Dynamic Epistemic Logic Displayed

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Abstract. We introduce a display calculus for the logic of Epistemic Actions and Knowledge (EAK) of Baltag-Moss-Solecki. This calculus is *cut-free* and *complete* w.r.t. the standard Hilbert-style presentation of EAK, of which it is a *conservative extension*, given that—as is common to display calculi—it is defined on an expanded language in which all logical operations have adjoints. The additional dynamic operators do not have an interpretation in the standard Kripke semantics of EAK, but do have a natural interpretation in the final coalgebra. This proof-theoretic motivation revives the interest in the global semantics for dynamic epistemic logics pursued among others by Baltag [4], Cîrstea and Sadrzadeh [8].

1 Introduction

Dynamic logics form a large family of nonclassical logics, and perhaps the one enjoying the widest range of applications. Indeed, they are designed to formalize *change* caused by *actions* of diverse nature: updates on the memory state of a computer, displacements of moving robots in an environment, measurements in models of quantum physics, belief revisions, knowledge updates, etc. In each of these areas, formulas express properties of the model encoding the present state of affairs, as well as the pre- and post-conditions of a given action. Actions are semantically represented as *transformations* of one model into another, encoding the state of affairs after the action has taken place. Languages for dynamic logics are expansions of classical propositional logic with *dynamic operators*, parametrized with actions; dynamic operators are modalities interpreted in terms of the transformation of models corresponding to their action-parameters.

However, when dynamic logics feature both dynamic and 'static' modalities, as in the case of the Dynamic Epistemic Logics, they typically lose many desirable properties, such as the closure under uniform substitution. This and other difficulties make their algebraic and proof-theoretic treatment not straightforward, and indeed, the existing proposals appeal to technical solutions which do not meet some of the requirements commonly sought for in proof-theoretic semantics [21, 22]. In [2], a tableaux calculus is introduced, which is labelled, and restricted to the logic of Public Announcements (PAL); in [15] and [16], sequent calculi are presented, covering truthful and arbitrary public announcements respectively, which are again labelled. In [5] and [9], sequent calculi are defined, which are nested; these calculi are sound and complete w.r.t. a certain algebraic semantics which is more general than the standard Kripke semantics for the logic of Baltag-Moss-Solecki; they manipulate sequents whose succedents are unary, and in which three types of objects feature on a par (formulas, agents and actions); finally, two different entailment relations occur, for actions and propositions, respectively, which need to be brought together by means of rules of hybrid type.

In the present paper, we bring into focus that (at least one aspect of) the difficulties hinted at above is the following. Whereas the interpretation of the *adjoints* of static modal operators is equally available in standard models and in the final coalgebra, this is no longer the case for dynamic modalities. In particular, Section 2 will emphasize that dynamic modalities do not in general come in adjoint pairs w.r.t. the standard Kripke semantics. In other words, display postulates (cf. Section 2) are not sound for dynamic modalities w.r.t. to the standard semantics. However, the soundness of these display postulates will be shown w.r.t. the final coalgebra semantics.

After reviewing dynamic epistemic logic (EAK) in Section 3, we define the Belnap's style display calculus D.EAK in Section 4. In Section 5, we outline the proofs that D.EAK is sound w.r.t. the final coalgebra semantics, complete w.r.t. the well known Hilbert-style presentation of EAK, and that the cut rule is eliminable. In Section 6 we briefly discuss why D.EAK is a conservative extension of EAK, and we outline some ongoing research directions.

2 Coalgebraic semantics of dynamic logics

Modal formulas A are interpreted in Kripke models $\mathbb{M} = (W, R, V)$ as subsets of their domains W, and we write $[\![A]\!]_M \subseteq W$ for their interpretation. Equivalently, we can describe the interpretation of A in each Kripke model via the final coalgebra[4] $\mathbb{Z}$ first by defining $[\![A]\!]_{\mathbb{Z}}$ to be the set of elements of $\mathbb{Z}$ satisfying A, and then by recovering $[\![A]\!]_M \subseteq W$ as

$$[\![A]\!]_{\mathbb{M}} = f^{-1}([\![A]\!]_{\mathbb{Z}}), \tag{1}$$

where f is the unique homomorphism $\mathbb{M} \to \mathbb{Z}$. This construction works essentially because, in the category of models/Kripke structures/coalgebras, homomorphisms (i.e. functional bisimulations) preserve the satisfaction/validity of modal formulas. Bisimulation invariance is also enjoyed by formulas of such dynamic logics as EAK (cf. Section 3). Hence, for these dynamic logics, both Kripke semantics and the final coalgebra semantics are equivalently available. However, so far the community has not warmed up to adopting the final coalgebra semantics for dynamic logic, Baltag's [4], and Cîrstea and Sadrzadeh's [8] being among the few proposals exploring this setting. This is unlike the case of standard modal logic, in which the coalgebraic option has taken off, to the point that it has given rise to a field in its own right. In the present section, we offer new reasons to consider the final coalgebra semantics for dynamic logic; indeed, we bring to the fore one aspect in which the final coalgebra semantics for dynamic logics is more advantageous than the standard semantics.

The interpretation of dynamic modalities is given in terms of the *actions* parametrizing them. Actions can be semantically represented as transformations of Kripke models,

[4] Here we rely on the theorem of [1] that the final coalgebra $\mathbb{Z}$ exists. Moreover, even if the carrier of $\mathbb{Z}$ is a proper class, it is still the case that subsets of $\mathbb{Z}$ correspond precisely to 'modal predicates', that is, predicates that are invariant under bisimilarity, see [14].

i.e., as relations between states of different Kripke models. From the viewpoint of the final coalgebra, any action symbol α can then be interpreted as a binary relation $\alpha_{\mathbb{Z}}$ on the final coalgebra $\mathbb{Z}$. In this way, the following well known fact becomes immediately applicable to the final coalgebra model:

Proposition 1. *Every relation $R \subseteq X \times Y$ gives rise to the modal operators*

$$\langle R\rangle, [R] : PY \to PX \text{ and } \langle R^{\circ}\rangle, [R^{\circ}] : PX \to PY$$

defined as follows: for every $V \subseteq X$ and every $U \subseteq Y$,

$$\langle R\rangle U = \{x \in X \mid \exists y \,.\, xRy \;\&\; y \in U\} \qquad [R]U = \{x \in X \mid \forall y \,.\, xRy \;\Rightarrow\; y \in U\}$$
$$\langle R^{\circ}\rangle V = \{y \in Y \mid \exists x \,.\, xRy \;\&\; x \in V\} \qquad [R^{\circ}]V = \{y \in Y \mid \forall x \,.\, xRy \;\Rightarrow\; x \in V\}.$$

These operators come in adjoint pairs:

$$\langle R\rangle U \subseteq V \;\text{ iff }\; U \subseteq [R^{\circ}]V \tag{2}$$
$$\langle R^{\circ}\rangle V \subseteq U \;\text{ iff }\; V \subseteq [R]U. \tag{3}$$

Let $\langle\alpha_{\mathbb{Z}}\rangle, [\alpha_{\mathbb{Z}}], \langle{\alpha_{\mathbb{Z}}}^{\circ}\rangle, [{\alpha_{\mathbb{Z}}}^{\circ}]$ be the semantic modal operators given by Proposition 1 in the special case where $X = Y$ is the carrier Z of $\mathbb{Z}$; they respectively provide a natural interpretation in the final coalgebra $\mathbb{Z}$ for the four connectives $\langle\alpha\rangle, [\alpha], \underbrace{\widehat{\alpha}}, \overline{\underline{\alpha}}$, parametric in the action symbol α. As a direct consequence of the adjunctions (2), (3), the following display postulates, which are so crucial for the present work, are sound under this interpretation (cf. Section 5.1 for more details on this interpretation).

$$\frac{\{\alpha\}X \vdash Y}{X \vdash \overbrace{\underbrace{\alpha}}\, Y}\;{\scriptstyle \{\alpha\} \atop \overbrace{\underbrace{\alpha}}} \qquad \frac{X \vdash \{\alpha\}Y}{\overbrace{\underbrace{\alpha}}\, X \vdash Y}\;{\scriptstyle \overbrace{\underbrace{\alpha}} \atop \{\alpha\}}$$

On the other hand, standard Kripke models are not in general closed under (the interpretations of) α and α°. As a direct consequence of this fact, we can show that e.g. the display postulate $\left({\{\alpha\} \atop \overbrace{\underbrace{\alpha}}}\right)$ is not sound in some Kripke models M for any interpretation of formulas of the form $\overline{\underline{\alpha}}\,B$ in M.

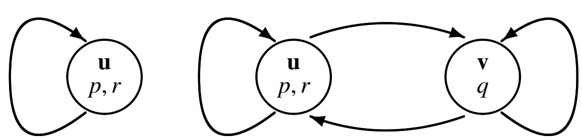

Fig. 1. The models M^{α} and M.

Indeed, consider the model M represented on the right-hand side of the picture above; let the action α be the public announcement (cf. [3]) of the atomic proposition r, and let $A := \Box p$ and $B := q$; hence M^{α} is the submodel on the left-hand side of the picture. Let $i : M^{\alpha} \hookrightarrow M$ be the submodel injection map. Clearly, $[\![\Box p]\!]_M = \varnothing$, which implies that the inclusion $[\![A]\!]_M \subseteq [\![\,\overline{\underline{\alpha}}\,B]\!]_M$ trivially holds for any interpretation of $\overline{\underline{\alpha}}\,B$ in M; however, $i[[\![\Box p]\!]_{M^{\alpha}}] = \{u\}$, hence $[\![\langle\alpha\rangle\Box p]\!]_M = [\![\alpha]\!]_M \cap i[[\![\Box p]\!]_{M^{\alpha}}] = V(r) \cap \{u\} = \{u\} \nsubseteq \{v\} = [\![q]\!]_M$, which falsifies the inclusion $[\![\langle\alpha\rangle A]\!]_M \subseteq [\![B]\!]_M$. This proves our claim.

3 The logic of epistemic actions and knowledge

In the present section, the relevant preliminaries on the syntax and semantics of the logic of epistemic actions and knowledge (EAK) [3] will be given, which are different but equivalent to the original version [3], and follow the presentation in [13, 17].

Let AtProp be a countable set of proposition letters. The set $\mathcal{L}$ of formulas A of (the single-agent[5] version of) the logic of epistemic actions and knowledge (EAK) and the set $\mathsf{Act}(\mathcal{L})$ of the *action structures* α *over* $\mathcal{L}$ are built simultaneously as follows:

$$A := p \in \mathsf{AtProp} \mid \neg A \mid A \vee A \mid \Diamond A \mid \langle\alpha\rangle A \quad (\alpha \in \mathsf{Act}(\mathcal{L})).$$

An *action structure over* $\mathcal{L}$ is a tuple $\alpha = (K, k, \alpha, Pre_\alpha)$, such that K is a finite nonempty set, $k \in K$, $\alpha \subseteq K \times K$ and $Pre_\alpha : K \to \mathcal{L}$. Notice that α denotes *both* the action structure *and* the accessibility relation of the action structure. Unless explicitly specified otherwise, occurrences of this symbol are to be interpreted contextually: for instance, in $j\alpha k$, the symbol α denotes the relation; in M^α, the symbol α denotes the action structure. Of course, in the multi-agent setting, each action structure comes equipped with *a collection* of accessibility relations indexed in the set of agents, and then the abuse of notation disappears.

Sometimes we will write $Pre(\alpha)$ for $Pre_\alpha(k)$. Let $\alpha_i = (K, i, \alpha, Pre_\alpha)$ for every action structure $\alpha = (K, k, \alpha, Pre_\alpha)$ and every $i \in K$. Intuitively, the actions α_i for $k\alpha i$ encode the uncertainty of the (unique) agent about the action that is actually taking place. The standard stipulations hold for the defined connectives $\top$, $\bot$, $\wedge$, $\to$ and $\leftrightarrow$.

Models for EAK are relational structures $M = (W, R, V)$ such that W is a nonempty set, $R \subseteq W \times W$, and $V : \mathsf{AtProp} \to \mathcal{P}(W)$ is a map. The evaluation of the static fragment of the language is standard. For every Kripke frame $\mathcal{F} = (W, R)$ and every $\alpha \subseteq K \times K$, let the Kripke frame $\coprod_\alpha \mathcal{F} := (\coprod_K W, R \times \alpha)$ be defined[6] as follows: $\coprod_K W$ is the $|K|$-fold coproduct of W (which is set-isomorphic to $W \times K$), and $R \times \alpha$ is the binary relation on $\coprod_K W$ defined as

$$(w, i)(R \times \alpha)(u, j) \quad \text{iff} \quad wRu \ \text{ and } \ i\alpha j.$$

For every model $M = (W, R, V)$ and every action structure $\alpha = (K, k, \alpha, Pre_\alpha)$, let

$$\coprod_\alpha M := (\coprod_K W, R \times \alpha, \coprod_K V)$$

be such that its underlying frame is defined as detailed above, and $(\coprod_K V)(p) := \coprod_K V(p)$ for every $p \in \mathsf{AtProp}$. Finally, the *update* of M with the action structure α is the submodel $M^\alpha := (W^\alpha, R^\alpha, V^\alpha)$ of $\coprod_\alpha M$ the domain of which is the subset

$$W^\alpha := \{(w, j) \in \coprod_K W \mid M, w \Vdash Pre_\alpha(j)\}.$$

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[5] The multi-agent generalization of this simpler version is straightforward, and consists in taking the indexed version of the modal operators, axioms, and interpreting relations (both in the models and in the action structures) over a set of agents.

[6] We will of course apply this definition to relations α which are part of the specification of some action structure; in these cases, the symbol α in $\coprod_\alpha \mathcal{F}$ will be understood as the action structure. This is why the abuse of notation turns out to be useful.

Given this preliminary definition, formulas of the form $\langle\alpha\rangle A$ are evaluated as follows:

$$M, w \Vdash \langle\alpha\rangle A \quad \text{iff} \quad M, w \Vdash Pre_\alpha(k) \text{ and } M^\alpha, (w,k) \Vdash A.$$

Proposition 2 ([3, Theorem 3.5]). *EAK is axiomatized completely by the axioms and rules for the minimal normal modal logic K plus the following axioms:*

1. $\langle\alpha\rangle p \leftrightarrow (Pre(\alpha) \wedge p)$;
2. $\langle\alpha\rangle\neg A \leftrightarrow (Pre(\alpha) \wedge \neg\langle\alpha\rangle A)$;
3. $\langle\alpha\rangle(A \vee B) \leftrightarrow (\langle\alpha\rangle A \vee \langle\alpha\rangle B)$;
4. $\langle\alpha\rangle\Diamond A \leftrightarrow (Pre(\alpha) \wedge \bigvee\{\Diamond\langle\alpha_i\rangle A \mid k\alpha i\})$.

An immediate and well known consequence of the theorem above is that every $\mathcal{L}$-formula is EAK-equivalent to some formula in the static fragment of $\mathcal{L}$. This implies in particular that $\mathcal{L}$-formulas are invariant under standard bisimulation, and this fact extends of course to the multi-agent version.

The representation of actions as action structures is just one possible approach. Here we prefer to keep a black-box perspective on actions, and to identify agents $\mathtt{a}$ with the indistinguishability relation they induce on actions; so, in the remainder of the paper, the role of the action-structures α_i for $k\alpha i$ will be played by actions β such that $\alpha\mathtt{a}\beta$.

4 EAK displayed

In the present section, the display calculus D.EAK for the logic EAK (cf. section 3) is introduced piecewise: in the next subsection, display calculi will be presented which are multi-modal versions of display-style sequent calculi proposed in the literature for the (bi-)intuitionistic versions of basic and tense normal modal logic [11,21]. This presentation is modular w.r.t. intuitionistic logic: namely, for the sake of a more straightforward extension to the intuitionistic counterparts of PAL and EAK [13, 17], it takes the connectives in the language of IEAK as first-class citizens; the classical base is captured by adding the two *Grishin rules* (see below) to the system. In section 4.2, the rules for the dynamic connectives are introduced. The calculus D.EAK consists of all the rules in the two subsections.

The language $\mathcal{L}$(m-IK) of the multi-modal version of Fischer Servi's intuitionistic modal logic IK features one pair of modal connectives for each element $\mathtt{a}$ in a set $\mathtt{A}$ of agents, and consists of *formulas* built from a set of atomic propositions $\{p, q, r, \ldots\}$ and one constant $\bot$, according to the following BNF grammar:

$$A := p \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Diamond_{\mathtt{a}} A \mid \Box_{\mathtt{a}} A\,.$$

The language $\mathcal{L}$(tm-IK) of the "tense-like" version of m-IK is obtained by expanding $\mathcal{L}$(m-IK) with one pair of (adjoint) modalities $\blacklozenge_{\mathtt{a}}$ and $\blacksquare_{\mathtt{a}}$, for each $\mathtt{a}$ in $\mathtt{A}$.

The language $\mathcal{L}$(btm-IK) of the bi-intuitionistic version of tm-IK is obtained by expanding the language of tm-IK with $\top$ and one extra propositional connective $\succ\!\!-$, referred to as *subtraction* or *disimplication*,[7] which behaves as the dual intuitionistic implication. The reader is referred to [18] for an axiomatic presentation of bi-intuitionistic logic and to [11, 12] for its relative display calculi.

[7] Formulas $A \succ\!\!- B$ are classically equivalent to $\neg A \wedge B$.

4.1 The static fragment

Display calculi typically involve sequents $X \vdash Y$, where X and Y are *structures*, built from formulas A (in the present case, $A \in \mathcal{L}$(m-IK) (resp. $\mathcal{L}$(tm-IK), $\mathcal{L}$(btm-IK))) and the structural constant I by means of *structural connectives* (or *proxies*), according to the following BNF grammar[8]:

$$X := \mathrm{I} \mid A \mid X > X \mid X \,;X \mid \bullet_{a} X \mid \circ_{a} X\,.$$

Each structural connective is associated with a pair of logical connectives, as follows:

Proxies	$>$		$;$		I		$\circ_{a}$		$\bullet_{a}$		Structural symbols
Connectives	$\succ\!\!-$	$\rightarrow$	$\wedge$	$\vee$	$\top$	$\bot$	$\Diamond_{a}$	$\Box_{a}$	$\blacklozenge_{a}$	$\blacksquare_{a}$	Operational symbols

moreover, structural connectives form adjoint pairs by definition (which will be witnessed in the ensuing display postulates), as follows:

$$;\ \dashv\ > \qquad >\ \dashv\ ; \qquad \circ_{a} \dashv \bullet_{a} \qquad \bullet_{a} \dashv \circ_{a}$$

The display calculi D.m-IK, D.tm-IK and D.btm-IK are defined by means of rules which are classified as *structural* and as *operational* rules. The structural rules below only concern structural connectives, and are common to the three of them (where the structures $X^{-\alpha}$ and $Y^{-\alpha}$ are dynamic-proxy-free):[9]

$$\frac{}{p \vdash p}\,Id \qquad \begin{array}{c} X \vdash Y \\ \hline\hline \mathrm{I} \vdash X > Y \end{array}\,\mathrm{I}_{L} \qquad \frac{X \vdash Z}{Y^{-\alpha} \vdash X > Z}\,W_{L} \qquad \frac{X\,;X \vdash Y}{X \vdash Y}\,C_{L} \qquad \frac{\mathrm{I} \vdash X}{\mathrm{I} \vdash \circ_{a} X}\,{\scriptstyle \mathrm{I} \atop \circ_{a}}$$

$$\frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\,Cut \qquad \begin{array}{c} Y \vdash X \\ \hline\hline X > Y \vdash \mathrm{I} \end{array}\,\mathrm{I}_{R} \qquad \frac{Z \vdash Y}{Y > Z \vdash X^{-\alpha}}\,W_{R} \qquad \frac{Y \vdash X\,;X}{Y \vdash X}\,C_{R} \qquad \frac{\mathrm{I} \vdash X}{\mathrm{I} \vdash \bullet_{a} X}\,{\scriptstyle \mathrm{I} \atop \bullet_{a}}$$

$$\frac{Y\,;X \vdash Z}{X\,;Y \vdash Z}\,E_{L} \qquad \frac{X\,;(Y\,;Z) \vdash W}{(X\,;Y)\,;Z \vdash W}\,A_{L} \quad \frac{Y \vdash \circ_{a} X > \circ_{a} Z}{Y \vdash \circ_{a}(X > Z)}\,{\scriptstyle > \atop \circ_{a}} \qquad \begin{array}{c} X\,;Y \vdash Z \\ \hline\hline Y \vdash X > Z \end{array}\,{\scriptstyle ; \atop >} \qquad \begin{array}{c} \circ_{a} X \vdash Y \\ \hline\hline X \vdash \bullet_{a} Y \end{array}\,{\scriptstyle \circ_{a} \atop \bullet_{a}}$$

$$\frac{Z \vdash X\,;Y}{Z \vdash Y\,;X}\,E_{R} \qquad \frac{W \vdash (Z\,;Y)\,;X}{W \vdash Z\,;(Y\,;X)}\,A_{R} \quad \frac{Y \vdash \bullet_{a} X > \bullet_{a} Z}{Y \vdash \bullet_{a}(X > Z)}\,{\scriptstyle > \atop \bullet_{a}} \qquad \begin{array}{c} Z \vdash Y\,;X \\ \hline\hline Y > Z \vdash X \end{array}\,{\scriptstyle > \atop ;} \qquad \begin{array}{c} X \vdash \circ_{a} Y \\ \hline\hline \bullet_{a} X \vdash Y \end{array}\,{\scriptstyle \bullet_{a} \atop \circ_{a}}$$

The operational rules govern the introduction of the logical connectives: here below are the ones which are common to the three calculi:

$$\frac{}{\bot \vdash \mathrm{I}}\,\bot_{L} \quad \frac{\mathrm{I} \vdash X}{\top \vdash X}\,\top_{L} \qquad \frac{A\,;B \vdash Z}{A \wedge B \vdash Z}\,\wedge_{L} \qquad \frac{B \vdash Y \qquad A \vdash X}{B \vee A \vdash Y\,;X}\,\vee_{L} \quad \frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash X > Y}\,\rightarrow_{L}$$

$$\frac{X \vdash \mathrm{I}}{X \vdash \bot}\,\bot_{R} \quad \frac{}{\mathrm{I} \vdash \top}\,\top_{R} \quad \frac{X \vdash A \qquad Y \vdash B}{X\,;Y \vdash A \wedge B}\,\wedge_{R} \qquad \frac{Z \vdash B\,;A}{Z \vdash B \vee A}\,\vee_{R} \qquad \frac{Z \vdash A > B}{Z \vdash A \rightarrow B}\,\rightarrow_{R}$$

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[8] Notice that, in the context of the full calculus, the variables X, Y, Z, W appearing in the rules in the present subsection are to be interpreted as structures of the full language of D.EAK, unless explicitly indicated otherwise with symbols such as $X^{-\alpha}$.

[9] The weakening rules W_L and W_R are equivalent to the standard ones via the Display Postulates $\left({; \atop >}\right)$ and $\left({> \atop ;}\right)$; in these rules, the principal structure appears 'in display'; besides making an easier life in the proof of the cut elimination, we believe that this feature of W_L and W_R is more in line with the general design principles of display calculi. Notice also that the presence of the rules E_L and E_R makes it possible for us to dispense with the structural connective $<$ and its relative rules, such as $A\,;B \vdash C / A \vdash C < B$.

Here below, from the left to right, are the operational rules completing D.m-IK (1^{st} and 2^{nd} column), D.tm-IK (4^{th} and 5^{th} column), and D.btm-IK (3^{rd} column):

$$\frac{\circ_{\mathsf{a}} A \vdash X}{\Diamond_{\mathsf{a}} A \vdash X}\,\Diamond_{\mathsf{a}L} \quad \frac{A \vdash X}{\Box_{\mathsf{a}} A \vdash \circ_{\mathsf{a}} X}\,\Box_{\mathsf{a}L} \quad \frac{A > B \vdash Z}{A \succ\!\!- B \vdash Z}\,\succ\!\!-_L \quad \frac{\bullet_{\mathsf{a}} A \vdash X}{\blacklozenge_{\mathsf{a}} A \vdash X}\,\blacklozenge_{\mathsf{a}L} \quad \frac{A \vdash X}{\blacksquare_{\mathsf{a}} A \vdash \bullet_{\mathsf{a}} X}\,\blacksquare_{\mathsf{a}L}$$

$$\frac{X \vdash A}{\circ_{\mathsf{a}} X \vdash \Diamond_{\mathsf{a}} A}\,\Diamond_{\mathsf{a}R} \quad \frac{X \vdash \circ_{\mathsf{a}} A}{X \vdash \Box_{\mathsf{a}} A}\,\Box_{\mathsf{a}R} \quad \frac{A \vdash X \qquad Y \vdash B}{X > Y \vdash A \succ\!\!- B}\,\succ\!\!-_R \quad \frac{X \vdash A}{\bullet_{\mathsf{a}} X \vdash \blacklozenge_{\mathsf{a}} A}\,\blacklozenge_{\mathsf{a}R} \quad \frac{X \vdash \bullet_{\mathsf{a}} A}{X \vdash \blacksquare_{\mathsf{a}} A}\,\blacksquare_{\mathsf{a}R}$$

Finally, the classical versions of each of these calculi can be obtained from the above ones e.g. by adding the following *Grishin's* structural rules [12, 21]:

$$\frac{X > (Y\,;Z) \vdash W}{(X > Y)\,;Z \vdash W}\,Gri_L \qquad \frac{W \vdash X > (Y\,;Z)}{W \vdash (X > Y)\,;Z}\,Gri_R$$

4.2 The dynamic fragment

The calculi introduced in the present subsection involve sequents $X \vdash Y$, where X and Y are *structures*, built from formulas $A \in \mathcal{L}$(m-IEAK) (resp. $\mathcal{L}$(tm-IEAK), $\mathcal{L}$(btm-IEAK)) and the structural constant I according to the following BNF grammar:

$$X := \mathrm{I} \mid A \mid X\,;X \mid X > X \mid \bullet_{\mathsf{a}} X \mid \circ_{\mathsf{a}} X \mid \{\alpha\}X \mid \widetilde{\underset{\smile}{\alpha}}\,X.$$

Hence, the structural language above expands the one of the previous subsection with structural connectives $\{\alpha\}$ and $\widetilde{\underset{\smile}{\alpha}}$ for each action $\alpha \in \Gamma$; these are by definition adjoint to each other as follows: $\{\alpha\} \dashv \widetilde{\underset{\smile}{\alpha}}$ and $\widetilde{\underset{\smile}{\alpha}} \dashv \{\alpha\}$. The proxy $\{\alpha\}$ is associated with the logical connectives $[\alpha]$ and $\langle\alpha\rangle$, and thus it occurs in the operational rules concerning them. Likewise, new logical connectives $\widehat{\underset{\vee}{\alpha}}$ and $\overline{\underline{\alpha}}$ are introduced which stand in an analogous relation with $\widetilde{\underset{\smile}{\alpha}}$, and which are adjoint to $[\alpha]$ and $\langle\alpha\rangle$ as follows: $\langle\alpha\rangle \dashv \overline{\underline{\alpha}}$ and $\widehat{\underset{\vee}{\alpha}} \dashv [\alpha]$. As discussed in section 2, these new connectives have a natural interpretation in the final coalgebra, but not in the standard semantics.

$\{\alpha\}$		$\widetilde{\underset{\smile}{\alpha}}$	
$[\alpha]$	$\langle\alpha\rangle$	$\widehat{\underset{\vee}{\alpha}}$	$\overline{\underline{\alpha}}$

The display calculi D.m-IEAK, D.tm-IEAK and D.btm-IEAK are defined by adding the rules of the present subsection to D.m-IK, D.tm-IK and D.btm-IK, respectively; the display calculus D.EAK is obtained by adding the Grishin rules to D.m-IK. The rules in the present subsection come in four groups: pure and contextual structural and operational rules. Here follow the pure structural rules; the dynamic *display postulates* appear in the 5^{th} column below:

$$\frac{}{\{\alpha\}\,p \vdash p}\,atom_L \qquad \frac{}{p \vdash \{\alpha\}\,p}\,atom_R \qquad \frac{X \vdash Y}{\{\alpha\}X \vdash \{\alpha\}Y}\,balance$$

$$\frac{\{\alpha\}Y > \{\alpha\}Z \vdash X}{\{\alpha\}(Y > Z) \vdash X}\,\tfrac{\{\alpha\}}{>} \quad \frac{\{\alpha\}X\,;\{\alpha\}Y \vdash Z}{\{\alpha\}(X\,;Y) \vdash Z}\,\tfrac{\{\alpha\}}{;} \quad \frac{\widetilde{\underset{\smile}{\alpha}}\,Y > \widetilde{\underset{\smile}{\alpha}}\,X \vdash Z}{\widetilde{\underset{\smile}{\alpha}}\,(Y > X) \vdash Z}\,\tfrac{\widetilde{\underset{\smile}{\alpha}}}{>} \quad \frac{\widetilde{\underset{\smile}{\alpha}}\,X\,;\widetilde{\underset{\smile}{\alpha}}\,Y \vdash Z}{\widetilde{\underset{\smile}{\alpha}}\,(X\,;Y) \vdash Z}\,\tfrac{\widetilde{\underset{\smile}{\alpha}}}{;} \quad \frac{\{\alpha\}X \vdash Y}{X \vdash \widetilde{\underset{\smile}{\alpha}}\,Y}\,\tfrac{\{\alpha\}}{\widetilde{\underset{\smile}{\alpha}}}$$

$$\frac{Y \vdash \{\alpha\}X > \{\alpha\}Z}{Y \vdash \{\alpha\}(X > Z)}\,\tfrac{>}{\{\alpha\}} \quad \frac{Z \vdash \{\alpha\}Y\,;\{\alpha\}X}{Z \vdash \{\alpha\}(Y\,;X)}\,\tfrac{;}{\{\alpha\}} \quad \frac{Y \vdash \widetilde{\underset{\smile}{\alpha}}\,X > \widetilde{\underset{\smile}{\alpha}}\,Z}{Y \vdash \widetilde{\underset{\smile}{\alpha}}\,(X > Z)}\,\tfrac{>}{\widetilde{\underset{\smile}{\alpha}}} \quad \frac{Z \vdash \widetilde{\underset{\smile}{\alpha}}\,Y\,;\widetilde{\underset{\smile}{\alpha}}\,X}{Z \vdash \widetilde{\underset{\smile}{\alpha}}\,(Y\,;X)}\,\tfrac{;}{\widetilde{\underset{\smile}{\alpha}}} \quad \frac{Y \vdash \{\alpha\}X}{\widetilde{\underset{\smile}{\alpha}}\,Y \vdash X}\,\tfrac{\widetilde{\underset{\smile}{\alpha}}}{\{\alpha\}}$$

Here below are the pure operational rules:

$$\frac{\{\alpha\}A \vdash X}{\langle\alpha\rangle A \vdash X}\,\langle\alpha\rangle_L \quad \frac{X \vdash A}{\{\alpha\}X \vdash \langle\alpha\rangle A}\,\langle\alpha\rangle_R \quad \frac{A \vdash X}{[\alpha]A \vdash \{\alpha\}X}\,[\alpha]_L \quad \frac{X \vdash \{\alpha\}A}{X \vdash [\alpha]A}\,[\alpha]_R$$

The *contextual* rules encode inferences which can be performed only in the presence of a given assumption (in the case at hand, the preconditions of the action parametrizing a dynamic proxy). Here below the contextual structural rules:

reduce *swap-in* *swap-out*

$$\frac{Pre(\alpha)\,;\{\alpha\}A \vdash X}{\{\alpha\}A \vdash X}\,r_L \quad \frac{Pre(\alpha)\,;\{\alpha\}\circ_{\mathsf{a}} X \vdash Y}{Pre(\alpha)\,;\circ_{\mathsf{a}}\{\beta\}_{\alpha\mathsf{a}\beta}\, X \vdash Y}\,s\text{-}in_L \quad \frac{\big(Pre(\alpha)\,;\circ_{\mathsf{a}}\{\beta\}\,X \vdash Y \mid \alpha\mathsf{a}\beta\big)}{Pre(\alpha)\,;\{\alpha\}\circ_{\mathsf{a}} X \vdash \textbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big)}\,s\text{-}out_L$$

$$\frac{X \vdash Pre(\alpha) > \{\alpha\}A}{X \vdash \{\alpha\}A}\,r_R \quad \frac{Y \vdash Pre(\alpha) > \{\alpha\}\circ_{\mathsf{a}} X}{Y \vdash Pre(\alpha) > \circ_{\mathsf{a}}\{\beta\}_{\alpha\mathsf{a}\beta}\, X}\,s\text{-}in_R \quad \frac{\big(Y \vdash Pre(\alpha) > \circ_{\mathsf{a}}\{\beta\}\,X \mid \alpha\mathsf{a}\beta\big)}{\textbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big) \vdash Pre(\alpha) > \{\alpha\}\circ_{\mathsf{a}} X}\,s\text{-}out_R$$

The *swap-in* rules are unary and should be read as follows: if the premise holds, then the conclusion holds relative to any action β such that $\alpha\mathsf{a}\beta$. The *swap-out* rules do not have a fixed arity; they have as many premises[10] as there are actions β such that $\alpha\mathsf{a}\beta$; in the conclusion, the symbol $\textbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big)$ refers to a string $(\cdots(Y\,;Y)\,;\cdots\,;Y)$ with n occurrences of Y, where n is the number of actions β such that $\alpha\mathsf{a}\beta$. Finally, the contextual operational rules:

reverse

$$\frac{Pre(\alpha)\,;\{\alpha\}A \vdash X}{Pre(\alpha)\,;[\alpha]A \vdash X}\,rev_L \quad \frac{X \vdash Pre(\alpha) > \{\alpha\}A}{X \vdash Pre(\alpha) > \langle\alpha\rangle A}\,rev_R$$

5 Soundness, completeness and cut elimination

5.1 Soundness in the final coalgebra

In the present section, we outline the soundness of D.EAK w.r.t. the final coalgebra semantics. Structures will be translated into formulas, and formulas will be interpreted as subsets of the final coalgebra, as discussed in section 2. In order to translate structures as formulas, proxies need to be translated as logical connectives; to this effect, any given occurrence of a proxy is translated as one or the other of its associated logical connectives, according to which side of the sequent the given occurrence can be displayed on as main connective [6, 21], as reported in Table 1.

Sequents $A \vdash B$ will be interpreted as inclusions $[\![A]\!]_Z \subseteq [\![B]\!]_Z$; rules $(A_i \vdash B_i \mid i \in I)/C \vdash D$ will be interpreted as implications of the form "if $[\![A_i]\!]_Z \subseteq [\![B_i]\!]_Z$ for every $i \in I$, then $[\![C]\!]_Z \subseteq [\![D]\!]_Z$". As for rules not involving $\widetilde{\underline{\alpha}}$, we will rely on the following observation, which is based on the invariance of EAK-formulas under bisimulation (cf. Section 3):

[10] The *swap-out* rule could indeed be infinitary if action structures were allowed to be infinite, which in the present setting, as in [3], is not the case.

Table 1. Translation of proxies into logical connectives

Main connective	if displayed in antecedent	if displayed in succedent
I	$\top$	$\bot$
$A\,;B$	$A \wedge B$	$A \vee B$
$A > B$	$A \succ\!\!- B$	$A \rightarrow B$
$\circ A$	$\Diamond A$	$\Box A$
$\bullet A$	$\blacklozenge A$	$\blacksquare A$
$\{\alpha\}A$	$\langle\alpha\rangle A$	$[\alpha]A$
$\overset{\frown}{\underset{\smile}{\alpha}}A$	$\overset{\frown}{\underset{\vee}{\alpha}}A$	$\overset{\frown}{\underline{\alpha}}A$

Lemma 1. *The following are equivalent for all EAK-formulas A and B:*
(1) $[\![A]\!]_Z \subseteq [\![B]\!]_Z$*;*
(2) $[\![A]\!]_M \subseteq [\![B]\!]_M$ *for every model M.*

Proof. The direction from (2) to (1) is clear; conversely, fix a model M, and let $f : M \rightarrow Z$ be the unique arrow; then (1) immediately implies that $[\![A]\!]_M = f^{-1}([\![A]\!]_Z) \subseteq f^{-1}([\![B]\!]_Z) = [\![B]\!]_M$.

In the light of the lemma above, and using the translations provided in Table 1, the soundness of unary rules $A \vdash B/C \vdash D$ not involving $\overset{\frown}{\underset{\smile}{\alpha}}$, such as *balance*, $\langle\alpha\rangle_R$ and $[\alpha]_L$, can be straightforwardly checked as implications of the form "if $[\![A]\!]_M \subseteq [\![B]\!]_M$ on every model M, then $[\![C]\!]_M \subseteq [\![D]\!]_M$ on every model M". As an example, let us check the soundness of *balance*: Let A, B be EAK-formulas such that $[\![A]\!]_M \subseteq [\![B]\!]_M$ on every model M. Let us fix a model M, and show that $[\![\langle\alpha\rangle A]\!]_M \subseteq [\![[\alpha]B]\!]_M$. As discussed in [13, Subsection 4.2], the following identities hold in any standard model:

$$[\![\langle\alpha\rangle A]\!]_M = [\![Pre(\alpha)]\!]_M \cap \iota_k^{-1}[i[[\![A]\!]_{M^\alpha}]], \tag{4}$$

$$[\![[\alpha]A]\!]_M = [\![Pre(\alpha)]\!]_M \Rightarrow \iota_k^{-1}[i[[\![A]\!]_{M^\alpha}]], \tag{5}$$

where the map $i : M^\alpha \rightarrow \coprod_\alpha M$ is the submodel embedding, and $\iota_k : M \rightarrow \coprod_\alpha M$ is the embedding of M into its k-colored copy. Letting $g(-) := \iota_k^{-1}[i[-]]$, we need to show

$$[\![Pre(\alpha)]\!]_M \cap g([\![A]\!]_{M^\alpha}) \subseteq [\![Pre(\alpha)]\!]_M \Rightarrow g([\![B]\!]_{M^\alpha}).$$

This is a direct consequence of the Heyting-valid implication "if $b \leq c$ then $a \wedge b \leq a \rightarrow c$", the monotonicity of g, and the assumption that $[\![A]\!]_M \subseteq [\![B]\!]_M$ holds on every model, hence on M^α.

Actually, for all rules $(A_i \vdash B_i \mid i \in I)/C \vdash D$ not involving $\overset{\frown}{\underset{\smile}{\alpha}}$ except *balance*, $\langle\alpha\rangle_R$ and $[\alpha]_L$, stronger soundness statements can be proven of the form "for every model M, if $[\![A_i]\!]_M \subseteq [\![B_i]\!]_M$ for every $i \in I$, then $[\![C]\!]_M \subseteq [\![D]\!]_M$" (this amounts to the soundness w.r.t. the standard semantics). This is the case for all display postulates not involving $\overset{\frown}{\underset{\smile}{\alpha}}$, the soundness of which boils down to the well known adjunction conditions holding in every model M. As to the remaining rules not involving $\overset{\frown}{\underset{\smile}{\alpha}}$, thanks to the following general principle of *indirect (in)equality*, the stronger soundness condition above boils down to the verification of inclusions which interpret validities of IEAK [13], and hence, a fortiori, of EAK. Same arguments hold for the Grishin rules, except that their soundness boils down to classical but not intuitionistic validities.

Lemma 2. ***(Principle of indirect inequality)*** *Tfae for any preorder P and all $a, b \in P$:*
(1) $a \leq b$;
(2) $x \leq a$ implies $x \leq b$ for every $x \in P$;
(3) $b \leq y$ implies $a \leq y$ for every $y \in P$.

As an example, let us verify *s-out$_L$*: fix a model M, fix EAK-formulas A and B, and assume that for every action β, if $\alpha \mathsf{a} \beta$ then $[\![Pre(\alpha)]\!]_M \cap [\![\Diamond_{\mathsf{a}}\langle\beta\rangle A]\!]_M \subseteq [\![B]\!]_M$, i.e., that $[\![Pre(\alpha)]\!]_M \cap \bigcup\{[\![\Diamond_{\mathsf{a}}\langle\beta\rangle A]\!]_M \mid \alpha \mathsf{a} \beta\} \subseteq [\![B]\!]_M$; we need to show that $[\![Pre(\alpha)]\!]_M \cap [\![\langle\alpha\rangle\Diamond_{\mathsf{a}} A]\!]_M \subseteq [\![B]\!]_M$. By the principle of indirect inequality, it is enough to show that

$$[\![\langle\alpha\rangle\Diamond_{\mathsf{a}} A]\!]_M \subseteq [\![Pre(\alpha)]\!]_M \cap \bigcup\{[\![\Diamond_{\mathsf{a}}\langle\beta\rangle A]\!]_M \mid \alpha \mathsf{a} \beta\},$$

which is true (cf. Proposition 2). Finally, the soundness of the rules which do involve $\overset{\frown}{\underset{\smile}{\alpha}}$ remains to be shown. The soundness of the display postulates immediately follows from Proposition 1. As an example, let us verify the soundness of $\left(\frac{\overset{\frown}{\underset{\smile}{\alpha}}}{>}\right)$: translating the structures into formulas, and applying the principle of indirect inequality, it boils down to verifying that $[\![\,\widehat{\underset{\smile}{\alpha}}\,(A \succ\!\!- B)]\!]_Z \subseteq [\![\,\overline{\underset{\smile}{\alpha}}\,A]\!]_Z \succ\!\!- [\![\,\widehat{\underset{\smile}{\alpha}}\,B]\!]_Z$ for all EAK-formulas A and B. Since, in Z, $\widehat{\underset{\smile}{\alpha}}$ and $\overline{\underset{\smile}{\alpha}}$ are respectively interpreted as $\langle\alpha^\circ\rangle$ and $[\alpha^\circ]$, this inclusion can be rewritten as

$$\langle\alpha^\circ\rangle([\![A]\!]_Z \succ\!\!- [\![B]\!]_Z) \subseteq [\alpha^\circ][\![A]\!]_Z \succ\!\!- \langle\alpha^\circ\rangle[\![B]\!]_Z,$$

where $A \succ\!\!- B$ can be interpreted classically, i.e. as $\neg A \wedge B$. The straightforward verification that this is an instance of a principle valid in every frame is left to the reader.

5.2 Completeness

For the completeness of D.EAK, it is enough to show that all the axioms of EAK are derivable in D.EAK. Due to space restrictions, here we only report on the derivations of $\langle\alpha\rangle\Diamond_{\mathsf{a}} A \leftrightarrow Pre(\alpha) \wedge \bigvee\{\Diamond_{\mathsf{a}}\langle\beta\rangle A \mid \alpha \mathsf{a} \beta\}$. For ease of notation, we assume that the actions β such that $\alpha \mathsf{a} \beta$ form the set $\{\beta_i \mid 1 \leq i \leq n\}$.

$$\dfrac{Pre(\alpha) \vdash Pre(\alpha) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\{\beta_1\}A \vdash \langle\beta_1\rangle A}}{\circ_{\mathsf{a}}\{\beta_1\}A \vdash \Diamond_{\mathsf{a}}\langle\beta_1\rangle A}}{Pre(\alpha)\,;\circ_{\mathsf{a}}\{\beta_1\}A \vdash \Diamond_{\mathsf{a}}\langle\beta_1\rangle A} \quad \cdots \quad \dfrac{\dfrac{\dfrac{A \vdash A}{\{\beta_n\}A \vdash \langle\beta_n\rangle A}}{\circ_{\mathsf{a}}\{\beta_n\}A \vdash \Diamond_{\mathsf{a}}\langle\beta_n\rangle A}}{Pre(\alpha)\,;\circ_{\mathsf{a}}\{\beta_n\}A \vdash \Diamond_{\mathsf{a}}\langle\beta_n\rangle A}}{Pre(\alpha)\,;\{\alpha\}\circ_{\mathsf{a}} A \vdash ;\!\left(\Diamond_{\mathsf{a}}\langle\beta_i\rangle A\right)}\,\textit{s-out}}{Pre(\alpha)\,;\{\alpha\}\circ_{\mathsf{a}} A \vdash \bigvee\left(\Diamond_{\mathsf{a}}\langle\beta_i\rangle A\right)}}{\{\alpha\}\circ_{\mathsf{a}} A \vdash \bigvee\left(\Diamond_{\mathsf{a}}\langle\beta_i\rangle A\right)}\,r}{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Pre(\alpha)\,;\{\alpha\}\circ_{\mathsf{a}} A \vdash Pre(\alpha) \wedge \bigvee\left(\Diamond_{\mathsf{a}}\langle\beta_i\rangle A\right)}{\{\alpha\}\circ_{\mathsf{a}} A \vdash Pre(\alpha) \wedge \bigvee\left(\Diamond_{\mathsf{a}}\langle\beta_i\rangle A\right)}\,r}{\circ_{\mathsf{a}} A \vdash \overline{\underset{\smile}{\alpha}}\,Pre(\alpha) \wedge \bigvee\left(\Diamond_{\mathsf{a}}\langle\beta_i\rangle A\right)}}{\Diamond_{\mathsf{a}} A \vdash \overline{\underset{\smile}{\alpha}}\,Pre(\alpha) \wedge \bigvee\left(\Diamond_{\mathsf{a}}\langle\beta_i\rangle A\right)}}{\{\alpha\}\Diamond_{\mathsf{a}} A \vdash Pre(\alpha) \wedge \bigvee\left(\Diamond_{\mathsf{a}}\langle\beta_i\rangle A\right)}}{\langle\alpha\rangle\Diamond_{\mathsf{a}} A \vdash Pre(\alpha) \wedge \bigvee\left(\Diamond_{\mathsf{a}}\langle\beta_i\rangle A\right)}}$$

$$
\begin{array}{ccc}
A \vdash A & \cdots & A \vdash A \\
\hline
\circ_a A \vdash \Diamond_a A & \cdots & \circ_a A \vdash \Diamond_a A \\
\hline
\{\alpha\}\circ_a A \vdash \langle\alpha\rangle\Diamond_a A & \cdots & \{\alpha\}\circ_a A \vdash \langle\alpha\rangle\Diamond_a A \\
\hline
Pre(\alpha)\,;\{\alpha\}\circ_a A \vdash \langle\alpha\rangle\Diamond_a A & \cdots & Pre(\alpha)\,;\{\alpha\}\circ_a A \vdash \langle\alpha\rangle\Diamond_a A \\
\hline
Pre(\alpha)\,;\circ_a\{\beta_1\}A \vdash \langle\alpha\rangle\Diamond_a A \;\; s\text{-}in & \cdots & Pre(\alpha)\,;\circ_a\{\beta_n\}A \vdash \langle\alpha\rangle\Diamond_a A \;\; s\text{-}in \\
\hline
\circ_a\{\beta_1\}A \vdash Pre(\alpha) > \langle\alpha\rangle\Diamond_a A & \cdots & \circ_a\{\beta_n\}A \vdash Pre(\alpha) > \langle\alpha\rangle\Diamond_a A \\
\hline
\{\beta_1\}A \vdash \bullet_a(Pre(\alpha) > \langle\alpha\rangle\Diamond_a A) & \cdots & \{\beta_n\}A \vdash \bullet_a(Pre(\alpha) > \langle\alpha\rangle\Diamond_a A) \\
\hline
\langle\beta_1\rangle A \vdash \bullet_a(Pre(\alpha) > \langle\alpha\rangle\Diamond_a A) & \cdots & \langle\beta_n\rangle A \vdash \bullet_a(Pre(\alpha) > \langle\alpha\rangle\Diamond_a A) \\
\hline
\circ_a\langle\beta_1\rangle A \vdash Pre(\alpha) > \langle\alpha\rangle\Diamond_a A & \cdots & \circ_a\langle\beta_n\rangle A \vdash Pre(\alpha) > \langle\alpha\rangle\Diamond_a A \\
\hline
\Diamond_a\langle\beta_1\rangle A \vdash Pre(\alpha) > \langle\alpha\rangle\Diamond_a A & \cdots & \Diamond_a\langle\beta_n\rangle A \vdash Pre(\alpha) > \langle\alpha\rangle\Diamond_a A \\
\hdashline
\end{array}
$$

$$
\begin{array}{c}
\bigvee\left(\Diamond_a\langle\beta_i\rangle A\right) \vdash \boldsymbol{;}\left(Pre(\alpha) > \langle\alpha\rangle\Diamond_a A\right) \\
\hdashline
\bigvee\left(\Diamond_a\langle\beta_i\rangle A\right) \vdash Pre(\alpha) > \langle\alpha\rangle\Diamond_a A \\
\hline
Pre(\alpha)\,;\bigvee\left(\Diamond_a\langle\beta_i\rangle A\right) \vdash \langle\alpha\rangle\Diamond_a A \\
\hline
Pre(\alpha)\wedge\bigvee\left(\Diamond_a\langle\beta_i\rangle A\right) \vdash \langle\alpha\rangle\Diamond_a A
\end{array}
$$

5.3 Cut-elimination

In the present subsection, we outline the proof of the cut eliminability of D.EAK following the original strategy devised by Gentzen (cf. [20]). Without loss of generality, we consider a derivation π of the sequent $X \vdash Y$ in D.EAK which contains a unique application of Cut as the last rule (let us refer to this application as Cut*), and we show that a derivation of the same sequent exists in which Cut is not applied. We proceed by induction on the set of tuples (ρ, δ), ordered lexicographically, where ρ is the complexity of the cut formula in Cut* (the *rank* of Cut*), and δ is the sum of the maximal lengths of branches in the subdeductions of the premises of Cut* (the *degree* of Cut*). In the base case, Cut* can be directly eliminated by exhibiting a cut-free proof π' with the same conclusion. This is more in general the case when the cut formula is atomic.

The inductive step consists in transforming π into a derivation π' with the same conclusion and one or more applications of Cut with lower rank or with same rank but lower degree. The typical situation in the original Gentzen proof is that, when the cut formula is not atomic and is not *principal*[11] in at least one of the premises, the transformation involves one or more Cut-applications of same rank and lower degree than Cut*, whereas when the cut formula is not atomic and is principal in both premises, the transformation involves one or more Cut-applications of lower rank than Cut*, as illustrated, e.g., in the following transformation:

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array}}{\{\alpha\}X \vdash \langle\alpha\rangle A} \quad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ \{\alpha\}A \vdash Y\end{array}}{\langle\alpha\rangle A \vdash Y}}{\{\alpha\}X \vdash Y}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array} \quad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ \{\alpha\}A \vdash Y\end{array}}{A \vdash \widetilde{\alpha}\, Y}}{X \vdash \widetilde{\alpha}\, Y}}{\{\alpha\}X \vdash Y}
$$

This regularity breaks down when the Cut-formula is principal in both premises and has been introduced by means of an application of either contextual rules *reverse*. In

[11] An occurrence of a formula in a node of a derivation is *principal* if that occurrence has been introduced by means of the last rule applied in the subdeduction ending in that node.

this case, such a simple transformation as the one above is not available, and we need to consider all the possible ways in which the *proxy* $\{\alpha\}$ has been introduced in the subdeduction of each premise of Cut*. The proxy $\{\alpha\}$ might have been introduced by $\langle\alpha\rangle_L$, $atom_R$, *balance*, and $\left(\frac{\underline{\alpha}}{\{\alpha\}}\right)$ (applied bottom-up) on the left premise, and by $[\alpha]_R$, $atom_L$, *balance*, and $\left(\frac{\{\alpha\}}{\underline{\alpha}}\right)$ (applied bottom-up) on the right premise. This creates 16 subcases (each of which can be subdivided into simpler and more complicated instances), of which we illustrate just two (in their least complicated incarnations), as examples: the following one produces a Cut application of lower rank:

$$
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\,\pi_1\\ B \vdash A\end{array}}{[\alpha]B \vdash \{\alpha\}A}\\
\vdots\,\pi_1^*\\
X \vdash \{\alpha\}A
\end{array}
}{X \vdash [\alpha]A}
\quad
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\,\pi_2\\ A \vdash C\end{array}}{\{\alpha\}A \vdash \langle\alpha\rangle C}\\
\vdots\,\pi_2^*\\
Pre(\alpha)\,;\{\alpha\}A \vdash Y
\end{array}
}{Pre(\alpha)\,;[\alpha]A \vdash Y}
}{[\alpha]A \vdash Pre(\alpha) > Y}
}{X \vdash Pre(\alpha) > Y}
\quad\rightsquigarrow\quad
\begin{array}{c}
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ B \vdash A\end{array}\quad\begin{array}{c}\vdots\,\pi_2\\ A \vdash C\end{array}}{B \vdash C}}{\{\alpha\}B \vdash \langle\alpha\rangle C}\\
\vdots\,\pi_2^*\\
Pre(\alpha)\,;\{\alpha\}B \vdash Y
\end{array}
}{Pre(\alpha)\,;[\alpha]B \vdash Y}
}{[\alpha]B \vdash Pre(\alpha) > Y}\\
\vdots\,\pi_1^*\\
X \vdash Pre(\alpha) > Y
\end{array}
$$

the next one produces a Cut application of same rank and lower degree:

$$
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\,\pi_1\\ A \vdash A\end{array}}{A \vdash \{\alpha\}A}\\
\vdots\,\pi_1^*\\
X \vdash \{\alpha\}A
\end{array}
}{X \vdash [\alpha]A}
\quad
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\,\pi_2\\ A \vdash C\end{array}}{\{\alpha\}A \vdash \langle\alpha\rangle C}\\
\vdots\,\pi_2^*\\
Pre(\alpha)\,;\{\alpha\}A \vdash Y
\end{array}
}{Pre(\alpha)\,;[\alpha]A \vdash Y}
}{[\alpha]A \vdash Pre(\alpha) > Y}
}{X \vdash Pre(\alpha) > Y}
\quad\rightsquigarrow\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\,\pi_1\\ A \vdash A\end{array}}{A \vdash \{\alpha\}A}\\
\vdots\,\pi_1^*\\
X \vdash \{\alpha\}A
\end{array}
}{X \vdash [\alpha]A}
\quad
\dfrac{\begin{array}{c}\vdots\,\pi_2\\ A \vdash C\end{array}}{[\alpha]A \vdash \{\alpha\}C}
}{X \vdash \{\alpha\}C}
}{X \vdash Pre(\alpha) > \{\alpha\}C}
}{X \vdash Pre(\alpha) > \langle\alpha\rangle C}
}{Pre(\alpha)\,;X \vdash \langle\alpha\rangle C}\\
\vdots\,\pi_2^*\\
Pre(\alpha)\,;(Pre(\alpha)\,;X) \vdash Y
\end{array}
}{(Pre(\alpha)\,;Pre(\alpha))\,;X \vdash Y}
}{Pre(\alpha)\,;Pre(\alpha) \vdash Y < X}
}{Pre(\alpha) \vdash Y < X}
}{Pre(\alpha)\,;X \vdash Y}
}{X \vdash Pre(\alpha) > Y}
$$

6 Conclusions, conservativity, and further directions

Besides the cut-elimination, the results in the present paper can be summarized by the following chain of inclusions between consequence relations, where K is the class of standard Kripke models:

$$\models_{\mathsf{K}} = \vdash_{EAK} \subseteq \vdash_{D.EAK} \subseteq \models_{Z}\ .$$

D.EAK conservatively extends EAK. Of course, the language of the latter two consequence relations is an expansion of the language of the former two. To be able to claim that D.EAK adequately captures EAK, we need to show that $\vdash_{D.EAK}$ is a conservative extension of $\vdash_{EAK}$. To see this, let A, B be EAK-formulae such that $A \vdash_{D.EAK} B$. By the soundness of D.EAK w.r.t. the final coalgebra semantics, this implies that $[\![A]\!]_Z \subseteq [\![B]\!]_Z$, which, by Lemma 1, implies that $[\![A]\!]_M \subseteq [\![B]\!]_M$ for every Kripke model M, which, by the completeness of EAK w.r.t. the standard Kripke semantics, implies that $A \vdash_{EAK} B$.

Proof-theoretic semantics for EAK. The rules of EAK enjoy the following requirements, which are well known in the literature of proof-theoretic semantics [21, 22]: the fundamental structural rules of D.EAK are 'eliminable': i.e., *Id* can be restricted to atomic formulas, and *Cut* can be removed without affecting the set of theorems. The operational rules enjoy the properties of *separation*: each of them introduces exactly one connective, and of *symmetry*: for each connective, its left-introduction rules and its right-introduction rules form nonempty and disjoint sets. All of them but the *reverse* rules also enjoy *explicitness*, which can be reformulated as follows: the side structures occur unrestricted. However, the offending side substructure is limited to the formula $Pre(\alpha)$, which can always be derived, e.g. via weakening. Hence, we conjecture that this offense is essentially harmless. An entirely satisfactory motivation that D.EAK provides proof-theoretic semantics for the connectives of EAK is work in progress.

Intuitionistic coalgebraic semantics. We wish to develop the intuitionistic version of these results. This requires to work in the setting of the final coalgebra for the Vietoris functor on discrete Esakia spaces (S4-frames and p-morphisms).

Cut-elimination *á la* Belnap. Our proof of cut elimination, which is very lengthy and could only be sketched in the present paper, follows the methodology of Gentzen's original proof. A shorter and more insightful route to the same result consists in either applying Belnap's meta-theorem for cut elimination [6] for display calculi, or some suitable extension of it. In the latter case, this strengthening would be essentially analogous to extension of Belnap's meta-theorem to linear logic [7, 19], and is the focus of current investigation.

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