Nothing but the Truth

August 30, 2011

Abstract

A curious feature of Belnap's "useful four-valued logic", also known as first-degree entailment (FDE), is that the overdetermined value B (both true and false) is treated as a designated value. Although there are good theoretical reasons for this, it seems prima facie more plausible to have only one of the four values designated, namely T (exactly true). This paper follows this route and investigates the resulting logic, which we call Exactly True Logic.

Keywords: Relevant logic, first degree entailment, Belnap's four-valued logic, De Morgan lattices, in- and overcomplete data

1 Belnap's useful four-valued logic

Nuel Belnap introduced the logic of first-degree entailment, FDE, in two influential papers, both including important contributions by Belnap's student and collaborator J. Michael Dunn.

One of the essays is entitled "A useful four-valued logic" [3], the other "How a computer should think" [2]. The two papers have been merged into chapter 81 of [1], which is the most easily accessible source[1].

There are two main concerns driving this logic. The first is to give a basis for a notion of relevant entailment. The other, as is obvious from the title of the second paper, is to give computers some rules for processing information.

Relevant logicians are after a notion of entailment that does not allow for inferences like *excluded middle* ($B \vDash A \vee \neg A$) and *ex contradictione quodlibet* ($A \wedge \neg A \vDash B$).

The problem with these inferences is that the premises have nothing at all to do with the conclusion; they are, to use the terminology that gave relevant logic its name, *irrelevant* to the truth or falsity of the conclusion.

[1] A scan of the chapter is available for download on Belnap's homepage: http://www.pitt.edu/~belnap/papers.html

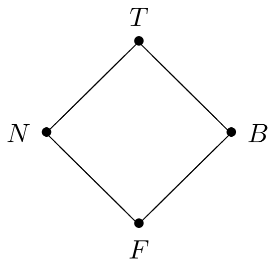


Figure 1: The lattice **4**

Classical logicians endorse *ex contradictione quodlibet* because entailment is defined in terms of truth preservation in all models. Since contradicitions have no classical models, the condition for the validity of $A \wedge \neg A \models B$ is vacuously satisfied. A straightforward way to get rid of this inference is therefore to provide models for $A \wedge \neg A$. This can be done by adding to the two classical truth values a third one, B, and treat it as an additional designated value (a designated value is one that the consequence relation is supposed to preserve from premisses to conclusion). A fourth value, N, will serve to get rid of the *excluded middle* type of inferences. This is achieved by treating this value as undesignated and thus obtaining models in which $A \vee \neg A$ fails to receive a designated value.

These four values can be arranged nicely in the lattice shown in Figure 1. We will view conjunction as the meet, disjunction as the join and negation as an operator that flips T and F but has B and N as fixed points.

As we said, the consequence relation should transmit the values T and B. That is, an inference is valid iff under every valuation, if the premisses receive a designated value, then the conclusion will receive a designated value as well.

It is not hard to see that there are no valid inferences in which there isn't at least one propositional parameter that occurs in both the premisses and the conclusion. This is quite satisfying, as "parameter sharing" is one of the principal ways in which the vague notion of relevance has been cashed out.

However, for this to be more than a mathematical trick, it would be nice to be given an informal interpretation of what those four values are supposed to *mean*. This is the point where the second motivation for Belnap's logic comes into play. Belnap thinks of the values as recording what information a computer has received about different statements. The computer is given input in the form of statements that are labeled as true or false. As many people are supposed to be building up the database of the computer, it is not impossible that one person might enter a statement as true, while another enters it as false. We then end up with four possibilities for each statement:

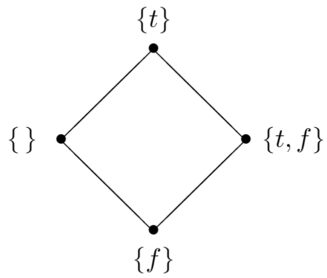
N The computer has received no information pertaining to the statement;

F The computer has received the information that the statement is false;

T The computer has received the information that the statement is true;

B The computer has received the information that the statement is true and that it is false.

Given these interpretations for the four values, how can we justify the choice of designated values, T and B? To make Belnap's argument more conspicuous, it is convenient to relabel the four values in a manner suggested by M. Dunn. For the remainder of this section we represent them as members of the power set of the two basic truth values t and f: N corresponds to the empty set {}, F to {f}, T to {t} and B to {t,f}.



Each value now clearly represents the pieces of information that have been given to the computer. Now, what should we want to be preserved in this logic? Belnap actually gives two alternatives. First and most common is the request that the consequence relation preserve truth. This is the standard idea throughout logic, and also under the computer interpretation in particular this seems a sound request: you hope to feed the computer true information and to get back true information. As in the set formulation there are two sets that contain t, namely {t} and {t,f}, it seems natural to take these as designated values.

However, interestingly, Belnap also briefly considers a second property to be preserved. He suggests we might want to preserve non-falsity: if the premisses do not contain f in their truth value, then neither will the conclusion. Again, the motivation for this choice seems clear: those statements that you tell the computer are false are those that you would not want to assume to hold in your further deliberations, so the computer should make sure that no false statements will show up in the consequences it draws from input that isn't told false. Looking up the truth values that would get designated under this requirement, we find {t} and {}.

Taking another look at the lattice will show that the two choices of designated values are perfectly symmetrical, and it is therefore no surprise that the two logics thus characterized are identical with respect to the inferences they validate (they also agree on their tautologies: there are none).

2 Exactly True Logic

Very well. Either of the two features that one might wish to preserve, truth or non-falsity, leads to the same logic. What more could we ask for?

In our opinion, a very plausible answer to this question is that we should ask for a consequence relation that preserves *truth-and-non-falsity.* We glue this property together with hyphens because at one point Belnap describes FDE as preserving truth *and* non-falsity in a different sense from ours [1, p. 519]:

> Now for an account which is close to the informal considerations underlying our understanding of the four values as keeping track of markings with told True and told False: say that the inference from A to B is valid, or that A entails B, if the inference never leads us from told True to the absence of told True (preserves Truth), and also never leads us from the absence of told False to told False (preserves non-Falsity). Given our system of markings, this is hardly to ask too much.

FDE preserves truth and non-falsity in the following sense: if all premises are true, so is the conclusion, and if all premises are non-false, so is the conclusion.

The requirement we propose, however, is as follows: if all premises are true and not false, then so is the conclusion. Very simply put, we note with Belnap that "told true" is good, "told false" is bad (see [1, p. 516]). To elaborate a bit more, if you want to put your faith in some statements but not others, based on what the computer tells you about its database, then you should be reluctant to do so in the cases of propositions that are "told false", no matter whether they are "told true" or not.

Furthermore, if you let the computer draw inferences for you, you should have him choose those pieces of information that you would be willing to put your faith in and reason from those to others that you would be equally willing to rely on.

In sum, the prudent computer should choose those pieces of information that are univocally supported and infer conclusions that are similarly univocally supported. Of course, the only truth value that both contains t and does not contain f is {t}. Given the computer interpretation, designating only this value seems like the natural choice[2].

Let us now go back to our original four names, T, N, B and F. Talking "across" the two variants of semantics, the difference between Belnap and our proposal

[2] The only source we are aware of that notes this possibility is the SEP entry on Many Valued Logic by S. Gottwald [8], who notes that "for computer science applications it is natural to have [{t}] as the only designated [value]". However, Prof. Gottwald informed us in private communication that neither he himself nor anyone he knew of has investigated the resulting logic.

is that he wants to designate t, while we want to designate T. In [1, p. 512] he discusses the difference between the two values and suggests to read t as "told at least true" and T as "told exactly true" in circumstances where confusion between the two threatens. We adopt this terminology in calling the new logic we propose *Exactly True Logic* (ETL).

What happens to logical consequence if we designate only T? For one thing, a contradiction will now never take a designated value. Therefore, ETL validates *ex contradictione quodlibet*, and thus fails to fulfill the needs of relevant logicians. Maybe this is the reason why this logic has not been studied in more detail, at least by relevant logicians.

Another reason why the new system seems to have been overlooked is that one might well think that the new logic will coincide with a known one, such as strong Kleene, classical logic or FDE itself. This, however, is not the case. Even though the new logic validates *ex contradictione quodlibet*, $A \wedge \neg A \vDash C$, just as strong Kleene does, the following inference (valid in strong Kleene)

$$(A \wedge \neg A) \vee (B \wedge \neg B) \vDash C$$

fails in ETL. For a counterexample, take $v(A) = B$, $v(B) = N$ and $v(C) = F$. It is easy to check that under this valuation, the premise will be assigned value T.

This is a most unusual feature. For one thing, it allows for theories that contain disjunctions, but cannot consistently be expanded by *either* disjunct! As far as we know, this feature does not have a standard name in the literature, so we will dub it *anti-primeness*.

Similarly, the inference from $A \vDash C$ and $B \vDash C$ to $A \vee B \vDash C$ fails. Presumably, this will not make it easy to find a nice sequent calculus for this logic. Instead, we give a Hilbert-style proof system in the next section.

3 A Hilbert system for ETL

We are going to prove that ETL is axiomatized by the Hilbert-style calculus defined by the rules shown in Table 1. By "Hilbert-style" we mean a calculus having possibly axioms (in fact ours has none) and only rules of the form

$$\Gamma \vdash \varphi$$

where Γ is a finite set of formulas and φ is a formula, rather than meta-logical Gentzen-style rules of the form

$$\frac{\Gamma \vdash \varphi}{\Gamma' \vdash \varphi'}$$

Although from a proof-theoretic point of view Gentzen-style calculi (especially those that enjoy cut elimination and the subformula property) are usually preferred to Hilbert-style calculi, from an algebraic logic point of view it is just the opposite. The reason is that using a Hilbert-style presentation it is quite easy to characterize, in any algebraic model of a logic, the sets of elements that are closed under the rules of the logic (i.e., the designated elements or, in algebraic logic jargon, the "logical filters" of the logic [7, 6]). As we shall see, this choice will enable us to obtain a quite simple algebraic proof of completeness that relies on a known completeness result for FDE.

As mentioned above, ETL has no tautologies, therefore any calculus for it will have no axioms. Our calculus is an extension of the one introduced by Font [5] for FDE, which consists of rules (R1) to (R15) shown in Table 1.

The somewhat unusual disjunctive form of (R10) to (R15) is motivated by technical reasons. Notice, however, that these rules are (strictly) stronger than the ones which may look more natural, as the following result shows [5, Proposition 3.2]:

Proposition 3.1. *The following rules follow from (R1) to (R15):*

(a) The rule $(Ri^{+})\ \dfrac{\varphi}{\psi}$ *, for each one of the rules* $(Ri)\ \dfrac{\varphi \vee r}{\psi \vee r}$ *where* $i \in \{10, \ldots, 15\}$.

(b) The rule $\dfrac{\varphi \wedge r}{\psi \wedge r}$ *in the same cases.*

Proof. (a) From φ by (R4) we obtain $\varphi \vee \psi$. Then we apply (Ri) to get $\psi \vee \psi$ and by (R6) we obtain ψ.

(b) From $\varphi \wedge r$ by (R1) we obtain φ. Now using (a) we get ψ. Also from $\varphi \wedge r$, by (R2), follows r. Thus applying (R3) we get $\psi \wedge r$. □

Let us also note that we could not have added the rule

$$\frac{(p \wedge (\neg p \vee q)) \vee r}{q \vee r}$$

instead of (R16), because this rule is not sound w.r.t. the semantics of ETL. To see this, consider a valuation h such that $h(p) = N$ and $h(q) = h(r) = B$. In this case we have

$$h((p \wedge (\neg p \vee q)) \vee r) = T$$

but $h(q \vee r) = B$.

Font [5, Theorem 3.11] proves that his calculus is complete with respect to the semantics of FDE and we are going to see that adding (R16) is enough to prove completeness with respect to the semantics of ETL. This will also imply that ETL a proper strengthening of FDE. In order to obtain this result we need to introduce some terminology and auxiliary lemmas.

(R1) $\dfrac{p \wedge q}{p}$	(R2) $\dfrac{p \wedge q}{q}$	(R3) $\dfrac{p \quad q}{p \wedge q}$
(R4) $\dfrac{p}{p \vee q}$	(R5) $\dfrac{p \vee q}{q \vee p}$	(R6) $\dfrac{p \vee p}{p}$
(R7) $\dfrac{p \vee (q \vee r)}{(p \vee q) \vee r}$	(R8) $\dfrac{p \vee (q \wedge r)}{(p \vee q) \wedge (p \vee r)}$	(R9) $\dfrac{(p \vee q) \wedge (p \vee r)}{p \vee (q \wedge r)}$
(R10) $\dfrac{p \vee r}{\neg\neg p \vee r}$	(R11) $\dfrac{\neg\neg p \vee r}{p \vee r}$	(R12) $\dfrac{\neg(p \vee q) \vee r}{(\neg p \wedge \neg q) \vee r}$
(R13) $\dfrac{(\neg p \wedge \neg q) \vee r}{\neg(p \vee q) \vee r}$	(R14) $\dfrac{\neg(p \wedge q) \vee r}{(\neg p \vee \neg q) \vee r}$	(R15) $\dfrac{(\neg p \vee \neg q) \vee r}{\neg(p \wedge q) \vee r}$
(R16) $\dfrac{p \wedge (\neg p \vee q)}{q}$		

Table 1: A Hilbert-style calculus for ETL

As an algebra, the four-element Belnap lattice $\mathbf{4} = \langle \{\mathrm{F}, \mathrm{T}, \mathrm{B}, \mathrm{N}\}, \wedge, \vee, \neg \rangle$ belongs to the class known as *De Morgan lattices*, i.e. distributive lattices with a unary operator $\neg$ satisfying the following equations:

$$\neg(x \wedge y) \approx \neg x \vee \neg y$$
$$\neg(x \vee y) \approx \neg x \wedge \neg y$$
$$x \approx \neg\neg x.$$

De Morgan lattices form a variety that is known to be generated by $\mathbf{4}$. This implies that an equation $x \approx y$ is valid in $\mathbf{4}$ iff it is valid in all De Morgan lattices. In what follows, the expression $x \leq y$ will be used as an abbreviation for the equation $x \wedge y \approx x$. Also, we will denote by $\vDash_{ETL}$ the semantic consequence relation associated with ETL.

Lemma 3.2. *Let $\varphi_1, \ldots, \varphi_n, \varphi$ be propositional formulas. The following are equivalent:*

(i) $\{\varphi_1, \ldots, \varphi_n\} \vDash_{ETL} \varphi$

(ii) $\mathbf{4}$ *satisfies* $\varphi_1 \wedge \ldots \wedge \varphi_n \leq \neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \vee \varphi$

Proof. (i)⇒(ii). Let h be a valuation on $\mathbf{4}$ and let us abbreviate $\psi := \varphi_1 \wedge \ldots \wedge \varphi_n$. Assume (i). If $h(\psi) = \mathrm{F}$, we are done. If $h(\psi) = \mathrm{N}$, then $h(\neg\psi) = \mathrm{N}$ as well and

obviously $\mathrm{N} \leq \mathrm{N} \vee a$ for any element $a \in \mathbf{4}$. The same reasoning applies to the case where $h(\psi) = \mathrm{B}$. Finally, if $h(\psi) = \mathrm{T}$, then, by (i), $h(\varphi) = \mathrm{T}$ as well and since $\mathrm{T} \leq \neg\mathrm{T} \vee \mathrm{T} = \mathrm{T}$, we are done.
(ii)⇒(i). Assume $h(\varphi_i) = \mathrm{T}$ for $1 \leq i \leq n$. Then $h(\psi) = \mathrm{T}$, so by (ii) we have $\mathrm{T} \leq \neg\mathrm{T} \vee h(\varphi) = h(\varphi)$, which obviously implies $h(\varphi) = \mathrm{T}$. □

Taking into account the fact that the lattice **4** generates the variety of De Morgan lattices, we see that (i) and (ii) are also equivalent to

(iii) the equation $\varphi_1 \wedge \ldots \wedge \varphi_n \leq \neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \vee \varphi$ is valid in the class of De Morgan lattices.

The next lemma uses some fundamental results of algebraic logic (see [6] for proofs and more details). Let us introduce the terminology that we shall need. We consider logical matrices as models of logics, by a *logical matrix* meaning a pair $\langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is an algebra and $D \subseteq A$ is a subset of designated elements. One says that a matrix is a *model* of a logic $\mathcal{L}$ when $\Gamma \vdash_{\mathcal{L}} \varphi$ implies that, for any valuation h on $\mathbf{A}$, if $h(\gamma) \in D$ for all $\gamma \in \Gamma$, then $h(\varphi) \in D$. In particular, this implies that D is closed under any rule $\varphi \vdash \psi$ of $\mathcal{L}$, i.e. for all valuations h, if $h(\varphi) \in D$, then $h(\psi) \in D$.

A *matrix congruence* of a matrix $\langle \mathbf{A}, D \rangle$ is a congruence of $\mathbf{A}$ such that whenever two elements $a, b \in \mathbf{A}$ are related and $a \in D$, then $b \in D$ as well. Any matrix has a greatest logical congruence; we say that a matrix is *reduced* when it has just one logical congruence (which needs to be the identity).

It is easy to see that any logic $\mathcal{L}$ is complete with respect to the class of all its matrix models, in the sense that $\Gamma \vdash_{\mathcal{L}} \varphi$ iff, for any model $\langle \mathbf{A}, D \rangle$ of $\mathcal{L}$, $h(\gamma) \in D$ for all $\gamma \in \Gamma$ implies $h(\varphi) \in D$. More interestingly, it is known that any logic is complete (in the above sense) with respect to the class of its reduced matrix models. This implies in particular that, when trying to disprove something, it is sufficient to look at reduced models. In fact, if $\Gamma \nvdash_{\mathcal{L}} \varphi$, then there is some reduced matrix $\langle \mathbf{A}, D \rangle$ for $\mathcal{L}$ and some valuation h such that $h(\gamma) \in D$ for all $\gamma \in \Gamma$ but $h(\varphi) \notin D$.

Lemma 3.3. *Let $\mathcal{L}$ be an extension of FDE that satisfies (R16). Then $\mathcal{L}$ is also an extension of ETL.*

Proof. In order to show that $\mathcal{L}$ is an extension of ETL, suppose $\varphi \nvdash_{\mathcal{L}} \psi$ but $\varphi \vDash_{ETL} \psi$. As said above, this implies that there is some reduced matrix $\langle \mathbf{A}, D \rangle$ for $\mathcal{L}$ and a valuation h such that $h(\varphi) \in D$ and $h(\psi) \notin D$. Since $\mathcal{L}$ is obviously an extension of FDE, $\langle \mathbf{A}, D \rangle$ will be a model of FDE. As proved in [5, Theorem 3.14], this implies that $\mathbf{A}$ is a De Morgan lattice and D is a lattice filter. Moreover, by assumption D is closed under (R16). By Lemma 3.2, $\varphi \vDash_{ETL} \psi$ implies that the equation $\varphi \leq \neg\varphi \vee \psi$ is valid in any De Morgan lattice. Then $h(\varphi) \leq \neg h(\varphi) \vee h(\psi)$ and, since D is a lattice filter, we have that $\neg h(\varphi) \vee h(\psi) \in D$. But D is closed under (R16), so we should have $h(\psi) \in D$, which is against the hypothesis. □

From the previous lemma it is straightforward to obtain the following:

Theorem 3.4. *ETL is axiomatized by the Hilbert-style calculus defined by the rules of Table 1.*

Proof. Let $\mathcal{L}$ be the logic defined by the the rules of Table 1. To prove that $\vdash_{\mathcal{L}} \leq \vDash_{ETL}$ it suffices to check that the matrix $\langle \mathbf{4}, \{\mathrm{T}\}\rangle$ is a model of (R16), because we already know from [5] that it is a model of (R1) to (R15). In order to prove that $\vDash_{ETL} \leq \vdash_{\mathcal{L}}$, we can use Lemma 3.3, as obviously $\mathcal{L}$ is an extension of FDE that satisfies (R16). Therefore, $\vDash_{ETL} = \vdash_{\mathcal{L}}$. □

4 Disjunctive Syllogism

We would like to make some comments on the new rule (R16) we added and assess the relative merits of ETL and FDE.

(R16) is known as *disjunctive syllogism.* It may be interesting to note that this rule is one of the major areas of dissent in the relevantist camp[3]. On the face of it, it seems like a wholly plausible rule, quite unlike the objectionable *ex contradicitone quodlibet.* However, as a simple and famous argument shows, this rule, alongside conjunction elimination and disjunction introduction (rules (R1), (R2) and (R4) in our calculus) leads to the same effect as *ex contradictione quodlibet*:

1.	$A \wedge \neg A$	Hyp.
2.	A	1., $\wedge - E$
3.	$\neg A$	1., $\wedge - E$
4.	$\neg A \vee B$	3., $\vee - I$
5.	B	2., 4., Disjunctive Syllogism (R16)

A difficult choice has to be made if one wants to avoid such an inference. Even though $\wedge{-}E$, $\vee{-}I$ and even the transitivity of deduction have been questioned to this end, the usual move is to renounce disjunctive syllogism. Some relevantists were quick to adopt a valiant stance by arguing that disjunctive syllogism is indeed not a valid form of inference.[4] The issue is complex, but we think it is fair to say that it is at least not a straightforward matter to convince oneself that disjunctive syllogism should flout the intuitive constraints of relevance as we have outlined them in the first section of this paper.

ETL, then, has some syntactical interest for the relevant logician. It is the logic that results from adding to a certain type of Hilbert-style calculus for FDE the most salient rule of inference that relevantists might like to preserve but cannot have. However, it is clear that ETL won't do more than maybe pique their curiosity; it can't really tempt them, as *ex contradictione quodlibet* is valid.

[3]A good survey of the debate can be found in section 2.4 of [9].

[4]See [4] for an early example.

5 ETL vs. FDE

Thus, if ETL can be motivated, then only through our claim that it is better suited to the intuitive interpretation that Belnap offered us: if a data set contains contradictory information, then something went wrong. It is exactly the contradictory, corrupted pieces of information that show that not all is well with the data set. Thus, we are well advised to keep the underdetermined value N and the overdetermined value B apart; it is of interest to know whether no information or too much information has been given.

However, it is clear that if one decides to place one's trust in the data set, one should treat the corrupted data with suspicion. Yes, that a statement has value B means that someone claimed its truth. But it also means that someone (presumably someone else) has claimed its falsity. The idea that such statements are not deserving our acceptance, even if we are willing to accept the information recorded in the data set in general, has led us to the designation of T alone, and thus to ETL.

If ETL scores better than FDE in terms of intuitive interpretation, it might seem somewhat unattractive when the salient properties of its consequence relation are taken into account. It is not all that easy to give a good reason why we should want a logic that refuses to assign a designated value to a contradiction, but is more lenient when it comes to disjunctions of contradictions. The closely related feature we called "anti-primeness" above likewise does not seem enticing.

Indeed, we will not pretend otherwise: these are quite counterintuitive features. However, when it comes to a direct comparison between FDE and the new logic, we believe that this should not weigh too heavily against the latter. This is because what we see here is merely a slight exacerbation of an unintuitive feature that has been with FDE ever since it was proposed. The lattice will give out the value T for a disjunction of two statements with the values B and N (and, dually, the value F for a conjunction of those statements). In particular, the fact that a contradiction with the value B and a contradiction with the value N will receive value T when disjoined is a feature of the logical lattice, not of ETL in particular. Indeed, ETL can be seen to point to this initially perplexing semantic phenomenon via its peculiar proof theory.

As Belnap [1, p. 516] remarked, this peculiarity is pretty much inevitable if one wants to interpret conjunctions as meet and disjunctions as join operations on the lattice (see also [10, p. 927]). Moreover, he tried to assuage the initial implausibility by noting the following: If we are wondering about the value that a disjunction of an over- and an underdetermined statement should receive, we should first note that it is at least "told true", because one of the disjunctions is "told true". On the other hand, it is not "told false", because for that outcome we should require that both statements were "told false", and the underdetermined statement was not so told. Therefore, we end up with "exactly true" as the only suitable value.

If, following this type of reasoning, we can make our peace with the idea that a disjunction of an overdetermined and an underdetermined statement should receive the value "exactly true", then this explanation will likewise suffice to cover the peculiarities of ETL. Seen this way, it seems that ETL has an advantage over FDE when it comes to how well it fits the intended interpretation, simply because the natural thought that contradictory information should be handled with suspicion is captured.

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