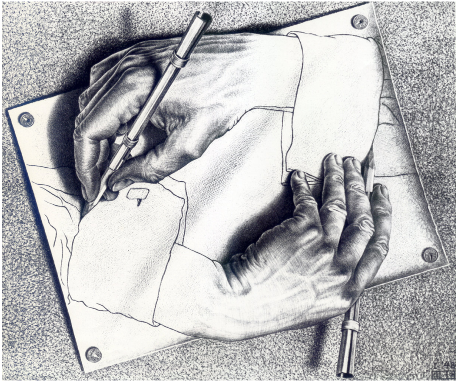
Jérôme Fortier, with Luigi Santocanale

Cuts in circular proofs



Delft – February 17, 2014

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- $\mu X.(1 + A \times X) = A^* =$ Finite words over A

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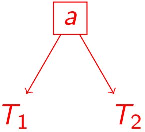
$$\begin{aligned} * &\mapsto \varepsilon \\ \langle a, w \rangle &\mapsto a \cdot w \end{aligned}$$

- $\mu X.(1 + A \times X \times X) =$ Finite binary labelled trees

$$* \mapsto \text{Empty tree}$$

$$\langle a, T_1, T_2 \rangle \mapsto$$



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A stream over an alphabet A is made of a head $a \in A$, and another stream called the tail.

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$$\begin{tikzcd}[column sep=small] & \boxed{a} \arrow[dl] \arrow[dr] & \\ T_1 & & T_2 \end{tikzcd} \qquad \mapsto \qquad \langle a, T_1, T_2 \rangle$$

Lattice μ-calculus

Lattice μ-terms are generated by the following grammar:

$$t := X \mid 1 \mid t \times t \mid 0 \mid t + t \mid \color{red}{\mu X.t} \mid \color{red}{\nu X.t}$$

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Functorial interpretation

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Definition

A category $\mathcal{C}$ is μ-bicomplete iff this interpretation makes sense in $\mathcal{C}$.

Game semantics (in $\mathcal{S}ets$)

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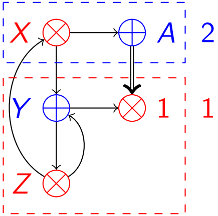

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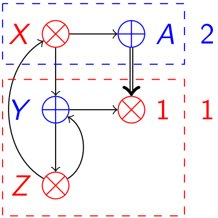
Parity games!



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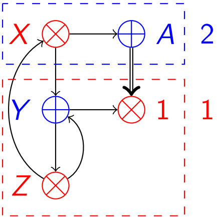

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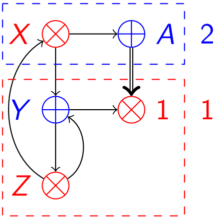

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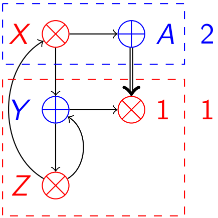

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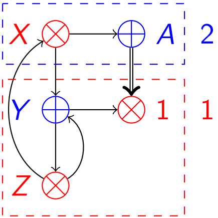

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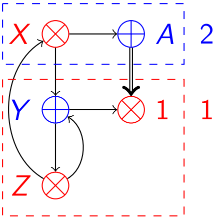

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Theorem (Santocanale, 2002)

Solutions for variable V in $\mathcal{S}(t)$ $\simeq$ The set of deterministic winning strategies for $\oplus$ from position V.

Therefore, we have a combinatorial (dynamic) characterization of the μ-defined objects.

Curry–Howard correspondence

Goal

Find a "good" (dynamic) syntax for expressing (and computing) functions (arrows) between objects of this kind.

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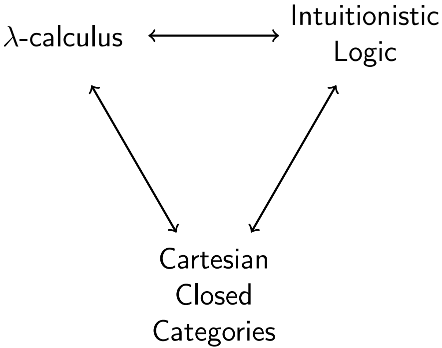

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$$\begin{tikzcd} \mu\text{-calculus} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Intuitionistic Logic} \arrow[dl, leftrightarrow] \\ & \text{Cartesian Closed Categories} & \end{tikzcd}$$

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\mu\text{-calculus} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Circular Proofs} \arrow[dl, leftrightarrow] \\
& \mu\text{-bicomplete Categories} &
\end{tikzcd}$$

Inference rules (Gentzen style)

Axioms:	$\dfrac{}{0 \vdash t}\,\texttt{LAx}$	$\dfrac{}{t \vdash 1}\,\texttt{RAx}$	$\dfrac{}{t \vdash t}\,\texttt{Id}$
Product: (conjunction)	$\dfrac{s_i \vdash t}{s_0 \times s_1 \vdash t}\,\texttt{L}\times_i$	$\dfrac{s \vdash t_0 \quad s \vdash t_1}{s \vdash t_0 \times t_1}\,\texttt{R}\times$	
Coproduct: (disjunction)	$\dfrac{s_0 \vdash t \quad s_1 \vdash t}{s_0 + s_1 \vdash t}\,\texttt{L}+$	$\dfrac{s \vdash t_i}{s \vdash t_0 + t_1}\,\texttt{R}+_i$	
Fixpoint:	$\dfrac{F(X) \vdash t}{X \vdash t}\,\texttt{LFix}$	$\dfrac{s \vdash F(X)}{s \vdash X}\,\texttt{RFix}$	if $X = F(X)$
Cut:	$\dfrac{r \vdash s \quad s \vdash t}{r \vdash t}\,\texttt{Cut}$		

Categorical interpretation

Axions:

$$\frac{}{0 \xrightarrow{?_t} t}\,\texttt{LAx} \qquad \frac{}{t \xrightarrow{!_t} 1}\,\texttt{RAx} \qquad \frac{}{t \xrightarrow{\mathrm{id}_t} t}\,\texttt{Id}$$

Product: (conjunction)

$$\frac{s_i \xrightarrow{f} t}{s_0 \times s_1 \xrightarrow{\mathtt{pr}_i \cdot f} t}\,\mathtt{L}\times_i \qquad \frac{s \xrightarrow{f} t_0 \quad s \xrightarrow{g} t_1}{s \xrightarrow{\langle f, g\rangle} t_0 \times t_1}\,\mathtt{R}\times$$

Coproduct: (disjunction)

$$\frac{s_0 \xrightarrow{f} t \quad s_1 \xrightarrow{g} t}{s_0 + s_1 \xrightarrow{\{f, g\}} t}\,\mathtt{L}+ \qquad \frac{s \xrightarrow{f} t_i}{s \xrightarrow{f \cdot \mathtt{in}_i} t_0 + t_1}\,\mathtt{R}+_i$$

Fixpoint:

$$\frac{F(X) \xrightarrow{f} t}{X \xrightarrow{\zeta_X^{-1} \cdot f} t}\,\texttt{LFix} \qquad \frac{s \xrightarrow{f} F(X)}{s \xrightarrow{f \cdot \zeta_X} X}\,\texttt{RFix} \qquad \text{if } X =_\mu F(X)$$

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$$\begin{prooftree}
\AxiomC{}
\RightLabel{$\mathtt{RAx}$}
\UnaryInfC{$1 \vdash 1$}
\RightLabel{$\mathrm{R}+_0$}
\UnaryInfC{$1 \vdash 1 + N$}
\RightLabel{$\mathtt{RFix}$}
\UnaryInfC{$1 \vdash N$}
\AxiomC{$\boxed{N \vdash N}$}
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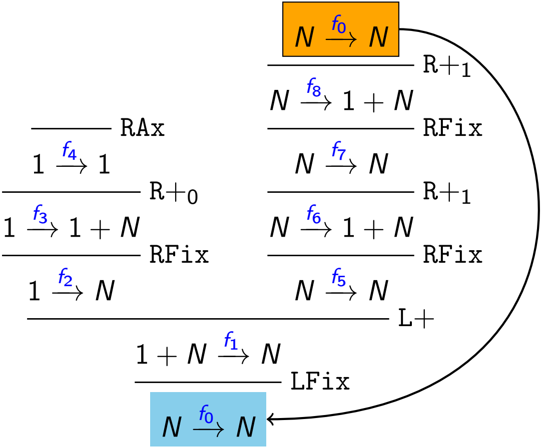
\end{prooftree}$$



$$
\left\{
\begin{array}{rcl}
f_8 & = & f_0 \cdot \mathtt{in}_1 \\
f_7 & = & f_8 \cdot \zeta_N \\
f_6 & = & f_7 \cdot \mathtt{in}_1 \\
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\\
f_4 & = & !_1 \\
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\\
f_1 & = & \{f_2, f_5\} \\
f_0 & = & \zeta_N^{-1} \cdot f_1
\end{array}
\right\}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\mathtt{RAx}$}
\UnaryInfC{$1 \xrightarrow{f_4} 1$}
\RightLabel{$\mathtt{R+}_0$}
\UnaryInfC{$1 \xrightarrow{f_3} 1+N$}
\RightLabel{$\mathtt{RFix}$}
\UnaryInfC{$1 \xrightarrow{f_2} N$}
\AxiomC{$\boxed{N \xrightarrow{f_0} N}$}
\RightLabel{$\mathtt{R+}_1$}
\UnaryInfC{$N \xrightarrow{f_8} 1+N$}
\RightLabel{$\mathtt{RFix}$}
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\RightLabel{$\mathtt{R+}_1$}
\UnaryInfC{$N \xrightarrow{f_6} 1+N$}
\RightLabel{$\mathtt{RFix}$}
\UnaryInfC{$N \xrightarrow{f_5} N$}
\RightLabel{$\mathtt{L+}$}
\BinaryInfC{$1+N \xrightarrow{f_1} N$}
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\end{prooftree}
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Solution: $f_0 = \mathtt{double}$

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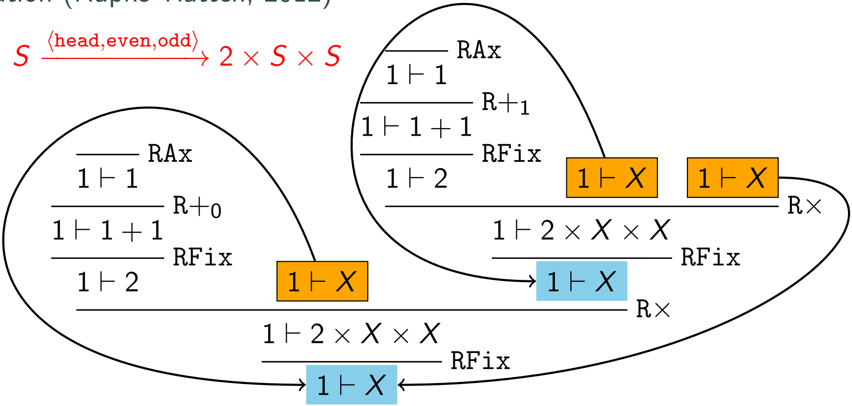

Non-valid circular proofs

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$$\boxed{s \vdash t}$$

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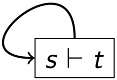

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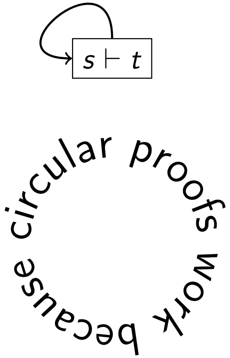

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$$\dfrac{\dfrac{\boxed{\text{God exists}} \qquad \dfrac{\dfrac{}{\text{Bible says it is the word of God}}\,\mathtt{Fact} \qquad \boxed{\text{What Bible says is true}}}{\text{Bible is the word of God}}\,\mathtt{Cut}}{\boxed{\text{What Bible says is true}}}\,\mathtt{Cut} \qquad \dfrac{}{\text{Bible says that God exists}}\,\mathtt{Fact}}{\boxed{\text{God exists}}}\,\mathtt{Cut}$$

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Definition

A circular proof is a finite pre-proof that satisfies the guard conditions.

Semantical results

Soundness Theorem (F.–S. 2013)

Every circular proof denotes a unique arrow of the free μ-bicomplete category $\mathcal{M}$.

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Except for this one!

$$\begin{tikzcd}
F(x) \arrow[r, dashed, "F(f)"] \arrow[d, "\zeta_x"'] & F(a) \arrow[d, "\alpha = [\![\Pi]\!]"] \\
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\end{tikzcd}$$

$$f = \alpha \cdot F(f) \cdot \xi_x^{-1}$$

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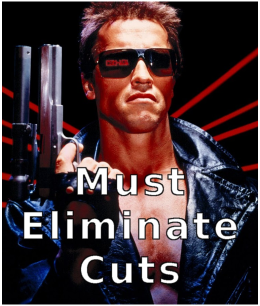
Cut-elimination



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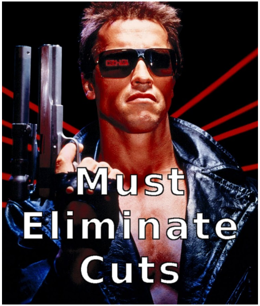
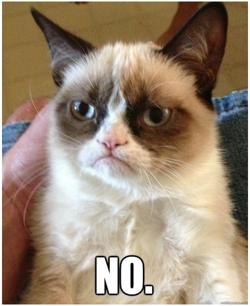




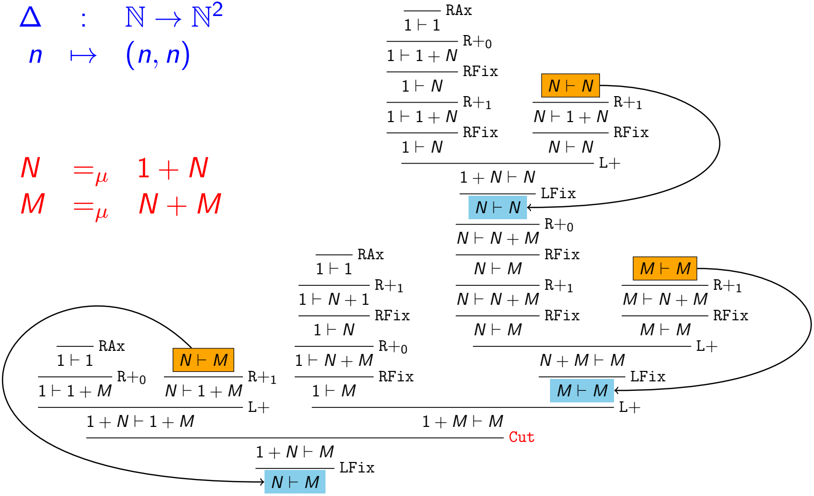
Theorem (Santocanale, 2001)

There is no cut-free circular proof whose interpretation in $\mathcal{S}ets$ is the diagonal $\Delta : \mathbb{N} \to \mathbb{N}^2$.

Diagonal map (with cuts)

$$\begin{array}{rcl} \Delta & : & \mathbb{N} \to \mathbb{N}^2 \\ n & \mapsto & (n, n) \end{array}$$

$$\begin{array}{rcl} N & =_\mu & 1 + N \\ M & =_\mu & N + M \end{array}$$



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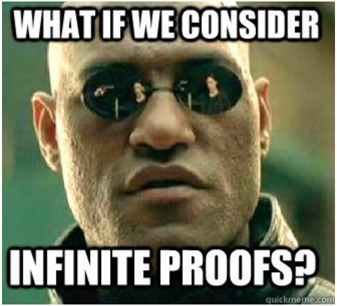

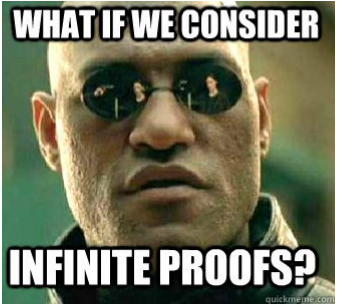

$\Longrightarrow$

Tape automaton

Strategy: "Push" the cuts away from the root.

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$\Downarrow$ Merge

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- Outputs a branch (chosen nondeterministically) of the cut-free infinite proof tree.

Commutative reductions (left)

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$$\dfrac{\dfrac{s_0 \vdash t_1 \qquad s_1 \vdash t_1}{s_0 + s_1 \vdash t_1}\,\texttt{L+} \qquad t_1 \vdash t_2 \qquad \cdots}{s_0 + s_1 \vdash t_n}\,\texttt{Cut}$$

Commutative reductions (left)

$$\dfrac{\dfrac{}{0 \vdash t_1}\,\texttt{LAx} \qquad t_1 \vdash t_2 \qquad \cdots}{0 \vdash t_n}\,\texttt{Cut} \quad \xRightarrow{\text{LFlip}} \quad \dfrac{}{0 \vdash t_n}\,\texttt{LAx}$$

$$\dfrac{\dfrac{F(X) \vdash t_1}{X \vdash t_1}\,\texttt{LFix} \qquad t_1 \vdash t_2 \qquad \cdots}{X \vdash t_n}\,\texttt{Cut} \quad \xRightarrow{\text{LFlip}} \quad \dfrac{\dfrac{F(X) \vdash t_1 \qquad t_1 \vdash t_2 \qquad \cdots}{X \vdash t_n}\,\texttt{Cut}}{X \vdash t_n}\,\texttt{LFix}$$

$$\dfrac{\dfrac{s_k \vdash t_1}{s_0 \times s_1 \vdash t_1}\,\mathrm{L}\times_k \qquad t_1 \vdash t_2 \qquad \cdots}{s_0 \times s_1 \vdash t_n}\,\texttt{Cut} \quad \xRightarrow{\text{LFlip}} \quad \dfrac{\dfrac{s_k \vdash t_1 \qquad t_1 \vdash t_2 \qquad \cdots}{s_k \vdash t_n}\,\texttt{Cut}}{s_0 \times s_1 \vdash t_n}\,\mathrm{L}\times_k$$

$$\dfrac{\dfrac{s_0 \vdash t_1 \qquad s_1 \vdash t_1}{s_0 + s_1 \vdash t_1}\,\mathrm{L}+ \qquad t_1 \vdash t_2 \qquad \cdots}{s_0 + s_1 \vdash t_n}\,\texttt{Cut} \quad \xRightarrow{\text{LFlip}} \quad \dfrac{\dfrac{s_0 \vdash t_1 \qquad t_1 \vdash t_2 \qquad \cdots}{s_0 \vdash t_n}\,\texttt{Cut} \qquad \dfrac{s_1 \vdash t_1 \qquad t_1 \vdash t_2 \qquad \cdots}{s_1 \vdash t_n}\,\texttt{Cut}}{s_0 + s_1 \vdash t_n}\,\mathrm{L}+$$

Commutative reductions (right)

$$\dfrac{\cdots \quad t_{n-2} \vdash t_{n-1} \quad \dfrac{}{t_{n-1} \vdash 1}\,\mathtt{RAx}}{t_0 \vdash 1}\,\mathtt{Cut} \quad \overset{\text{RFlip}}{\Longrightarrow} \quad \dfrac{}{t_0 \vdash 1}\,\mathtt{RAx}$$

$$\dfrac{\cdots \quad t_{n-2} \vdash t_{n-1} \quad \dfrac{t_{n-1} \vdash F(X)}{t_{n-1} \vdash X}\,\mathtt{RFix}}{t_0 \vdash X}\,\mathtt{Cut} \quad \overset{\text{RFlip}}{\Longrightarrow} \quad \dfrac{\dfrac{\cdots \quad t_{n-2} \vdash t_{n-1} \quad t_{n-1} \vdash F(X)}{t_0 \vdash F(X)}\,\mathtt{Cut}}{t_0 \vdash X}\,\mathtt{RFix}$$

$$\dfrac{\cdots \quad t_{n-2} \vdash t_{n-1} \quad \dfrac{t_{n-1} \vdash s_k}{t_{n-1} \vdash s_0 + s_1}\,\mathtt{R}+_k}{t_0 \vdash s_0 + s_1}\,\mathtt{Cut} \quad \overset{\text{RFlip}}{\Longrightarrow} \quad \dfrac{\cdots \quad t_{n-2} \vdash t_{n-1} \quad \dfrac{t_{n-1} \vdash s_k}{t_{n-1} \vdash s_k}\,\mathtt{Cut}}{t_0 \vdash s_0 + s_1}\,\mathtt{R}+_k$$

$$\dfrac{\cdots \quad t_{n-2} \vdash t_{n-1} \quad \dfrac{t_{n-1} \vdash s_0 \quad t_{n-1} \vdash s_1}{t_{n-1} \vdash s_0 \times s_1}\,\mathtt{R}\times}{t_0 \vdash s_0 \times s_1}\,\mathtt{Cut} \quad \overset{\text{RFlip}}{\Longrightarrow} \quad \dfrac{\dfrac{\cdots \quad t_{n-2} \vdash t_{n-1} \quad t_{n-1} \vdash s_0}{t_0 \vdash s_0}\,\mathtt{Cut} \quad \dfrac{\cdots \quad t_{n-2} \vdash t_{n-1} \quad t_{n-1} \vdash s_1}{t_0 \vdash s_1}\,\mathtt{Cut}}{t_0 \vdash s_0 \times s_1}\,\mathtt{R}\times$$

Elimination of identities

$$\dfrac{\cdots \quad t_{i-1} \vdash s \quad \dfrac{}{s \vdash s}\,\mathtt{Id} \quad s \vdash t_{i+2} \quad \cdots}{t_0 \vdash t_n}\,\mathtt{Cut}$$

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$$\Downarrow \text{IdElim}$$

$$\dfrac{\cdots \quad t_{i-1} \vdash s \quad s \vdash t_{i+2} \quad \cdots}{t_0 \vdash t_n}\,\mathtt{Cut}$$

$\text{IdElim}(M, i) = \text{Remove } u_i \text{ from } M.$

Essential reductions

Otherwise, $M = [\mathtt{R}\ldots\mathtt{RL}\ldots\mathtt{L}]$.

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$$\dfrac{\cdots \quad \dfrac{t_{i-1} \vdash s_0 \quad t_{i-1} \vdash s_1}{t_{i-1} \vdash s_0 \times s_1}\,\mathrm{R}\times \quad \dfrac{s_k \vdash t_{i+1}}{s_0 \times s_1 \vdash t_{i+1}}\,\mathrm{L}\times_k \quad \cdots}{t_0 \vdash t_n}\,\mathtt{Cut}$$

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Cut-elimination algorithm

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Theorem (F.–S., 2013)

For every input tape M, the internal phase halts!

Proof. Suppose it does not halt...

$$
\begin{array}{rcl}
M_1 & = & [\; u_{11} \quad u_{12} \quad u_{13} \;] \\
\text{Merge} \Downarrow & & \\
M_2 & = & [\; u_{21} \quad u_{22} \quad u_{23} \quad u_{24} \;] \\
\text{Merge} \Downarrow & & \\
M_3 & = & [\; u_{31} \quad u_{32} \quad u_{33} \quad u_{34} \quad u_{35} \;] \\
\text{Reduce} \Downarrow & & \\
M_4 & = & [\; u_{41} \quad u_{42} \quad u_{43} \quad u_{44} \quad u_{45} \;] \\
\text{IdElim} \Downarrow & & \\
M_5 & = & [\; u_{51} \quad u_{52} \quad u_{53} \quad u_{54} \;] \\
\text{Reduce} \Downarrow & & \\
M_6 & = & [\; u_{61} \quad u_{62} \quad u_{63} \quad u_{64} \;] \\
 & \vdots &
\end{array}
$$

Proof. Suppose it does not halt...

$$
\begin{tikzcd}
M_1 \arrow[d, Rightarrow, "\text{Merge}"'] & = & {[\; u_{11}} & u_{12} \arrow[d, "1"'] \arrow[dr, "2"] & {u_{13} \;]} & & \\
M_2 \arrow[d, Rightarrow, "\text{Merge}"'] & = & {[\; u_{21}} \arrow[d, "1"'] \arrow[dr, "2"] & u_{22} & u_{23} & {u_{24} \;]} & \\
M_3 \arrow[d, Rightarrow, "\text{Reduce}"'] & = & {[\; u_{31}} & u_{32} \arrow[d, "0"] & u_{33} \arrow[d, "0"] & u_{34} & {u_{35} \;]} \\
M_4 \arrow[d, Rightarrow, "\text{IdElim}"'] & = & {[\; u_{41}} & u_{42} & u_{43} & u_{44} & {u_{45} \;]} \\
M_5 \arrow[d, Rightarrow, "\text{Reduce}"'] & = & {[\; u_{51}} & u_{52} & u_{53} \arrow[d, "0"] & {u_{54} \;]} \arrow[d, "0"] & \\
M_6 & = & {[\; u_{61}} & u_{62} & u_{63} & {u_{64} \;]} & \\
 & \vdots & & & & &
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$$

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$$
\begin{tikzcd}
M_1 \arrow[d, Rightarrow, "\text{Merge}"'] & = & {[} & u_{11} \arrow[d, dotted, "\perp"'] & u_{12} \arrow[d, "1"'] \arrow[dr, "2"] & u_{13} \arrow[dr, dotted, "\perp"] & {]} & & \\
M_2 \arrow[d, Rightarrow, "\text{Merge}"'] & = & {[} & u_{21} \arrow[d, "1"'] \arrow[dr, "2"] & u_{22} \arrow[dr, dotted, "\perp"] & u_{23} \arrow[dr, dotted, "\perp"] & u_{24} \arrow[dr, dotted, "\perp"] & {]} & \\
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\end{tikzcd}
$$

$\vdots$

Proof. Suspose it does not halt...

$$
\begin{tikzcd}
 & & & * \arrow[dl, "1"'] \arrow[d, "2"] \arrow[dr, "3"] & & & \\
M_1 \arrow[d, Rightarrow, "\text{Merge}"'] & = & {[}\ u_{11} \arrow[d, dotted, "\perp"'] & u_{12} \arrow[d, "1"'] \arrow[dr, "2"] & u_{13}\ {]} \arrow[dr, dotted, "\perp"] & & \\
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M_5 \arrow[d, Rightarrow, "\text{Reduce}"'] & = & {[}\ u_{51} \arrow[d, dotted, "\perp"'] & u_{52} \arrow[d, dotted, "\perp"'] & u_{53} \arrow[d, "0"] & u_{54}\ {]} \arrow[d, "0"] & \\
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\end{tikzcd}
$$

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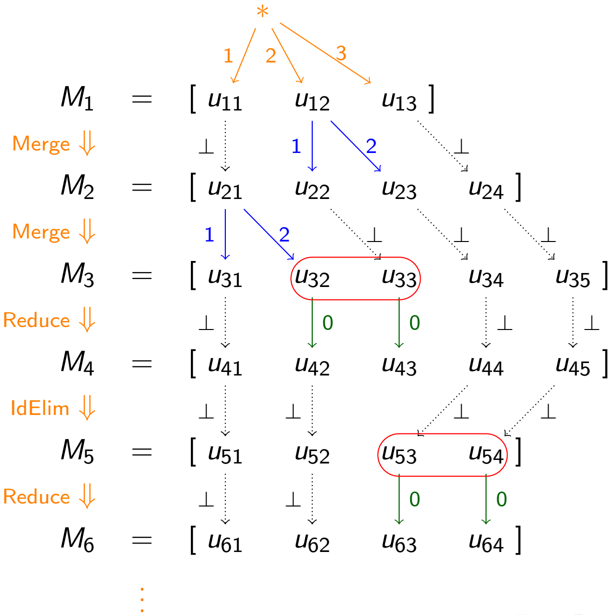

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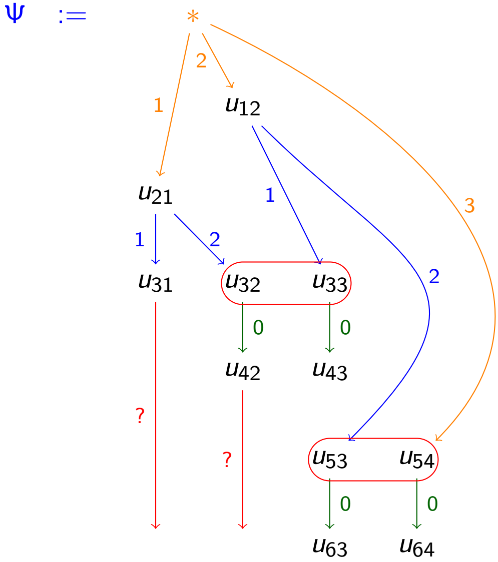

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Lemma (F.–S., 2013)

1. *The least infinite branch of Ψ is a ν-branch.*
2. *Let E be a nonempty collection of ν-branches and let $\gamma = \bigvee E$. Then γ is a ν-branch.*
3. *If β is a ν-branch, then there exists another ν-branch $\beta' \succ \beta$.*

Let

$$E = \text{All the } \nu\text{-branches}$$

So what?

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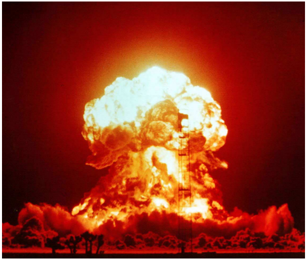
By 1 $E \neq \varnothing$. Let $\gamma = \bigvee E$. By 2, γ is a ν-branch. Hence by 3, there is another ν-branch $\gamma' \succ \gamma$. But then, $\gamma' \in E$ and therefore $\gamma' \preceq \bigvee E = \gamma$.



$\square$

Cut-eliminating infinite proof-trees

- We can cut eliminate a cut-free infinite proof against a fixed circular proof Π.
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Considering that for any μ-definable set X,

$$\begin{aligned} X &\simeq \text{Winning strategies for } \oplus \text{ in some game} \\ &\simeq \text{Cut-free infinite valid proofs of } 1 \vdash X, \end{aligned}$$

Cut-elimination is a generic algorithm for computing all the μ-definable functions.

Computability problems

- What are the set-theoretic functions that one can denote with circular proofs and computed with cut-elimination?

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Primitive recursive $\leq$ μ-definable $\leq$ Recursive

- Which of those bounds are strict?
- Is the Ackermann function definable?
- What about streams?

Computability problems

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- Philosophical question: What is the meaning of circularity in mathematical reasoning?

Merci!

Thank you!

Dank u!