\{\varphi(c_i) : i \in I\}$. The case in which $\varphi(p)$ is negative in p is treated similarly.

    The cases in which $\varphi(p) = \lhd\varphi_1(p), \varphi(p) = \blacklozenge\varphi_1(p), \varphi(p) = \blacktriangleleft \varphi_1(p)$, $\psi(p) = \Box\psi_1(p), \psi(p) = \rhd\psi_1(p), \psi(p) = \blacksquare\psi_1(p), \psi(p) = \blacktriangleright \psi_1(p)$ are similar to the previous case.

(6) As to the case in which $\varphi(p) = \varphi_1(p) - \varphi_2(p)$, if $\varphi(p)$ is positive in p, then $\varphi_1(p)$ is syntactically closed and positive in p, and $\varphi_2(p)$ is open and negative in p. By induction hypothesis, $\varphi_1(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\}$ and $\varphi_2(\bigwedge\{c_i : i \in I\}) = \bigvee\{\varphi_2(c_i) : i \in I\}$. Therefore, $\varphi(\bigwedge\{c_i : i \in I\}) = \varphi_1(\bigwedge\{c_i : i \in I\}) - \varphi_2(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\} - \bigvee\{\varphi_2(c_i) : i \in I\}$, and $\bigwedge\{\varphi(c_i) : i \in I\} = \bigwedge\{\varphi_1(c_i) - \varphi_2(c_i) : i \in I\}$. An argument similar to the one given in the previous item shows that $\mathcal{C} = \{\varphi_1(c_i) : i \in I\}$ is downward-directed and $\mathcal{O} = \{\varphi_2(c_i) : i \in I\}$ is upward-directed. Hence, Lemma B.10 implies that $\varphi(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\} - \bigvee\{\varphi_2(c_i) : i \in I\} = \bigwedge\{\varphi_1(c_i) - \varphi_2(c_{i'}) : i, i' \in I\}$. Hence, to show that $\varphi(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi(c_i) : i \in I\}$, it suffices to show that $\bigwedge\{\varphi_1(c_i) - \varphi_2(c_{i'}) : i, i' \in I\} = \bigwedge\{\varphi_1(c_i) - \varphi_2(c_i) : i \in I\}$. This proof is similar to the one for $\varphi(p) = \varphi_1(p) \vee \varphi_2(p)$. The case in which $\varphi(p)$ is negative in p is treated similarly.

    The case in which $\psi(p) = \psi_1(p) \to \psi_2(p)$ is similar to the previous case.

□

Corollary B.13. *Let φ be syntactically closed and ψ syntactically open, and let $\vec{p}, \vec{\mathbf{j}}, \vec{\mathbf{m}}$ be all the proposition variables, nominals, co-nominals, respectively, occurring in φ and ψ, $\vec{a} \in \mathbb{A}, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$. Then*

(1) $\varphi^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}) \in K(\mathbb{A}^\delta)$;
(2) $\psi^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}) \in O(\mathbb{A}^\delta)$.

Proof of Topological Ackermann Lemma. We prove the right-hand topological Ackermann Lemma, the left-hand topological Ackermann Lemma being similar.

$\Leftarrow$: By the monotonicity of $\beta_i(p)$ and the antitonicity of $\gamma_i(p)$ in p for $1 \leq i \leq n$, together with $\alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}) \leq a_0$ we have that $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})) \leq \beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}))$.

$\Rightarrow$: Suppose that $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}))$ for $1 \leq i \leq n$. Then, by Corollary B.13, $\alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}) \in K(\mathbb{A}^\delta)$, that is, $\alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}) = \bigwedge\mathcal{U}$, where $\mathcal{U} = \{u \in \mathbb{A} : \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}) \leq u\}$, making it the meet of a downward-directed set of clopen elements. Therefore, $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \bigwedge\mathcal{U}) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \bigwedge\mathcal{U})$ for all $1 \leq i \leq n$. Since β_i is syntactically closed and positive in p and γ_i is syntactically open and negative in p, Lemma B.12 implies that $\bigwedge\{\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, u) : u \in \mathcal{U}\} \leq \bigvee\{\gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, u) : u \in \mathcal{U}\}$ for $1 \leq i \leq n$. Again by Corollary B.13, $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, u) \in K(\mathbb{A}^\delta)$ and $\gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, u) \in O(\mathbb{A}^\delta)$ for $1 \leq i \leq n$, therefore by compactness, there is a finite set $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\bigwedge\{\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, u) : u \in \mathcal{U}_0\} \leq \bigvee\{\gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, u) : u \in \mathcal{U}_0\}$ for $1 \leq i \leq n$. Letting $a_0 = \bigwedge\mathcal{U}_0 \in \mathbb{A}$, we have that

$\alpha^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y}) = \bigwedge\mathcal{U} \leq \bigwedge\mathcal{U}_0 = a_0$, and by the monotonicity of $\beta_i(p)$ and the antitonicity of $\gamma_i(p)$ in p for $1 \leq i \leq n$, it follows that $\beta_i^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},a_0) \leq \beta_i^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},u)$ and $\gamma_i^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},u) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},a_0)$ for all $u \in \mathcal{U}_0$ and $1 \leq i \leq n$. Therefore, $\beta_i^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},a_0) \leq \bigwedge\{\beta_i^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},u) : u \in \mathcal{U}_0\} \leq \bigvee\{\gamma_i^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},u) : u \in \mathcal{U}_0\} \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a},\vec{x},\vec{y},a_0)$ for $1 \leq i \leq n$. $\square$