# Substructural fuzzy-relevance logic*

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October 8, 2015

**Abstract**

This paper proposes a new topic in substructural logic for use in research joining the fields of relevance and fuzzy logics. For this, we consider old and new relevance principles. We first introduce fuzzy systems satisfying an old relevance principle, i.e., Dunn's weak relevance principle. We present ways to obtain relevant companions of the weakening-free uninorm (based) systems introduced by Metcalfe and Montagna and fuzzy companions of the system **R** of relevant implication (without distributivity) and its neighbors. The algebraic structures corresponding to the systems are then defined, and completeness results are provided. We next consider fuzzy systems satisfying new relevance principles introduced by Yang. We show that the weakening-free uninorm (based) systems and some extensions and neighbors of **R** satisfy the new relevance principles.

## 1 Introduction

- The purpose of this paper: to extend the world of fuzzy logic to the realm of relevance logic, and vice versa.
- (Fuzzy logic, [3]) A (weakly implicative) logic **L** is said to be *fuzzy* if it is complete w.r.t. linearly ordered matrices (or algebras) and *core fuzzy* if it is complete w.r.t. *standard* algebras (i.e., algebras on the real unit interval $[0, 1]$).
- (Old relevance principles, [1, 4]) A system is said to be *strongly relevant* if it satisfies the *strong* relevance principle (SRP) in [1] that $\varphi \to \psi$ is a theorem only if $\varphi$ and $\psi$ share a propositional variable and *weakly relevant* if it satisfies the *weak* relevance principle (WRP) in [4] that $\varphi \to \psi$ is a theorem only if either (i) $\varphi$ and $\psi$ share a propositional variable or (ii) both $\neg\varphi$ and $\psi$ are theorems.
  − These principles work for relevance systems without propositional constants.

*NDJFL, 56(2015), 471-491.

- (New relevance principles, [7]) A system is said to be *strongly relevant* if it satisfies the new *strong* relevance principle (NSRP) in [7] that $\varphi \to \psi$ is a theorem only if $\varphi$ and $\psi$ either explicitly or strong implicitly share a propositional variable and *weakly relevant* if it satisfies the new *weak* relevance principle (NWRP) that $\varphi \to \psi$ is a theorem only if either (i) $\varphi$ and $\psi$ share either explicitly or strong implicitly share a propositional variable, or (ii) both $\neg\varphi$ and $\psi$ are theorems.
  – These principles work for relevance systems with and without propositional constants.

- – The system **R** (= $\mathbf{R}^0$) is strongly relevant in that it satisfies the principle SRP, and the system **RM** (= $\mathbf{RM}^0$) is weakly relevant in that it satisfies the principle WRP.[1] However, the system **UL** is neither strongly nor weakly relevant because it proves such formulas as ($\alpha$) $(\varphi \wedge \neg\varphi) \to (\psi \vee \neg\psi)$.
  – The weakening-free fuzzy systems in [6] are all relevant in the sense that they satisfy the principle NSRP or the principle NWRP since theorems such as ($\alpha$) strong implicitly share at least one propositional variable.

- We call the relevance principles in [1, 4] *old* relevance principles and the relevance principles in [7] *new* relevance principles. Here, we introduce logics being both *fuzzy* in Cintula's sense and *relevant* in the old and new senses.

# 2 Fuzzy-relevance logics (I)

In this section, we introduce several fuzzy-relevance systems satisfying the principle WRP and their corresponding non-fuzzy relevance systems.

## 2.1 Syntax

We base (fuzzy-)relevance logics on a countable propositional language with formulas *FOR* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\to, \&, \wedge, \vee$, and constants $\mathbf{f}, \mathbf{t}$, with defined connectives:

df1. $\neg\varphi := \varphi \to \mathbf{f}$, and
df2. $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

We moreover define $\varphi^n_{\mathbf{t}}$ as $\varphi_{\mathbf{t}} \& \ldots \& \varphi_{\mathbf{t}}$, $n$ factors, where $\varphi_{\mathbf{t}} := \varphi \wedge \mathbf{t}$, and similarly for $\varphi^n$.

[1] Here, we regard $\mathbf{R}^{\mathbf{t}}$ (the **R** with the constant **t**) as **R**. Often in the literature of relevance logic, **R** is used for the **t**-free fragment of $\mathbf{R}^{\mathbf{t}}$. One reason for that is that $\mathbf{R}^{\mathbf{t}}$ proves formulas such as ($\gamma$) $(\varphi \wedge \mathbf{t}) \to (\mathbf{t} \vee \psi)$ and so seems not to satisfy the old relevance principles (cf. see [5]). However, we have to mention that, in the literature of relevance logic (e.g., [2]), the constant **t** is interpreted as the conjunction of all true sentences. Thus, ($\gamma$) does implicitly satisfy SRP, and so the relevance principles in a sense do not fail in $\mathbf{R}^{\mathbf{t}}$. Hence, here we assume that such formulas satisfy SRP. We shall, in Sect. 3, introduce NSRP and NWRP as principles allowing implicit variable sharing.

We start with the following axiomatization of **RMAILL** (Relevant multiplicative additive intuitionistic linear logic) as the basic relevance logic defined here.[2]

**Definition 1.** ***RMAILL*** *consists of the following axiom schemes and rules:*

| | |
|---|---|
| *A1.* $\varphi \to \varphi$ | *(self-implication, SI)* |
| *A2.* $(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \psi) \to \psi$ | *($\wedge$-elimination, $\wedge$-E)* |
| *A3.* $((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$ | *($\wedge$-introduction, $\wedge$-I)* |
| *A4.* $\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$ | *($\vee$-introduction, $\vee$-I)* |
| *A5.* $((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$ | *($\vee$-elimination, $\vee$-E)* |
| *A6.* $(\varphi \& \psi) \to (\psi \& \varphi)$ | *(&-commutativity, &-C)* |
| *A7.* $(\varphi \& \boldsymbol{t}) \leftrightarrow \varphi$ | *(push and pop, PP)* |
| *A8.* $(\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \& \psi) \to \chi)$ | *(residuation, RE)* |
| *A9.* $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ | *(suffixing, SF)* |
| *A10.* $\varphi \vee \neg\varphi$ | *(excluded middle, EM)* |
| $\varphi \to \psi, \varphi \vdash \psi$ | *(modus ponens, mp)* |
| $\varphi, \psi \vdash \varphi \wedge \psi$ | *(adjunction, adj)* |

*Relevant uninorm logic* **RUL**, the basic relevant fuzzy logic defined here, is **RMAILL** extended with the "prelinearity" axiom scheme below.

**Definition 2.** ***RUL*** *is* ***RMAILL*** *plus*
*A11.* $(\varphi \to \psi)_{\boldsymbol{t}} \vee (\psi \to \varphi)_{\boldsymbol{t}}$ *($PL_{\boldsymbol{t}}$)*

Relevant fuzzy logics are defined by extending **RUL** with suitable axiom schemes as follows:

**Definition 3.** *A logic is an* axiomatic extension *(*extension *for short) of* ***L*** *if and only if (iff) it results from the addition of axiom scheme(s) to* ***L***. *In particular, the following are relevant fuzzy logics extending* ***RUL***:

- *Involutive RUL* ***RIUL*** *is* ***RUL*** *plus (DNE)* $\neg\neg\varphi \to \varphi$.
- *Idempotent RUL* ***RUML*** *is* ***RUL*** *plus (ID)* $(\varphi \& \varphi) \leftrightarrow \varphi$.
- *Involutive RUML* ***RIUML*** *is* ***RIUL*** *plus (ID) and (FP)* $\boldsymbol{t} \leftrightarrow \boldsymbol{f}$[3].

The system **LRW** is the $\mathbf{FL}_e$ with (DNE). Fuzzy relevance logics are defined by extending **LRW** or **RMAILL** with suitable axiom schemes as follows.

**Definition 4.** *The following are fuzzy relevance logics extending* ***LRW***:

- *Fuzzy LRW* ***FRW*** *is* ***LRW*** *plus A11 and (EM).*
- *Fuzzy LR* ***FR*** *is* ***FRW*** *plus (SIN)* $\varphi \to (\varphi \& \varphi)$.
- *Fuzzy LRM* ***FRM*** *(=* ***RM****) is* ***FR*** *plus (SDE)* $(\varphi \& \varphi) \to \varphi$.

For easy reference, we let Ls be the set of fuzzy-relevance logics defined previously.

[2]The systems **RMAILL** and **MAILL** are the $\mathbf{FL}_e$ with (EM) and $\mathbf{FL}_{e\perp}$, respectively, (cf. see [5]).

**Definition 5.** *Ls* = {***RUL***, ***RIUL*** *(=* ***FRW****),* ***RUML****,* ***RIUML****,* ***FR****,* ***RM*** *(=* ***FRM****)*}.

The relevant (local) deduction theorem (R(L)DT) for L is as follows:

**Proposition 1.** *Let $T$ be a theory over $L$ ($\in Ls$) and $\varphi, \psi$ be formulas.*

*(i)* (*RLDT*) $T \cup \{\varphi\} \vdash_L \psi$ *iff there is $n$ such that* $T \vdash_L \varphi^n_{\boldsymbol{t}} \to \psi$.

*(ii)* (*RDT*) *For L with (SIN),* $T \cup \{\varphi\} \vdash_L \psi$ *iff* $T \vdash_L \varphi_{\boldsymbol{t}} \to \psi$.

## 2.2 Semantics

Suitable algebraic structures for the (fuzzy-)relevance logics are obtained as varieties of residuated lattices in the sense of [5].

**Definition 6.** *A* pointed commutative residuated lattice *is a structure* $(A, t, f, \wedge, \vee, *, \to)$ *such that*[4]*:*

*(I)* $(A, \wedge, \vee)$ *is a lattice.*

*(II)* $(A, *, t)$ *is a commutative monoid.*

*(III)* $y \leq x \to z$ *iff* $x * y \leq z$, *for all* $x, y, z \in A$ *(residuation).*

*(IV)* $f$ *is an arbitrary element of* $A$.

As $\varphi^n$ in Sect. 2.1, by $x^n$, we denote $x * \ldots * x$, $n$ factors.
Note that the class of pointed commutative residuated lattices characterizes the system $\mathbf{FL}_e$. Thus, we henceforth call such residuated lattices *$FL_e$-algebras*.

**Definition 7.** *Let* $\neg x := x \to f$, *and* $x_t := x \wedge t$.

*(i)* (*RMAILL-algebra*) *An* RMAILL-algebra *is a pointed commutative residuated lattice satisfying the condition:*
(*EM*) $t \leq x \vee \neg x$.

*(ii)* (*RUL-algebra*) *An* RUL-algebra *is an RMAILL-algebra satisfying the condition:*
($PL_t$) $t \leq (x \to y)_t \vee (y \to x)_t$.

In an analogy to Definition 7, we can define algebras corresponding to the systems introduced in Definitions 3 and 4. As in Sect. 2.1, for brevity, by *L-algebra(s)*, we henceforth ambiguously express algebras corresponding to all L systems.

**Definition 8.** (*Evaluation*) *Let* $\mathcal{A}$ *be an L-algebra. An* $\mathcal{A}$-evaluation *is a function* $v : FOR \to \mathcal{A}$ *satisfying:* $v(\varphi \to \psi) = v(\varphi) \to v(\psi)$, $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$, $v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$, $v(\varphi \& \psi) = v(\varphi) * v(\psi)$, $v(\boldsymbol{t}) = t$, $v(\boldsymbol{f}) = f$, *(and hence* $v(\neg\varphi) = \neg v(\varphi)$*).*

[4] A lattice does not have to have top and bottom elements $\top$ and $\bot$, and so $t$ and $f$ need not be the same as $\top$ and $\bot$, respectively, in pointed commutative residuated lattices. Note that lattices having $\top$ and $\bot$ are called *bounded* lattices (see (I′) in Sect. 4).

**Definition 9.** ([3]) *Let $\mathcal{A}$ be an L-algebra, $T$ a theory, $\varphi$ a formula, and $\mathcal{K}$ a class of L-algebras.*

*(i)* (*Tautology*) *$\varphi$ is a $t$-*tautology *in $\mathcal{A}$, briefly an $\mathcal{A}$-*tautology *(or $\mathcal{A}$-*valid*), if $v(\varphi) \geq t$ for each $\mathcal{A}$-evaluation $v$.*

*(ii)* (*Model*) *An $\mathcal{A}$-evaluation $v$ is an $\mathcal{A}$-*model *of $T$ if $v(\varphi) \geq t$ for each $\varphi \in T$. By* Mod(T, $\mathcal{A}$)*, we denote the class of $\mathcal{A}$-models of $T$.*

*(iii)* (*Semantic consequence*) *$\varphi$ is a* semantic consequence *of $T$ w.r.t. $\mathcal{K}$, denoted by $T \models_{\mathcal{K}} \varphi$, if $Mod(T, \mathcal{A}) = Mod(T \cup \{\varphi\}, \mathcal{A})$ for each $\mathcal{A} \in \mathcal{K}$.*

**Definition 10.** (***L**-algebra*, [3]) *Let $\mathcal{A}$, $T$, and $\varphi$ be as in Definition 9. $\mathcal{A}$ is an* **L**-algebra *if, whenever $\varphi$ is L-provable in any $T$ (i.e., $T \vdash_L \varphi$, $L$ an L logic), it is a semantic consequence of $T$ w.r.t. $\{\mathcal{A}\}$ (i.e., $T \models_{\{\mathcal{A}\}} \varphi$, $\mathcal{A}$ a corresponding L-algebra). By* MOD(L)*, we denote the class of **L**-algebras; by* $\text{MOD}^l$(L)*, the class of linearly ordered **L**-algebras. Finally, we write $T \models_L \varphi$ and $T \models^l_L \varphi$ in place of $T \models_{MOD(L)} \varphi$ and $T \models_{MOD^l(L)} \varphi$, respectively.*

Cintula [3] defined *weakly implicative logic* (WIL) as a logic satisfying A1, (mp), transitivity ($\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$), and congruence w.r.t connectives and called a WIL **L** a *fuzzy* logic (i.e., a weakly implicative fuzzy logic, WIFL) if it is complete w.r.t. linearly ordered (corresponding) matrices. He also showed that, for a finitary WIL **L**, the following are equivalent:

(1) **L** is a fuzzy logic.

(2) **L** has the Linear Extension Property, i.e., for each theory $T$, if $T \nvdash \varphi$, then there is a consistent linear theory $T' \supseteq T$ such that $T' \nvdash \varphi$.

(3) **L** has the Prelinearity Property, i.e., for each theory $T$ if $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$, then $T \vdash \chi$.

(4) **L** has the Subdirect Decomposition Property, i.e., each ordered **L**-matrix is a subdirect product of linearly ordered **L**-matrices.

**Theorem 1.** *Let **L** be an RMAILL. Then **L** is a fuzzy logic iff, for each $n$, $\vdash_{\boldsymbol{L}} (\varphi \to \psi)^n_{\boldsymbol{t}} \vee (\psi \to \varphi)^n_{\boldsymbol{t}}$.*

Then, from Theorem 1, we establish the following corollaries:

**Corollary 1.** (*Strong completeness*) *Let $T$ be a theory over $L$ ($\in Ls$) and $\varphi$ a formula. $T \vdash_L \varphi$ iff $T \models^l_L \varphi$.*

**Corollary 2.** *L is a fuzzy logic (in Cintula's sense).*

**Theorem 2.** *(i) L does not satisfy SRP (in [1]).*

*(ii) L satisfies WRP (in [4]).*

**Corollary 3.** *L is a relevance logic (in the weak sense of [4]).*

**Corollary 4.** *L is both a fuzzy logic and a relevance logic.*

### 2.3 Substructural relevance logics (I)

Let **RMAILL**, **RMALL** (= **LRW** plus (EM)), **LR**, **RMAILML**, **LRM**, and **RMALML** be the systems excluding ($PL_{\mathbf{t}}$) from the systems **RUL**, **RIUL** (= **FRW**), **FR**, **RUML**, **RM**, and **RIUML**, respectively. We let $Ls^-$ be the set of these systems, i.e.,

**Definition 11.** $Ls^- = \{\boldsymbol{RMAILL}, \boldsymbol{RMALL}, \boldsymbol{LR}, \boldsymbol{RMAILML}, \boldsymbol{LRM}, \boldsymbol{RMALML}\}$.

Theorem 1 in [3] says that, for a WIL **L**, $T \vdash \varphi$ iff $T \models \varphi$, and so we obtain the following corollary:

**Corollary 5.** (*Strong completeness*) *For each theory $T$ over $L^-$ ($\in Ls^-$) and formula $\varphi$, $T \vdash_{L^-} \varphi$ iff $T \models_{L^-} \varphi$.*

Let us verify the relevance of $\mathrm{L}^-$ ($\in \mathrm{Ls}^-$).

**Proposition 2.** ***LRM*** *and* ***RMALML*** *each proves $\neg(\varphi \to \varphi) \to (\psi \to \psi)$.*

**Corollary 6.** *For $L^- \in \{\boldsymbol{LRM}, \boldsymbol{RMALML}\}$, $L^-$ does not satisfy SRP (in [1]).*

**Theorem 3.** *(i) For $L^- \in \{\boldsymbol{RMAILL}, \boldsymbol{RMALL}, \boldsymbol{RMAILML}, \boldsymbol{LR}\}$, $L^-$ satisfies SRP (in [1]).*

*(ii) For $L^- \in \{\boldsymbol{LRM}, \boldsymbol{RMALML}\}$, $L^-$ satisfies WRP (in [4]).*

**Corollary 7.** *$L^-$ is a relevance logic (in the strong or weak sense of [1, 4]).*

## 3 Propositional constants and new relevance principles

We briefly recall new strong and weak relevance principles introduced in [7], i.e., NSRP and NWRP, because they are unfamiliar to the readers. Before introducing new relevance principles, we introduce their weak versions and related fact in order to help the readers better understand them.

**Definition 12.** *([7])*

*(i)* (*The implicit strong relevance principle, ISRP*) *$\varphi \to \psi$ is a theorem only if $\varphi$ and $\psi$ implicitly share a propositional variable where the word "implicitly" means that we can identify a sharing variable by means of metadefinitions or interpretations such as df3 to df6, df3′, and df4′.*

*(ii)* (*The implicit weak relevance principle, IWRP*) *$\varphi \to \psi$ is a theorem only if (a) $\varphi$ and $\psi$ implicitly share a propositional variable or (b) both $\neg\varphi$ and $\psi$ are theorems.*

Let $\varphi \to \psi$ *satisfy* the relevance principle ISRP (IWRP resp) in a logic **L** if it is a theorem of **L** and its antecedent $\varphi$ and consequent $\psi$ implicitly share a propositional variable (or both the negation of its antecedent and its consequent are theorems). Then we can prove the following:

**Proposition 3.** (*cf.* [7])

*(i)* $(\varphi \wedge \boldsymbol{t}) \to (\boldsymbol{t} \vee \psi)$ *satisfies ISRP in* $\boldsymbol{R}$.

*(ii)* $((\varphi \to \boldsymbol{F})\ \&\ \varphi) \to \psi$ *satisfies ISRP in* $\boldsymbol{R}^{\boldsymbol{T}}$ *and* $\boldsymbol{UL}$.

*(iii)* $((\varphi \to \boldsymbol{F}) \wedge \varphi) \to \psi$ *satisfies ISRP in* $\boldsymbol{R}^{\boldsymbol{T}}$.

*(iv)* $((\varphi \to \varphi) \to \boldsymbol{f}) \to (\psi \to \psi)$ *satisfies ISRP in* $\boldsymbol{IUML}$.

*(v)* $(\varphi \wedge \neg\varphi) \to (\psi \vee \neg\psi)$ *satisfies IWRP in* $\boldsymbol{RM}$.

*(vi)* $\neg(\varphi \to \varphi) \to (\psi \to \psi)$ *satisfies IWRP in* $\boldsymbol{RM}$ *and* $\boldsymbol{RM}^{\boldsymbol{T}}$.

The principles ISRP and IWRP, however, do not prevent us from giving metadefinitions of propositional constants to the systems having their object-definitions. (Note that the constants **T** and **F** can still be interpreted as df5 (**T** = the disjunction of all sentences.) and df6 (**F** = the conjunction of all sentences), respectively, in CL.) Let a propositional constant be *strongly meta-definable* in a logic **L** if it is meta-definable but not object-definable in **L**, e.g., the constants **t** and **f** in **R** and **T** and **F** in $\mathbf{R}^{\mathbf{T}}$; let the antecedent and consequent of an implication *strong implicitly* share a propositional variable if we can establish variable sharing between them by virtue of strong metadefinitions.

**Definition 13.** *([7])*

*(i)* (*The new strong relevance principle, NSRP*) $\varphi \to \psi$ *is a theorem only if* $\varphi$ *and* $\psi$ *either explicitly or strong implicitly share a propositional variable.*

*(ii)* (*The new weak relevance principle, NWRP*) $\varphi \to \psi$ *is a theorem only if (i)* $\varphi$ *and* $\psi$ *either explicitly or strong implicitly share a propositional variable, or (ii) both* $\neg\varphi$ *and* $\psi$ *are theorems.*

# 4 Fuzzy-relevance logics (II)

## 4.1 Fuzzy-relevance logics with constants T, F

In this section, we introduce several substructural fuzzy-relevance systems satisfying new relevance principles, i.e., NSRP and NWRP. First, we provide axiomatizations of the L with constants **T**, **F**.

**Definition 14.** *(i)* $\boldsymbol{UL}$ *is* $\boldsymbol{RUL}$ *minus (EM) plus constants* $\boldsymbol{F}$, $\boldsymbol{T}$, *and*

*A12.* $\boldsymbol{F} \to \varphi$ *(ex falsum quodlibet, EF)*

*A13.* $\varphi \to \boldsymbol{T}$ *(verum ex quolibet, VE)*

*(ii)* $\boldsymbol{IUL}$ *(=* $\boldsymbol{FRW}^{\boldsymbol{T}}$*) is* $\boldsymbol{UL}$ *plus (DNE).*

*(iii)* $\boldsymbol{FR}^{\boldsymbol{T}}$ *is* $\boldsymbol{IUL}$ *plus (SIN).*

*(iv)* ***UML*** *is* ***UL*** *plus (ID).*

*(v)* $\boldsymbol{RM^{T}}$ *is* ***UML*** *plus (DNE).*

*(vi)* ***IUML*** *is* $\boldsymbol{RM^{T}}$ *plus (FP).*

**Definition 15.** $Ls^{\boldsymbol{T}} = \{$***UL***, ***IUL*** *(=* $\boldsymbol{FRW^{T}}$*)*, $\boldsymbol{FR^{T}}$, ***UML***, $\boldsymbol{RM^{T}}$, ***IUML***$\}$.

**Proposition 4.** *(i) L (∈ Ls) proves*
*(1)* $(\varphi \wedge \boldsymbol{t}) \to (\boldsymbol{t} \vee \psi)$

*(ii)* $L^{\boldsymbol{T}}$ *(∈* $Ls^{\boldsymbol{T}}$*) proves*
*(1)* $(\varphi \wedge \neg\varphi) \to (\psi \vee \neg\psi)$
*(2)* $((\varphi \to \boldsymbol{F})\ \&\ \varphi) \to \psi$

*(iii)* $\boldsymbol{FR^{T}}$, ***UML***, $\boldsymbol{RM^{T}}$, *and* ***IUML*** *each proves*
*(1)* $((\varphi \to \boldsymbol{F}) \wedge \varphi) \to \psi$

*(iv)* $\boldsymbol{RM^{T}}$ *and* ***IUML*** *each proves*
*(1)* $\neg(\varphi \to \varphi) \to (\psi \to \psi)$

First, note that, using the standard technique, we can provide algebraic completeness results for $\mathrm{L}^{\mathbf{T}}$ (∈ $\mathrm{Ls}^{\mathbf{T}}$). For this, it suffices to note that $\mathrm{L}^{\mathbf{T}}$-algebras are obtained as varieties of pointed *bounded* commutative residuated lattices, i.e., for $\mathrm{L}^{\mathbf{T}}$-algebras, it suffices to replace the condition (I) in Definition 6 with

(I′) $(A, \top, \bot, \wedge, \vee)$ is a *bounded* lattice with top element $\top$ and bottom element $\bot$.

Since the condition (I′) can be defined in equations, it is clear that the class of $\mathrm{L}^{\mathbf{T}}$-algebras forms a variety. Then, as in [6], we can show that $\mathrm{L}^{\mathbf{T}}$ is complete w.r.t. an algebraic semantic given by a variety of $\mathrm{L}^{\mathbf{T}}$-algebras. Moreover, as in Sect. 2.2, we can prove the following:

**Theorem 4.** *Let* ***L*** *be a multiplicative additive intuitionistic linear logic. Then* ***L*** *is a fuzzy logic iff, for each n,* $\vdash_{\boldsymbol{L}} (\varphi \to \psi)^n_{\boldsymbol{t}} \vee (\psi \to \varphi)^n_{\boldsymbol{t}}$.

**Corollary 8.** (*Strong completeness*) *Let T be a theory over* $L^{\boldsymbol{T}}$ *(∈* $Ls^{\boldsymbol{T}}$*) and* $\varphi$ *a formula.* $T \vdash_{L^{\boldsymbol{T}}} \varphi$ *iff* $T \models^{l}_{L^{\boldsymbol{T}}} \varphi$.

**Corollary 9.** $L^{\boldsymbol{T}}$ *is a fuzzy logic (in Cintula's sense).*

**Proposition 5.** *(i)* $(\varphi \wedge \boldsymbol{t}) \to (\boldsymbol{t} \vee \psi)$ *satisfies NSRP in L (∈ Ls) and* $L^{\boldsymbol{T}}$ *(∈* $Ls^{\boldsymbol{T}}$*).*

*(ii)* $(\varphi \wedge (\varphi \to \boldsymbol{f})) \to (\psi \vee (\psi \to \boldsymbol{f}))$ *satisfies NSRP in* $L^{\boldsymbol{T}}$ *(∈* $Ls^{\boldsymbol{T}}$*).*

*(iii)* $((\varphi \to \boldsymbol{F})\ \&\ \varphi) \to \psi$ *satisfy NSRP in* $L^{\boldsymbol{T}}$ *(∈* $Ls^{\boldsymbol{T}}$*).*

*(iv)* $((\varphi \to \boldsymbol{F}) \wedge \varphi) \to \psi$ *satisfy NSRP in* $L^{\boldsymbol{T}} \in \{\boldsymbol{FR^T}, \boldsymbol{UML}, \boldsymbol{RM^T}, \boldsymbol{IUML}\}$.

*(v)* $\neg(\varphi \to \varphi) \to (\psi \to \psi)$ *satisfies NSRP in* ***IUML***.

*(vi)* $\neg(\varphi \to \varphi) \to (\psi \to \psi)$ *satisfies NWRP in* $\boldsymbol{RM^T}$.

**Theorem 5.** *(i) For* $L^{\boldsymbol{T}} \in Ls^{\boldsymbol{T}} \setminus \{\boldsymbol{RM^T}\}$, $L^{\boldsymbol{T}}$ *satisfies NSRP.*

*(ii)* $\boldsymbol{RM^T}$ *satisfies NWRP.*

**Corollary 10.** *(i) For* $L^{\boldsymbol{T}} \in Ls^{\boldsymbol{T}} \setminus \{\boldsymbol{RM^T}\}$, $L^{\boldsymbol{T}}$ *is a relevance logic (in the new strong sense of [7]).*

*(ii)* $\boldsymbol{RM^T}$ *is a relevance logic (in the new weak sense of [7]).*

**Corollary 11.** $L^{\boldsymbol{T}}$ *is both a fuzzy logic and a relevance logic.*

## 4.2 Substructural relevance logics (II)

Let **MAILL**, **MALL** (= $\mathbf{LRW^T}$), $\mathbf{LR^T}$, **MAILML**, $\mathbf{LRM^T}$, and **MALML** be the systems eliminating ($PL_{\mathbf{t}}$) from the systems **UL**, **IUL** (= $\mathbf{FRW^T}$), $\mathbf{FR^T}$, **UML**, $\mathbf{RM^T}$, and **IUML**, respectively. We let $\mathrm{Ls}^{\mathbf{T}-}$ be the set of these systems, i.e.,

**Definition 16.** $Ls^{\boldsymbol{T}-} = \{\boldsymbol{MAILL}, \boldsymbol{MALL}, \boldsymbol{LR^T}, \boldsymbol{MAILML}, \boldsymbol{LRM^T}, \boldsymbol{MALML}\}$.

Theorem 1 in [3] says that, for a WIL **L**, $\mathrm{T} \vdash \varphi$ iff $\mathrm{T} \models \varphi$, and so we obtain the following corollary:

**Corollary 12.** (*Strong completeness*) *For each theory* $T$ *over* $L^{\boldsymbol{T}}$ ($\in Ls^{\boldsymbol{T}-}$) *and formula* $\varphi$, $T \vdash_{L^{\boldsymbol{T}-}} \varphi$ *iff* $T \models_{L^{\boldsymbol{T}-}} \varphi$.

Let us verify the relevance of $\mathrm{L}^{\mathbf{T}-}$ ($\in \mathrm{Ls}^{\mathbf{T}-}$).

**Proposition 6.** $\boldsymbol{LRM^T}$ *and* ***MALML*** *each proves* $\neg(\varphi \to \varphi) \to (\psi \to \psi)$.

**Corollary 13.** $\boldsymbol{LRM^T}$ *does not satisfy NSRP (in [7]).*

**Theorem 6.** *(i) For* $L^{\boldsymbol{T}-} \in \{\boldsymbol{MAILL}, \boldsymbol{MALL}, \boldsymbol{LR^T}, \boldsymbol{MAILML}, \boldsymbol{MALML}\}$, $L^{-}$ *satisfies NSRP (in [7]).*

*(ii)* $\boldsymbol{LRM^T}$ *satisfies NWRP (in [7]).*

**Corollary 14.** $L^{\boldsymbol{T}-}$ *is a relevance logic (in the new strong or weak sense of [7]).*

# 5 Concluding remarks

We introduced several fuzzy-relevance logics with and without constants **T**, **F** and provided completeness results for them by showing that such logics are WI-FLs. We furthermore proved that they satisfy old and new relevance principles. In addition, we considered relevance logics obtained from the fuzzy-relevance logics by omitting prelinearity. All of the systems investigated here are extensions of the substructural logic $\mathbf{FL}_e$ and so they are all substructural logics. They also have the associative intensional conjunction (so called fusion) &. Therefore, such systems all can be called *associative (fuzzy-)relevance logics*.

The fuzzy-relevance logics without constants **T** and **F** are not characterized by models based on uninorms. Note that the uninorm-based systems introduced in [6] have constants **T** and **F**, and the systems with **T** and **F** investigated here are not relevant in the old senses. This implies that, as far as *uninorm* (based) systems have **T** and **F**, they cannot be relevant in the old senses and so are not fuzzy-relevance logics in the old senses.

# References


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