Generalized Ultraproduct and Kirman-Sondermann Correspondence for Vote Abstention *

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Abstract. The present paper refines Herzberg and Eckert's model-theoretic approach to aggregation. The proposed refinement is aimed at naturally accounting for vote abstention, and is technically based on a more general notion of ultraproduct than the one standardly occurring in model theory textbooks. Unlike the standard ultraproduct construction, which yields the empty model as soon as any one single coordinate features the empty model, this generalized ultraproduct construction faithfully reflects the indication of 'large sets'. Thus, our proposed refinement naturally accounts for those situations in which e.g. a voting round is non-null if and only if a 'large set' of voters actually participate in the vote. In the present setting, Arrow's impossibility theorem also covers 'elections with only two candidates'.
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1 Introduction

The present paper pertains to a line of research in social choice theory aimed at understanding the logical underpinning of Arrow's impossibility theorem [1] and also exploring its scope, as well as extending Arrow-type results to infinite electorates.

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This line of research originates in the work of Kirman and Sondermann [11], which characterizes the so-called Arrow-rational social welfare functions by establishing a bijective correspondence between them and the collection of ultrafilters over the set of individuals. The Kirman-Sonderman correspondence hinges on the fact that the decisive coalitions associated with any Arrow-rational social welfare function form an ultrafilter over the set of individuals. Herzberg and Eckert [8] gave a very elegant generalization of the Kirman-Sondermann correspondence in a model-theoretic setting by characterizing Arrow-rational social welfare functions as exactly those defined in terms of an ultraproduct construction parametrized by the ultrafilter of their associated decisive coalitions.

In the literature on social choice, there are several ways to treat abstention[4]. The first approach is to ignore any voters that abstain, and thus working in a variable domain model (see Pivato [14]). The second approach is to treat abstention as if the voters ranked all candidates equally. The third approach is to treat abstention as a separate type of input that may be elicited from a voter. This means that there are two types of inputs that voters may submit: ranking of candidates or abstention. Our method belongs to the third approach.

In the present paper, the results in [8] are extended to a setting in which the assumption that every individual votes/expresses a judgment is dropped. Allowing the empty model to occur in profiles is a natural way to formalize the vote abstention of the corresponding individual. However, the standard model-theoretic notion of ultraproduct is not amenable to support this natural formalization of vote abstention, given that it is enough for a coordinate to be empty for the standard ultraproduct construction to yield the empty set/model. This would correspond to situations in which the abstention of one voter would be enough to declare the voting round null. While this is true in some situations, there are many settings (e.g. referenda) in which the voting round is declared null unless a certain *quorum* of voters is met. Technically, the contribution of the present paper is based on replacing the standard model-theoretic ultraproduct construction with a generalized one, introduced by Makkai [13] in a category-theoretic setting. The main advantage of Makkai's ultraproduct is that it yields the empty model unless nonempty models occur in each coordinate belonging to some member of its associated ultrafilter. In this respect, Makkai's ultraproduct reflects more faithfully than the standard one the indications of the 'large sets' of the ultrafilter.

We observe that, in the extended setting accounting for vote abstention, Arrow's impossibility theorem strengthens. Indeed, the usual assumption, also required in [8], on the existence of three non-isomorphic models of the theory is dropped, and replaced by the weaker requirement on the existence of two non-isomorphic

[4] Notice that we use the term "abstention" in a way which is different from how it is typically used in the social choice literature. In particular, abstention does not mean being indifferent between two options (this would correspond, in our setting, to allowing the model associated with any voter to be a partial but not necessarily linear order). By abstention, we mean that voters do not take part in the voting process altogether.

models. This allows us to extend e.g. Arrow's impossibility theorem [1] to a setting of elections with only two candidates (cf. discussions at the end of Section 2).
Finally, from a more methodological perspective, besides allowing for the extension of the results in [8] to a setting accounting for vote abstention, Makkai's ultraproduct construction lends itself to connecting the model-theoretic approach to judgment aggregation to the algebraic and category-theoretic approaches in [4], [7], [9] and [10]. Establishing these systematic connections is the focus of ongoing research.

Structure of the paper. In Section 2, preliminaries are collected about Arrow-rational aggregators, the first leg of the generalized Kirman-Sondermann correspondence is introduced, and the Arrow's impossibility theorem for vote abstention is briefly discussed. In Section 3, the generalized ultraproduct construction is introduced as a specialization of Makkai's general definition to the present model-theoretic setting. In Section 4, relevant properties are collected of the generalized ultraproduct construction. In Section 5, the second leg of the generalized Kirman-Sondermann correspondence is introduced, and the proof of the Kirman-Sondermann isomorphism is given. In Section 6, the case study of preference aggregation in the setting of vote abstention is discussed.

2 Arrow-rational aggregators

Fix a first-order language $\mathcal{L}$, consisting of identity $\approx$, constant symbols c for each element in a given non-empty set A and of relation symbols R each of which of finite arity $k = k(R)$. Let $\mathcal{S}$ denote the set of atomic $\mathcal{L}$-formulas, and $\mathcal{I}$ the Boolean closure of $\mathcal{S}$. Fix a consistent set T of universal $\mathcal{L}$-sentences, let Ω be the class of models M of T the domain of which coincides with the subset A^M of the interpretations in M of the constant symbols in $\mathcal{L}$. In what follows, we will always consider models up to isomorphism. Hence, models in Ω can be thought of as equivalence classes of isomorphic models. We let $|\Omega|$ denote the cardinality of Ω modulo isomorphism. We will denote $\mathcal{L}$-structures by $\mathcal{B}$, and elements in Ω by M, N, possibly with subscripts or superscripts. Sometimes, abusing notation, we will use M, N for elements in $\Omega \cup \{\varnothing\}$. We let $R, \ldots, c, \ldots$ denote the symbols in the language $\mathcal{L}$ and let $R^{\mathcal{B}}, \ldots, c^{\mathcal{B}}, \ldots$ denote the corresponding semantic object in the $\mathcal{L}$-structure $\mathcal{B}$. For each $\mathcal{L}$-structure $\mathcal{B}$, let $A^{\mathcal{B}} := \{c^{\mathcal{B}} \mid \text{c constant symbol in } \mathcal{L}\}$. We let $|M|$ denote the domain of M. As usual, for any model M and formula λ, we write $M \models \lambda$ to indicate that λ is true of M.
The extra assumption that the universe of each model M in Ω is the set $A^M = \{c^M \mid \text{c is a constant symbol in } \mathcal{L}\}$ guarantees the following

Fact 1. *Any two models $M_1, M_2 \in \Omega \cup \{\varnothing\}$ such that $M_1 \models \lambda$ iff $M_2 \models \lambda$ for any $\lambda \in \mathcal{I}$ are isomorphic.*

Proof. The claim trivially holds both when M_1 and M_2 coincide with the empty set, and when only one of the two coincides with the empty set (in the latter case the assumptions do not hold: indeed, the sentence $c \approx c$ for any constant symbol c holds of the nonempty model and does not hold of the empty model). If M_1 and M_2 are both nonempty, then their domains bijectively correspond: indeed, $|M_1| = A^{M_1} \cong A^{M_2} = |M_2|$. By definition, this bijective correspondence identifies the interpretations of all constant symbols. Since by assumption $M_1 \models R(c_1, \ldots c_k)$ iff $M_2 \models R(c_1, \ldots c_k)$ for any relation symbol R and all constant symbols $c_1, \ldots c_k$, it is a straightforward verification that this correspondence identifies also the interpretations of each relation symbol.

Fix a non-empty set I, which we will think of as the set of *individuals*. The subsets of I will be referred to as *coalitions*. Elements $\underline{M} \in (\Omega \cup \{\varnothing\})^I$ are the *profiles*. For any such profile, and any $\lambda \in \mathcal{I}$, the *coalition supporting* λ *given* $\underline{M}$ is the set $C(\underline{M}, \lambda) := \{i \in I \mid M_i \models \lambda\}$.
An *aggregator* is a partial map $f : (\Omega \cup \{\varnothing\})^I \to \Omega \cup \{\varnothing\}$. The domain of f is denoted $dom(f)$.

Definition 1. *(cf. [8], definition before Remark 3.3) An aggregator f is **Arrow-rational** if it satisfies the following conditions:*

(A1) ***Universal Domain**: $dom(f) = (\Omega \cup \{\varnothing\})^I$.*
(A2) ***Generalized Pareto Principle**: for any $\underline{M} \in dom(f)$ and any $\lambda \in \mathcal{I}$,*

$$\text{if } f(\underline{M}) \models \lambda, \text{ then } C(\underline{M}, \lambda) \neq \varnothing.$$

(A3) ***Generalized Systematicity**: for all $\underline{M}, \underline{N} \in dom(f)$ and all $\lambda, \mu \in \mathcal{I}$,*

$$\text{if } C(\underline{M}, \lambda) = C(\underline{N}, \mu), \text{ then } f(\underline{M}) \models \lambda \text{ iff } f(\underline{N}) \models \mu.$$

The collection of Arrow-rational aggregators is denoted by $\mathcal{AR}$.

Definition 2. (Decisive coalition) *For any aggregator f, a coalition $C \subseteq I$ is f-*decisive *if, for any $\lambda \in \mathcal{I}$ and any $\underline{M} \in dom(f)$,*

$$\text{if } C = C(\underline{M}, \lambda), \text{ then } f(\underline{M}) \models \lambda.$$

Let $\mathcal{D}_f$ denote the set of the f-decisive coalitions.

The following lemma is an immediate consequence of the definitions involved:

Lemma 1. *For any aggregator f satisfying (A3), any $\underline{M} \in \mathrm{dom}(f)$ and $\lambda \in \mathcal{I}$,*

$$C(\underline{M}, \lambda) \in \mathcal{D}_f \quad \text{iff} \quad f(\underline{M}) \models \lambda.$$

The following lemma shows that the assignment $f \mapsto \mathcal{D}_f$ defines a map $\Lambda : \mathcal{AR} \to \beta I$, where βI denotes the set of ultrafilters over I. The map Λ provides one direction of the generalized Kirman-Sondermann correspondence we aim at obtaining. The following lemma is a variant of Lemma 5.3 in [8], which assumes the aggregator to be weakly Arrow-rational[5] instead of Arrow-rational, as is done

[5] An aggregator is *weakly Arrow-rational* if it satisfies conditions of (A2), (A3) of Definition 1 and the following condition (A1'): there exist models $M_1, M_2, M_3 \in \Omega$ s.t. $\{M_1, M_2, M_3\}^I \subseteq \mathrm{dom}(f)$, and M_1, M_2, M_3 respectively are models of three pairwise inconsistent $\mathcal{L}$-sentences.

here. Another perhaps more interesting difference is that here we assume that there are at least two non-isomorphic models in Ω, whereas Lemma 5.3 in [8] assumes the existence of at least three non-isomorphic models in Ω. The proof of this lemma can be found in an expanded version of the present paper [2].

Lemma 2. *For any $f \in \mathcal{AR}$, the collection $\mathcal{D}_f$ is an ultrafilter over I.*[6]

Notice that there are significant cases in which the lemma above is not implied by Lemma 5.3 in [8]. The reason is that, in significant cases, Arrow-rationality does not imply weak Arrow-rationality. Indeed, it was shown in [8, Remark 3.2] that condition (A1) implies condition (A1') if $\mu, \nu \in \mathcal{S}$ exist such that $\mu \wedge \nu$, $\mu \wedge \neg\nu$ and $\neg\mu \wedge \nu$ are each consistent with T. In this case, three pairwise different models M_1, M_2, M_3 exist in Ω such that $M_1 \models \mu \wedge \nu$, $M_2 \models \mu \wedge \neg\nu$ and $M_3 \models \neg\mu \wedge \nu$, which then makes (A1) sufficient for (A1'). However, let us provide a significant example in which such μ and ν do not exist, and Arrow-rationality does not imply weak Arrow-rationality. Indeed, let $\mathcal{L}$ consist of one binary relation symbol $<$ and two constant symbols a and b. Let T be the $\mathcal{L}$-theory that says that $<$ is a strict linear order and that there are exactly two alternatives a and b (this example models elections with only two candidates). Then, up to isomorphism, there are exactly two models for T. Hence, in this case, condition (A1) does not imply condition (A1'). Moreover, the assumptions of the lemma above are satisfied by this example, whereas those of Lemma 5.3 in [8] are not.

2.1 Arrow-type impossibility for vote abstention

Definition 3. *An aggregator $f : (\Omega \cup \{\varnothing\})^I \to \Omega \cup \{\varnothing\}$ is* dictatorial *if there exists some $i \in I$ such that $f(\underline{M}) = M_i$ for any profile $\underline{M}$.*

Lemma 3. *Any aggregator $f : (\Omega \cup \{\varnothing\})^I \to \Omega \cup \{\varnothing\}$ satisfying (A3) and such that $\mathcal{D}_f$ is a principal ultrafilter is dictatorial.*

Proof. Let $i_0 \in I$ be the generator of $\mathcal{D}_f$. It is enough to show that $f(\underline{M})$ is isomorphic to M_{i_0} for any profile $\underline{M}$. By Fact 1, it is enough to show that $f(\underline{M}) \models \lambda$ iff $M_{i_0} \models \lambda$ for any $\lambda \in \mathcal{I}$. Indeed, by Lemma 1,

$$f(\underline{M}) \models \lambda \ \text{ iff } \ C(\underline{M}, \lambda) \in \mathcal{D}_f \ \text{ iff } \ M_{i_0} \models \lambda.$$

As an immediate consequence of the lemmas above we obtain:

[6] Recall that, for every non-empty set I, a *filter* $\mathcal{D}$ over I is a collection of subsets of I which is closed under supersets and intersection of finitely many members. A filter $\mathcal{D}$ is *proper* if $\varnothing \notin \mathcal{D}$. An *ultrafilter* over I is a maximal proper filter. Maximality can be equivalently characterized by the following conditions: (a) for any $X \subseteq I$, if $X \notin \mathcal{D}$ then $I \setminus X \in \mathcal{D}$; (b) for all $X, Y \subseteq \mathcal{D}$, if $X \cup Y \in \mathcal{D}$, then either $X \in \mathcal{D}$ or $Y \in \mathcal{D}$. An ultrafilter $\mathcal{D}$ over I is *principal* if it is of the form $\{X \subseteq I \mid i_0 \in X\}$ for some $i_0 \in I$, and is *nonprincipal* otherwise. An immediate consequence is that, if I is finite, all ultrafilters over I are principal.

Corollary 1. *If T is a universal $\mathcal{L}$-theory such that $|\Omega| \geq 2$, then any Arrow-rational aggregator $f : (\Omega \cup \{\varnothing\})^I \to \Omega \cup \{\varnothing\}$ such that the ultrafilter $\mathcal{D}_f$ is principal is dictatorial.*

The assumption $|\Omega| \geq 2$ in the statement of the corollary above is needed in order to apply Lemma 2. As is well known, in the standard setting of Arrow's theorem, the analogous corollary fails for $|\Omega| = 2$, the majority rule being a counterexample. However, notice that, in the present setting in which aggregators are maps $f : (\Omega \cup \{\varnothing\})^I \to \Omega \cup \{\varnothing\}$, the majority rule is not guaranteed anymore to define an aggregator. Indeed, let $I = \{i_1, i_2, i_3\}$ and $A = \{a, b\}$. Let T be the universal theory of two-element linear orders (cf. Section 6).
Then Ω consists, up to isomorphism, of the models M_a (the one in which a is preferred to b, that is, in which Rab is true), and M_b (the one in which b is preferred to a, that is, in which Rba is true). No universal aggregator $f : (\Omega \cup \{\varnothing\})^I \to \Omega \cup \{\varnothing\}$ satisfies the following condition:

$$f(\underline{M}) \models \lambda \quad \text{iff} \quad |\{i \mid M_i \models \lambda\}| > |\{i \mid M_i \models \neg\lambda\}|. \tag{2.1}$$

Indeed, consider the input $\underline{M} = (M_a, M_b, \varnothing)$ and the sentences Rab, Rba and $a \equiv a$. Clearly, $\{i \mid M_i \models Rab\} = \{i_1\}$, $\{i \mid M_i \models Rba\} = \{i_2\}$ and $\{i \mid M_i \models a \equiv a\} = \{i_1, i_2\}$. If f satisfies (2.1), this implies that $f(\underline{M}) \models \neg Rab$, $f(\underline{M}) \models \neg Rba$ and $f(\underline{M}) \models a \equiv a$. However, none of $M_a, M_b, \varnothing$ satisfy the three sentences simultaneously, therefore f cannot be well-defined at $\underline{M} = (M_a, M_b, \varnothing)$, and thus f cannot be universal.

3 Generalized ultraproduct construction

The remainder of the paper is aimed at providing a setting which incorporates the Arrow-type impossibility result for vote abstention as a special case. Towards this aim, in the present section a construction is introduced which, for each (ultra)filter $\mathcal{D}$ over I and each profile $\underline{M} \in (\Omega \cup \{\varnothing\})^I$, yields an $\mathcal{L}$-model $U(\underline{M}, \mathcal{D})$. This construction amounts to the specialization of Makkai's ultraproduct construction (cf. [13, Section 1.3]) from a more general category-theoretic setting to the model-theoretic setting of interest here. In the remainder of this subsection we fix a set I and an (ultra)filter $\mathcal{D}$ over I.
We find it useful to make use of the following auxiliary definition: for any I-indexed family of sets $\underline{S} = \{S_i \mid i \in I\}$, let the *generalized union product of $\underline{S}$* be defined as follows:

$$GUP_{\mathcal{D}}(\underline{S}) := \coprod_{J \in \mathcal{D}} \prod_{j \in J} S_j = \bigcup \{\{(s_i)_{i \in J} \mid s_i \in S_i\} \mid J \in \mathcal{D}\}.$$

Notice that we are not excluding S_i to be empty for some $i \in I$. This definition naturally applies also to I-indexed families $\underline{R} = \{R_i \mid i \in I\}$ where R_i is a k-ary relation (for a fixed $k \geq 1$) on a given set S_i for each $i \in I$:[7]

$$GUP_{\mathcal{D}}(\underline{R}) := \coprod_{J \in \mathcal{D}} \prod_{j \in J} R_j = \bigcup \{\{(\overline{s}_i)_{i \in J} \mid \overline{s}_i \in R_i\} \mid J \in \mathcal{D}\}.$$

[7] In this case, we will say that $\underline{R}$ is a family of k-ary relations over $\underline{S}$.

The definition above also applies when $k = 0$, if we regard any element $c_i \in S_i$ as a 0-ary relation R_i on S_i.[8] Under this stipulation, I-indexed families $\underline{R}$ of 0-ary relations can be identified with I-indexed sequences $\underline{c} = (x_i)_{i\in I}$ such that for every $i \in I$

$$x_i = \begin{cases} c_i & \text{if } c_i \in S_i \\ * & \text{if } S_i = \varnothing, \end{cases}$$

where $* \notin \bigcup_{i\in I} S_i$. Then, for every $J \in \mathcal{D}$, the product set $\prod_{j\in J} R_j$ reduces to the sequence $(x_j)_{j\in J}$, and hence

$$GUP_{\mathcal{D}}(\underline{c}) := \coprod_{J\in\mathcal{D}} \prod_{j\in J} R_j = \bigcup\{(x_j)_{j\in J} \mid J \in \mathcal{D}\}.$$

For the sake of readability, we will drop the subscripted $\mathcal{D}$ when this causes no confusion. Clearly, $GUP(\underline{c}) \cap GUP(\underline{S}) \neq \varnothing$ iff some $J \in \mathcal{D}$ exists such that $c_j \in S_j$ for every $i \in J$.
Notice that if $\underline{R}$ is an I-indexed family of k-ary relations over $\underline{S}$, then $GUP(\underline{R})$ is not a k-ary relation on $GUP(\underline{S})$. Fortunately, this situation can be remedied as follows. For any set S and $k \geq 1$, let S^k denote the k-ary universal relation on S. The following isomorphism holds for any $J \in \mathcal{D}$ and any $k \geq 1$:

$$\sigma_J : \prod_{j\in J}(S_j)^k \longrightarrow (\prod_{j\in J} S_j)^k$$

which maps the J-indexed array $(\overline{s}_j)_{j\in J}$ of k-tuples $\overline{s}_j = (s_1^j, \ldots, s_k^j) \in (S_j)^k$ to the k-tuple of J-indexed arrays $((s_1^j)_{j\in J}, \ldots, (s_k^j)_{j\in J})$. Since $\prod_{j\in J} R_j \subseteq \prod_{j\in J}(S_j)^k$, the σ_J-direct image of $\prod_{j\in J} R_j$ is a k-ary relation:

$$\sigma_J[\prod_{j\in J} R_j] \subseteq (\prod_{j\in J} S_j)^k.$$

Hence, $GUP(\underline{R})$ induces the k-ary relation

$$GUP'(\underline{R}) := \bigcup\{\sigma_J[\prod_{j\in J} R_j] \mid J \in \mathcal{D}\} \subseteq (GUP(\underline{S}))^k.$$

Consider the equivalence relation on $GUP(\underline{S})$[9] defined as follows:

$$(s_j)_{j\in J} \equiv^{\mathcal{D}}_{\underline{S}} (t_h)_{h\in H} \quad \text{iff} \quad \{i \in J \cap H \mid s_i = t_i\} \in \mathcal{D}.$$

Definition 4. *(cf. [13], Section 1.3) For any profile $\underline{M} \in (\Omega \cup \{\varnothing\})^I$, the* ***generalized ultraproduct*** *of $\underline{M}$ over $\mathcal{D}$ is the $\mathcal{L}$-model $U = U(\underline{M}, \mathcal{D})$ specified as follows:*

[8] Regarding elements $c \in S$ as 0-ary relations on S departs from the usual convention in model theory, according to which 0-ary relations are truth-values.

[9] For ease of notation, we will often drop the subscript in $\equiv^{\mathcal{D}}_{\underline{S}}$ and rely on the context for its correct interpretation.

– *the universe* $|U(\underline{M},\mathcal{D})|$ *of* $U(\underline{M},\mathcal{D})$ *is*

$$U(\underline{S},\mathcal{D}) := GUP(\underline{S})/{\equiv^{\mathcal{D}}_{\underline{S}}},$$

where $\underline{S} = \{|M_i| \mid i \in I\}$*;*

– *for any constant symbol* c*,*

$$c^U = c^{U(\underline{M},\mathcal{D})} := [(x_j)_{j\in J}]_{\equiv^{\mathcal{D}}_{\underline{S}}};$$

where $(x_j)_{j\in J} \in GUP(\underline{c}) \cap GUP(\underline{S})$*, and* $\underline{c} = (x_i)_{i\in I}$ *such that for every* $i \in I$

$$x_i = \begin{cases} c^{M_i} & \text{if } M_i \neq \varnothing \\ * & \text{otherwise };\end{cases}$$

– *for any* k*-ary relation symbol* R $(k \geq 1)$*, the* k*-ary relation* $R^U = R^{U(\underline{M},\mathcal{D})}$ *on* U *is defined as follows:*

$$([(s_1^j)_{j\in J_1}]_{\equiv^{\mathcal{D}}_{\underline{S}}}, \ldots, [(s_k^j)_{j\in J_k}]_{\equiv^{\mathcal{D}}_{\underline{S}}}) \in R^U \qquad \text{iff} \qquad ((t_1^j)_{j\in J}, \ldots (t_k^j)_{j\in J}) \in GUP'(\underline{R})$$

for some $J \in \mathcal{D}$ *and some* $(t_1^j)_{j\in J}, \ldots, (t_k^j)_{j\in J}$ *such that, for every* $1 \leq \ell \leq k$*,*

$$(t_\ell^j)_{j\in J} \equiv^{\mathcal{D}}_{\underline{S}} (s_\ell^j)_{j\in J_\ell}.$$

Notice that the elements of $GUP(\underline{c}) \cap GUP(\underline{S})$ are all identified by $\equiv^{\mathcal{D}}_{\underline{S}}$, so c^U is well-defined. Notice also that c^U is defined only if $GUP(\underline{c}) \cap GUP(\underline{S}) \neq \varnothing$, and as discussed early on, this is the case iff some $J \in \mathcal{D}$ exists such that $M_j \neq \varnothing$ for every $j \in J$. On the other hand, as we will discuss next (cf. Fact 2), this condition also characterizes the non-emptiness of $U(\underline{M},\mathcal{D})$.

4 Properties of the generalized ultraproduct construction

Let $\underline{S}$ be an I-indexed family of sets. For any ultrafilter $\mathcal{D}$ over I and any $J \in \mathcal{D}$, if $S_i = \varnothing$ for some $i \in J$, then $\prod_{i\in J} S_i = \varnothing$. Hence:

Fact 2. *For every* I*-indexed family of sets* $\underline{S}$ *and any ultrafilter* $\mathcal{D}$ *over* I*,*

$$GUP(\underline{S}) \neq \varnothing \quad \text{iff} \quad \text{some } J \in \mathcal{D} \text{ exists s.t. } S_i \neq \varnothing \text{ for all } i \in J.$$

Recall that if $\mathcal{D}$ is a principal ultrafilter, $\mathcal{D}$ is generated by the singleton $\{i_0\}$ for some individual $i_0 \in I$, which can be identified with the *dictator*. The following fact is an immediate consequence of the fact above:

Fact 3. *For every profile* $\underline{M}$ *and any principal ultrafilter* $\mathcal{D}$ *over* I*,*

$$U(\underline{M},\mathcal{D}) = \varnothing \qquad \text{iff} \qquad M_{i_0} = \varnothing. \tag{4.1}$$

Definition 4 generalizes the following

Definition 5. *For any (ultra)filter $\mathcal{D}$ on I, and any profile $\underline{M} \in \Omega^I$, the* ***standard ultraproduct*** *of $\underline{M}$ over $\mathcal{D}$ is the $\mathcal{L}$-model $U' = U'(\underline{M}, \mathcal{D})$ specified as follows:*

- *the universe $|U'(\underline{M}, \mathcal{D})|$ of $U'(\underline{M}, \mathcal{D})$ is*
$$\prod_{i \in I} M_i / \sim_{\mathcal{D}},$$
where for any $(s_i)_{i\in I}, (t_i)_{i \in I} \in \prod_{i\in I} M_i$,
$$(s_i)_{i\in I} \sim_{\mathcal{D}} (t_i)_{i\in I} \quad \textit{iff} \quad \{i \in I \mid s_i = t_i\} \in \mathcal{D};$$
- *for any constant symbol c,*
$$c^{U'} := [\underline{c}]_{\sim_{\mathcal{D}}}$$
where $\underline{c} = (c^{M_i})_{i\in I}$;
- *for any k-ary relation symbol R ($k \geq 1$), the k-ary relation $R^{U'} = R^{U'(\underline{M},\mathcal{D})}$ on U' is defined as follows:*
$$\big([(s^i_1)_{i\in I}]_{\sim_{\mathcal{D}}}, \ldots, [(s^i_k)_{i\in I}]_{\sim_{\mathcal{D}}}\big) \in R^{U'} \quad \textit{iff} \quad \{i \in I \mid (s^i_1, \ldots, s^i_k) \in R^{M_i}\} \in \mathcal{D}.$$

The definition above is in general different from Definition 4. Indeed, if $M_i = \varnothing$ for some $i \in I$, then $U'(\underline{M}, \mathcal{D}) = \varnothing$, while $U(\underline{M}, \mathcal{D})$ does not need to be empty (cf. Fact 2). However, if $M_i \neq \varnothing$ for any $i \in I$, then the two constructions can be identified, as shown in the following

Fact 4. *For any (ultra)filter $\mathcal{D}$ on I, and any profile $\underline{M} \in (\Omega \cup \{\varnothing\})^I$, if $M_i \neq \varnothing$ for every $i \in I$ then $U(\underline{M}, \mathcal{D})$ and $U'(\underline{M}, \mathcal{D})$ are isomorphic.*

Proof. Clearly, for all $(y_i)_{i\in I}$ and $(y'_i)_{i\in I}$,
$$(y_i)_{i\in I} \equiv_{\mathcal{D}} (y'_i)_{i\in I} \quad \text{iff} \quad \{i \in I \mid y_i = y'_i\} \in \mathcal{D} \quad \text{iff} \quad (y_i)_{i\in I} \sim_{\mathcal{D}} (y'_i)_{i\in I}.$$

Moreover, for every $J \in \mathcal{D}$ and for every $(t_j)_{j\in J}$ there exists some $(y_i)_{i\in I}$ s.t. $(t_j)_{j\in J} \equiv_{\mathcal{D}} (y_i)_{i\in I}$: indeed, the assumption that $M_i \neq \varnothing$ for every $i \in I$ guarantees that there exists at least one I-indexed array defined as follows:
$$y_i = \begin{cases} t_i & \text{if } i \in K \\ \text{any } y \in M_i \neq \varnothing & \text{otherwise.} \end{cases}$$

By construction, $\{i \in I \cap J = J \mid y_i = t_i\} = J \in \mathcal{D}$, and hence $(t_j)_{j\in J} \equiv_{\mathcal{D}} (y_i)_{i\in I}$. From the facts above, it follows that the map $\varphi : |U(\underline{M}, \mathcal{D})| \to |U'(\underline{M}, \mathcal{D})|$ defined by the assignment $[(t_j)_{j\in J}]_{\equiv_{\mathcal{D}}} \mapsto [(y_i)_{i\in I}]_{\sim_{\mathcal{D}}}$ is well defined and has an inverse $\psi : |U'(\underline{M}, \mathcal{D})| \to |U(\underline{M}, \mathcal{D})|$ defined by the assignment $[(y_i)_{i\in I}]_{\sim_{\mathcal{D}}} \mapsto [(y_i)_{i\in I}]_{\equiv_{\mathcal{D}}}$. Moreover, these assignments identify $c^{U'}$ and c^U for every constant symbol c, and also identify R^U and $R^{U'}$ for every k-ary relation symbol R. Indeed, it can be easily verified that $\varphi(c^U) = c^{U'}$ and that
$$([(s^j_1)_{j\in J_1}]_{\equiv_{\mathcal{D}}}, \ldots, [(s^j_k)_{j\in J_k}]_{\equiv_{\mathcal{D}}}) \in R^U \text{ iff } (\varphi([(s^j_1)_{j\in J_1}]_{\equiv_{\mathcal{D}}}), \ldots, \varphi([(s^j_k)_{j\in J_k}]_{\equiv_{\mathcal{D}}})) \in R^{U'}.$$

The following is a restatement of [13, Theorem 1.3.1] specialized to the model-theoretic setting of our interest. The proof of this theorem appears in the extended version of the present paper (cf. [2]).

Theorem 5. *(Generalized Łos's Theorem). The following are equivalent for any formula* $\lambda(x_1, \ldots, x_n)$ *with* n *free variables and any profile* $\underline{M} \in (\Omega \cup \{\varnothing\})^I$*:*

- $U(\underline{M}, \mathcal{D}) \models \lambda\left([\underline{s_1}^{J_1}]_{\equiv_\mathcal{D}}, \ldots, [\underline{s_n}^{J_n}]_{\equiv_\mathcal{D}}\right)$;
- $\{i \in J_1 \cap \ldots \cap J_n \mid M_i \models \lambda(s_{1,i}, \ldots s_{n,i})\} \in \mathcal{D}$.

5 Generalized Kirman-Sondermann correspondence

The present section is aimed at introducing the second half of the generalized Kirman-Sondermann correspondence (the first half was discussed at the end of Section 2, before Lemma 2), and characterizing Arrow-rational aggregators in terms of the generalized ultraproduct construction introduced in the previous subsection. Recall that, for any $\mathcal{L}$-structure $\mathcal{B}$ with domain B and any $C \subseteq B$ such that $A^\mathcal{B} \subseteq C$, the *restriction* of $\mathcal{B}$ to C is the $\mathcal{L}$-structure the universe of which is C, which is obtained by restricting the interpretation of all relation symbols to C. For every $M \in \Omega$, let $res_A M$ denote the restriction of M to A^M. In what follows, we find it convenient to define $res_A M$ also when M is the empty model. If $M = \varnothing$, then we stipulate that $res_A M = \varnothing$.

Lemma 4. *For all* $\lambda \in \mathcal{I}$*,*

$$res_A U(\underline{M}, \mathcal{D}) \models \lambda \qquad \textit{iff} \qquad C(\underline{M}, \lambda) \in \mathcal{D}.$$

Proof. By the generalized Łos's theorem, $C(\underline{M}, \lambda) = \{i \in I \mid M_i \models \lambda\} \in \mathcal{D}$ iff $U(\underline{M}, \mathcal{D}) \models \lambda$. Since by assumption λ is quantifier-free, the latter condition is equivalent to $\mathrm{res}_A U(\underline{M}, \mathcal{D}) \models \lambda$.

Definition 6. *For every ultrafilter* $\mathcal{D}$ *over* I*, let* $f_\mathcal{D} : (\Omega \cup \{\varnothing\})^I \to \Omega \cup \{\varnothing\}$ *be defined by the assignment*

$$\underline{M} \mapsto res_A U(\underline{M}, \mathcal{D}).$$

By Łos's theorem, $U(\underline{M}, \mathcal{D}) \models T$ for every profile $\underline{M}$. Since T is a universal theory, this implies that $res_A U(\underline{M}, \mathcal{D}) \models T$, which shows that $f_\mathcal{D}$ is well defined. The following proposition shows that the assignment $\mathcal{D} \mapsto f_\mathcal{D}$ defines a map $\Phi : \beta I \to \mathcal{AR}$.

Proposition 1. *For every ultrafilter* $\mathcal{D}$ *over* I*, the aggregator* $f_\mathcal{D}$ *is Arrow-rational.*

Proof. Condition (A1) is verified by construction. As to (A2), fix a profile $\underline{M}$ and $\lambda \in \mathcal{I}$, and assume that $f_\mathcal{D}(\underline{M}) \models \lambda$, that is, $\mathrm{res}_A U(\underline{M}, \mathcal{D}) \models \lambda$. Then Lemma 4 implies that $C(\underline{M}, \lambda) \in \mathcal{D}$. Hence $C(\underline{M}, \lambda)$ must be nonempty, since $\mathcal{D}$ is an ultrafilter, and hence is proper. As to (A3), let $C(\underline{M}, \lambda) = C(\underline{N}, \mu)$ for some $\underline{M}, \underline{N}$ and $\lambda, \mu \in \mathcal{I}$. Hence, by Lemma 4,

$$f_\mathcal{D}(\underline{M}) \models \lambda \ \text{ iff } \ C(\underline{M}, \lambda) \in \mathcal{D} \ \text{ iff } \ C(\underline{N}, \mu) \in \mathcal{D} \ \text{ iff } \ f_\mathcal{D}(\underline{N}) \models \mu.$$

Next, we are going to show that the maps Λ and Φ defining the Kirman-Sondermann correspondence (cf. discussions before Lemma 2 and before Proposition 1) are inverse to one another.

Proposition 2. *For every $f \in \mathcal{AR}$, $f_{\mathcal{D}_f}$ and f can be identified up to isomorphism.*

Proof. By Lemmas 4 and 1,

$$\mathrm{res}_A U(\underline{M}, \mathcal{D}_f) \models \lambda \text{ iff } C(\underline{M}, \lambda) \in \mathcal{D} \text{ iff } f(\underline{M}) \models \lambda$$

for any profile $\underline{M}$ and any $\lambda \in \mathcal{I}$. Then the statement follows from Fact 1.

In the proof of the next proposition, we make crucial use of the assumption that at least two non-isomorphic models exist in Ω.

Proposition 3. *For every $\mathcal{D} \in \beta I$, $\mathcal{D}_{f_{\mathcal{D}}} = \mathcal{D}$.*

Proof. Fix $X \subseteq I$, and let us show that $X \in \mathcal{D}_{f_{\mathcal{D}}}$ iff $X \in \mathcal{D}$. By assumption, two non-isomorphic models M, N exist in $\Omega \cup \{\varnothing\}$. As shown in the proof of Proposition 2, this implies that $M \models \lambda$ and $N \not\models \lambda$ for some $\lambda \in \mathcal{I}$. Let us define the profile $\underline{M} \in (\Omega \cup \{\varnothing\})^I$ as follows: for any $i \in I$, let

$$M_i = \begin{cases} M & \text{if } i \in X \\ N & \text{if } i \notin X. \end{cases}$$

By construction, $C(\underline{M}, \lambda) = X$, and hence the required equivalence can be proved as follows:

$$\begin{aligned} C(\underline{M}, \lambda) \in \mathcal{D}_{f_{\mathcal{D}}} &\text{ iff } f_{\mathcal{D}}(\underline{M}) \models \lambda && \text{(Lemma 1)} \\ &\text{ iff } \mathrm{res}_A U(\underline{M}, \mathcal{D}) \models \lambda && \text{(Definition 6)} \\ &\text{ iff } C(\underline{M}, \lambda) \in \mathcal{D}. && \text{(Lemma 4)} \end{aligned}$$

The following is an immediate consequence of Propositions 2 and 3:

Theorem 6. (Kirman-Sondermann correspondence for vote abstention). *For any language $\mathcal{L}$, any universal $\mathcal{L}$-theory T with at least two non-isomorphic models, and any set I of individuals, the set $\mathcal{AR}$ of Arrow-rational aggregators (cf. Definition 1) and the set βI of the ultrafilters over I bijectively correspond via the map $\Lambda : \mathcal{AR} \to \beta I$ defined by the assignment $f \mapsto \mathcal{D}_f$. The inverse of Λ is the map $\Phi : \beta I \to \mathcal{AR}$, defined by the assignment $\mathcal{D} \mapsto f_{\mathcal{D}}$.*

6 Arrow-type impossibility theorem for vote abstention

By taking concrete universal theories T, the treatment developed so far specializes to concrete settings in social choice. As an example, in the present section, we capture and discuss the theory of preference aggregation in settings in which individuals might abstain from voting.
The case of preference aggregation over n candidates is modelled, as is done in [9], by taking $\mathcal{L}$ to be a language with n constant symbols $a_1, \ldots, a_n$ and one binary relation symbol R. Consider the following theory T_n:

– $\forall x(\neg Rxx)$ (irreflexivity);
– $\forall x \forall y \forall z(Rxy \wedge Ryz \to Rxz)$ (transitivity);
– $\forall x \forall y(Rxy \vee Ryx \vee x \approx y)$ (completeness);
– $\forall x(\neg x \approx x) \vee \forall x(x \approx a_1 \vee \ldots \vee x \approx a_n)$;
– $\forall x \forall y(x \approx a_j \wedge y \approx a_k \to \neg x \approx y)$ for $j \neq k$;

The first three sentences state that each model M_i of T is a linear order given by the individual i, and the last two items state that the domain of each model is either empty or consists of n pairwise distinct elements $a_1^M, \ldots, a_n^M$. Therefore, the aggregator $f : (\Omega \cup \{\varnothing\})^I \to \Omega \cup \{\varnothing\}$ aggregates a collection of linear orders (or empty order, corresponding to the voter abstention case) into a single linear order (or empty order).
When $n \geq 2$, it is easy to see that $|\Omega| \geq 2$; therefore Corollary 1 applies, yielding:

Theorem 7. *(Generalized Arrow impossibility theorem for preference aggregation). For T_n given above $(n \geq 2)$ and for any finite I, any Arrow-rational aggregator $f : (\Omega \cup \{\varnothing\})^I \to \Omega \cup \{\varnothing\}$ is dictatorial.*

The present setting for vote-abstention allows to prove a strengthened version of Arrow's impossibility theorem in preference aggregation which, unlike the standard one, holds e.g. also for 2-candidate elections. The technical reason for this is to be traced in the proof of Lemma 2, omitted in the present paper but available in [2], which is a variant of Lemma 5.3 in [8]. Indeed, given two non-isomorphic models, the empty model plays the role of the third one. As discussed after Corollary 1, the features of the present set up are such that the counterexamples to the analogous strengthening in the standard setting are not definable anymore.

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