Canonical extensions of ordered algebraic structures and relational completeness of some substructural logics*

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Abstract

In this paper we introduce canonical extensions of partially ordered sets and monotone maps and a corresponding discrete duality. We then use these to give a uniform treatment of completeness of relational semantics for various substructural logics with implication as the residual(s) of fusion.

1 Introduction

Canonical extensions were first introduced by Jónsson and Tarski for Boolean algebras with operators (BAOs) in their 1950's papers [14, 15]. Canonical extensions provide an algebraic formulation of what is otherwise treated via topological duality or relational methods. The theory of canonical extensions has since been simplified and generalized [9, 7, 10], leading to a widely applicable and transparent theory which is now ready to be applied even in the setting of partially ordered algebras. The only restriction is that the basic operations of the algebras to be considered either preserve or reverse the order in each coordinate. We will call such algebras *monotone poset expansions* (MPEs).

Rather than developing a complete and general theory of canonical extensions for MPEs at this stage, we have opted here to develop only what is necessary to solve a particular problem. In recent years a number of papers on completeness of various substructural logics with respect to relational semantics have been published [2, 1, 3, 16, 17]. Relational semantics have

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proved very useful for modal and intuitionistic logics. In addition, these fit well with the algebraic theory as they are closely related to topological duality for these varieties. The purpose of searching for relational semantics for substructural logics as well is of course that it is hoped that these semantics will prove a powerful tool in a similar manner in this setting as well. However, topological duality for non-distributive lattice based ordered algebras is vastly more complicated than its Boolean and distributive counterparts and this is where canonical extensions come in. Canonical extensions have been explored to some extent for non-distributive lattices [7], and the surprising fact is that the theory is very similar and essentially as smooth in this setting as in the Boolean setting.

The purpose of this paper then is to explore the potential of canonical extensions as a tool for the development of a smooth and transparent theory of relational methods for non-distributive lattice based ordered algebraic structures. To this end we will consider the setting from Dunn's paper [5]. There he attempted to provide a uniform approach to relational semantics for various implicative substructural logics by means of representation theorems for the corresponding algebras suggested by his *gaggle theory*, see [4] (further references and more can be found in [6]). In this work he considered several logics including the Lambeck calculus, linear logic, BCK logic, and relevance logic. While he achieved a uniform approach in the sense that the relational semantics obtained all arose through concrete representation of the algebras, he had to change his method of representation in ad hoc ways to fit the various logics.

We show here that using canonical extensions and the associated discrete duality we obtain complete relational semantics for each of the logics considered in [5].

We have organized the paper as follows: In section 2 we define the canonical extension of a poset, and derive some of its properties. In section 3 we treat extension of maps and define canonical extensions for MPEs. In section 4 we develop a discrete duality for a class of complete lattices that includes all lattices that arise as canonical extensions of posets. In section 5 we discuss the dual of additional binary operations of the type appropriate to our applications. Finally, in section 6 we use the results from the previous sections to obtain relational semantics for the substructural logics mentioned above.

It should be noted that apart from solving the problem at hand, the work herein gives the fundamental ideas for how to define topological dualities, possibly for MPEs in general, and definitely for LEs. In particular, the dual characterization of arbitrary morphisms in our discrete duality provides the relational part of the dual characterizations of arbitrary homomorphisms for topological duality of bounded lattices, a long standing open problem.

2 The canonical extension of a poset

Let P be a poset. Recall that a *filter* of P is a non-empty subset F of P satisfying:

1. F is an up-set, that is, if $x \in F$, $y \in P$, and $x \leq y$, then $y \in F$;
2. F is down-directed, that is, if x and y are in F, then there exists $z \in F$ with $z \leq x$ and $z \leq y$.

An ideal of P is defined dually. That is, I is an *ideal* of P provided I is a non-empty up-directed down-set of P. We denote by $\mathcal{F}$ and $\mathcal{I}$ the families of all filters and ideals of P, respectively.

We make the following definitions here:

Definition 2.1. Let P be a poset.

1. An *extension* of P is an order embedding $e : P \to Q$, i.e., for every $x, y \in P$, $x \leq y$ if and only if $e(x) \leq e(y)$. For ease of notation we will suppress e and call Q an extension of P and assume that P is a subposet of Q.
2. Given an extension Q of P, an element of Q is called *closed* provided it is the infimum in Q of some filter F of P. We denote the set of all closed elements of Q by $K(Q)$. Dually, an element of Q is called *open* provided it is the supremum in Q of some ideal I of P. We denote the set of all open elements of Q by $O(Q)$.
3. An extension Q of P is said to be *dense* provided each element of Q is both the supremum of all the closed elements below it and the infimum of all the open elements above it.
4. An extension Q of P is said to be *compact* provided that whenever D is a non-empty down-directed subset of P, U is a non-empty up-directed subset of P, and $\bigwedge_Q D \leq \bigvee_Q U$, then there are $x \in D$ and $y \in U$ with $x \leq y$.

We are now almost ready to give the abstract definition of the canonical extension of a poset. But first recall that a *completion* of a poset P is an extension Q of P which also happens to be a complete lattice.

Definition 2.2. Let P be a poset. A *canonical extension* of P is a dense and compact completion of P.

Remark 2.3. This definition exactly agrees with the definition of canonical extension for bounded lattices [7], bounded distributive lattices [9], and Boolean algebras [14]. As we shall see next, this definition provides an abstract characterization of the very natural completion of P obtained via the polarity given by the set of filters of P, the set of ideals of P, and the relation of non-disjointness between these. One may still question our choice of filters and ideals, if for no other reason then at least because there are several other possible choices that agree with this one in the case of bounded lattices. A note currently in preparation by the two last authors develops extensions of this type in greater generality and shows that the choice which leads to what is called canonical extension here is unique in allowing certain (algebraically) desirable properties such as preserving the existing finitary lattice structure, commuting with (Cartesian) product, and destroying all purely infinitary lattice structure. We may also mention that this choice can be viewed as natural from the point of view of abstract algebraic logic. This is the subject of an article in preparation by the third author. Finally, there certainly are situations in which one does not necessarily want to destroy all the infinitary structure of the original poset (e.g. dynamic modal logic) and then one might want to restrict the set of filters and ideals. However, this may lead to loss of preserved algebraic identities. Nevertheless, investigations in this direction would certainly be of interest.

The following proposition tends to cut our work in half when proving various results and is of intrinsic importance for applications in logic. For any poset P we denote by P^∂ the dual poset obtained by reversing the order on P.

Proposition 2.4. *Let P be a poset. If Q is a canonical extension of P, then Q^∂ is a canonical extension of P^∂.*

Proof. This follows from the fact that each of the defining properties: denseness, compactness, and being a completion are self-dual. $\square$

Theorem 2.5. *Let P be a poset. If P has a canonical extension then it is unique up to an isomorphism that fixes P.*

Proof. First we show that the set of closed elements of any canonical extension Q of P is reverse order isomorphic to the poset $(\mathcal{F}, \subseteq)$ of filters of P. First of all notice that the definition of closed elements yields a surjective order reversing map from $\mathcal{F}$ to $K(Q)$ given by $F \mapsto x_F = \bigwedge_Q F$. Now let $F, G \in \mathcal{F}$ with $x_F \geq x_G$. Let $p \in F$, then $x_F \leq p$ and thus $x_G \leq p$. We now have $D = G$ is a non-empty down-directed subset of P, $U = \{p\}$ is a non-empty up-directed subset of P, and $\bigwedge_Q G = x_G \leq p = \bigwedge_Q \{p\}$. By compactness we get $p \in G$ and we have shown that $F \subseteq G$.

By duality we then get that the set of open elements of any canonical extension Q of P is order isomorphic to the poset $(\mathcal{I}, \subseteq)$ of ideals of P via the order isomorphism $I \mapsto y_I = \bigvee_Q I$.

Next we show that the order structure on $K(Q) \cup O(Q)$ is uniquely determined by showing that for $F \in \mathcal{F}$ and $I \in \mathcal{I}$ we have:

1. $x_F \leq y_I$ if and only if $F \cap I \neq \emptyset$
2. $y_I \leq x_F$ if and only if for all $x \in I$ and for all $y \in F$, $x \leq y$.

To see that this is so notice that the 'if' part of 1 is trivial and the 'only if' part follows by compactness, whereas 2 simply is a reformulation of the definitions of supremum and infimum. Thus we have shown that for any two canonical extensions of P the union of the closed and the open elements of the two extensions are isomorphic via an isomorphism that fixes P.

Finally notice that denseness implies that any canonical extension is the Dedekind-MacNeille completion of the union of its closed and open elements. Now the uniqueness of the Dedekind-MacNeille completion of a poset combined with the uniqueness of the structure of the closed elements union the open elements yields the desired result. $\square$

Theorem 2.6. *Let P be a poset, then P has a canonical extension.*

Proof. The above uniqueness proof essentially spells out a construction of the canonical extension of P: First one defines a quasi-order on $\mathcal{F} \cup \mathcal{I}$ according to the two conditions given in the above proof. The corresponding partially ordered quotient will yield the open elements union the closed elements. Then one takes the Dedekind-MacNeille completion of this poset.

Alternatively, the polarity given by non-empty intersection between the filters and the ideals of P yields a Galois connection:

$$\Phi : \mathcal{P}(\mathcal{F}) \rightleftarrows \mathcal{P}(\mathcal{I}) : \Psi$$

where $\Phi(X) = \{I : F \in X \text{ implies } F \cap I \neq \emptyset\}$ and $\Psi(X) = \{F : I \in X \text{ implies } F \cap I \neq \emptyset\}$. The Galois closed subsets $\mathcal{G} = \{X \subseteq \mathcal{F} : \Psi(\Phi(X)) = X\}$ form a complete lattice. It is not too hard to show that the map $e : P \to \mathcal{G}$ given by $e(p) = \{F : p \in F\}$ is a canonical extension of P. We leave the details to the reader. $\square$

Remark 2.7. The description/construction of the canonical extension given in the proof of Theorem 2.5 is closely related to the construction of the canonical extension of BAOs given in [13], and the second construction corresponds to the one used for bounded lattices in [7]. Both constructions are clearly constructive unlike the more traditional constructions via topological duality. Below, we will see that the non-constructiveness comes in when we want to stipulate that the canonical extension has enough completely join and meet irreducible elements.

We will denote the canonical extension of a poset P by P^σ. The set of all completely join irreducible elements of a poset P we will denote by $J^\infty(P)$, the set of all completely meet irreducible elements of P we will denote by $M^\infty(P)$. We now give a few fundamental properties of canonical extension.

Theorem 2.8. *Let P, and Q be posets, then*

1. $(P^\partial)^\sigma = (P^\sigma)^\partial$.
2. $(P \times Q)^\sigma = P^\sigma \times Q^\sigma$.
3. *$J^\infty(P^\sigma)$ is join-dense in P^σ and $M^\infty(P^\sigma)$ is meet-dense in P^σ.*
4. *Any finite meets and joins existing in P are preserved in P^σ.*

Proof. 1 is just Proposition 2.4. To prove 2 first notice the projection onto either coordinate of an up- (down-)directed subset of the product is up- (down-)directed. Conversely, the product of two up- (down-)directed sets in the factors is up- (down-)directed in the product. Thus, viewing $P^\sigma \times Q^\sigma$ as an extension of $P \times Q$, the open (closed) elements are exactly the coordinate-wise open (closed) elements. Thus it is clear that $P^\sigma \times Q^\sigma$ is a dense completion of $P \times Q$. To see that it is also compact, we use again that the projection onto either coordinate of an up- (down-)directed subset of the product is up- (down-)directed. Then the coordinate-wise compactness yields the compactness of the product.

Statement 3 holds essentially because filters (ideals) are closed under unions of chains. Also notice that the axiom of choice is needed to prove 3. To show that $J^\infty(P^\sigma)$ is join-dense in P^σ, we just need to show that each closed element is the join of the completely join irreducible elements below it as we already know that the closed elements are join dense in P^σ. To this end let x be a closed element of P^σ. Let u be the join of 0 and all the completely join irreducible elements of P^σ that are below x. Then certainly $u \leq x$. Suppose however that $u < x$. Now we use the fact that the open elements of P^σ are meet-dense in P^σ to get $y \in O(P^\sigma)$ with $u \leq y$ but $x \nleq y$. Consider the set $S = \{v \in K(P^\sigma) : v \leq x \text{ but } v \nleq y\}$. Then S is non-empty since $x \in S$. Also if C is a non-empty chain in S, then we claim that $c_0 = \bigwedge C$ is again in S. First of all c_0 is a closed element. This is because $c_0 = \bigwedge\{p \in P : p \geq c \text{ for some } c \in C\}$ and the set $\{p \in P : p \geq c \text{ for some } c \in C\}$ is a filter of P since C is a non-empty chain. Secondly $c_0 \leq c \leq x$ for any one $c \in C$, so $c_0 \leq x$. Now suppose $c_0 \leq y$. But then $\bigwedge\{p \in P : p \geq c \text{ for some } c \in C\} \leq y$ and by compactness there is $p \in P$ and $c \in C$ so that $p \geq c$ and $p \leq y$. This implies that $c \leq y$ which contradicts the fact that $c \in S$. Thus $c_0 \nleq y$ and $c_0 \in S$ and therefore by Zorn's Lemma we may conclude that S has minimal elements. Let j be minimal in S. Since $j \nleq y$ we know that $j \neq 0$. We show that j is completely join irreducible. To this end suppose that $j = \bigvee T$ where T is a subset of P^σ. Again because the closed elements are join-dense in P^σ, we may assume that T is a set of closed elements. For each $t \in T$ with $t < j$ we know that $t \notin S$ by minimality of j. The only condition that can be violated is $t \nleq y$, and thus we know that for each $t \in T$ with $t < j$ we have $t \leq y$. It follows that $\bigvee\{t \in T : t < j\} \leq y$, and as $j \nleq y$, we conclude that $\bigvee\{t \in T : t < j\} \neq j$. As $j = \bigvee T$ there must be $t \in T$ with $t \nless j$, that is $t = j$. So we have shown that there is a completely join irreducible element in S. But j completely join irreducible and $j \leq x$ imply that $j \leq u$ by definition of u. This in turn implies that $j \leq y$ since $u \leq y$ by definition of y. But this contradicts $j \in S$ which implies that $j \nleq y$. We conclude that $u = x$ and that x is in fact the join of all the completely join irreducible elements below x.

Finally, to prove 4, let $x, y \in P$ and suppose $x \vee y$ exists in P and call this element z. Now let u denote the supremum of x and y in P^σ. Then $u \leq z$. We show that $z \leq u$ by showing that that every open element v above u also is above z. So let $v \in O(P^\sigma)$ with $u \leq v$. Then $x \leq v$ and $y \leq v$. The fact that v is open means that $I = \{p \in P : p \leq y\}$ is an ideal and that the join of this ideal is v. Since $x, y \in I$ there must be $p \in I$ with $x \leq p$ and $y \leq p$. Now, since z is the supremum of x and y in P, we must have that $z \leq p$ and thus $z \in I$. We conclude that $z \leq v$ and thus $z \leq u$ and $x \vee_P y = x \vee_{P^\sigma} y$. The corresponding statement for binary meets follows by duality. Notice also that if P has a zero then every open element of P^σ is above it since ideals are required to be non-empty. But then the fact that the opens are meet dense in P^σ implies that the zero of P is also the zero of P^σ. Dually for 1. $\square$

The key fact for the use of canonical extensions in studying relational semantics for various logics corresponding to BAOs and DLEs is that canonical extension produces a *concrete algebra*, that is, an algebra which may be viewed as the complex algebra of some relational structure. The corresponding fact for the underlying lattice is that it may be viewed as a complete field of sets in the Boolean case and a complete ring of sets in the DL case. This can of course not be true once we are outside the scope of distributive lattices. Nevertheless there is an analogue here, and the complete lattices obtained when taking the canonical extensions of posets are also concrete in some sense that we will describe in Section 4. Right here we give the abstract description of these 'concrete' lattices.

Definition 2.9. A complete lattice C is *perfect* provided $J^\infty(C)$ is join-dense in C and $M^\infty(C)$ is meet-dense in C, i.e. for every $x \in C$, $x = \bigvee\{j \in J^\infty(C) : j \leq x\}$ and $x = \bigwedge\{m \in M^\infty(C) : x \leq m\}$.

It should be clear that among Boolean algebras the ones that are perfect lattices are exactly the complete and atomic Boolean algebras, and among distributive lattices the ones that are perfect are exactly the complete, completely distributive lattices that are join generated by their completely join prime elements, or equivalently the doubly algebraic complete distributive lattices.

Corollary 2.10. *Let P be any poset, then P^σ is a perfect lattice.*

Proof. This is exactly the content of statement 3 in the above theorem. □

3 Canonical extensions of maps

In order to be able to define canonical extensions for monotone poset expansions, we now need to describe how we will extend additional operations from a poset to its canonical extension. First we define monotone operations and monotone poset expansions.

Definition 3.1. Each $\varepsilon \in \{1, \partial\}^n$ is called a monotonicity type. An n-ary operation $f : P^n \to P$ on a poset P is said to be *monotone* provided there is $\varepsilon \in \{1, \partial\}^n$ so that the order-variant $f : P^\varepsilon \to P$,where $P^\varepsilon = P^{\varepsilon_1} \times \ldots \times P^{\varepsilon_n}$ and P^1 is P and P^∂ is the order-dual of P, is order preserving. For more details of this notation see [11].

A *monotone poset expansion* (MPE) is a tuple $(P, (f_i)_{i \in I})$ where P is a poset and each f_i is an n_i-ary monotone operation on P. The corresponding sequence $(\varepsilon_i)_{i\in I}$ of monotonicity types for the individual operations is called the monotonicity type of the MPE $(P, (f_i)_{i\in I})$.

Given a monotone operation $f : P^n \to P$ on a poset P, we may consider it as an order preserving map $f : P^\varepsilon \to P$ whose domain is obtained from P by taking some combination of order duals and finite product. Since canonical extension of posets commutes with both of these, all we need to define is how to extend order preserving maps.

Definition 3.2. Let P and Q be posets, and $f : P \to Q$ an order preserving map. Define maps $f^\sigma, f^\pi : P^\sigma \to Q^\sigma$ by setting:

$$f^\sigma(u) = \bigvee\{\bigwedge\{f(p) : x \leq p \in P\} : u \geq x \in K(P^\sigma)\}$$
$$f^\pi(u) = \bigwedge\{\bigvee\{f(p) : y \geq p \in P\} : u \leq y \in O(P^\sigma)\}.$$

Remark 3.3. It should be clear to anyone familiar with the theory of canonical extensions that these definitions agree with the ones given in the bounded lattice, distributive bounded lattice, and Boolean cases. In the setting of bounded distributive lattices it was shown in [10] that these extensions satisfy universal properties with respect to certain topologies on canonical extensions. It was also shown in [10] and [7] that canonical extensions (both the sigma or the pi versions) are functorial for monotone DLEs and LEs. Of course the corresponding questions should also be considered and answered in this more general setting eventually. However, they are not central to the problem considered in this paper and we leave them for future work.

As in the lattice case, the following facts hold for these extensions:

Lemma 3.4. *For every order preserving map $f : P \to Q$, both f^σ and f^π are order preserving extensions of f. In addition $f^\sigma \leq f^\pi$ with equality holding on both $K(P^\sigma)$ and $O(P^\sigma)$. For $u \in P^\sigma$, $x \in K(P^\sigma)$, and $y \in O(P^\sigma)$ we have*

$$f^\sigma(u) = \bigvee\{f^\sigma(x) : u \geq x \in K(P^\sigma)\}$$
$$f^\sigma(x) = \bigwedge\{f(p) : x \leq p \in P\}$$
$$f^\pi(u) = \bigwedge\{f^\pi(y) : u \leq y \in O(P^\sigma)\}$$
$$f^\pi(y) = \bigvee\{f(p) : y \geq p \in P\}$$

and f^σ and f^π send closed elements to closed elements and open elements to open elements.

Proof. Let $x, x' \in K(P^\sigma)$ with $x \leq x'$, then as f is order preserving we have $\bigwedge\{f(p) : x \leq p \in P\} \leq \bigwedge\{f(p) : x' \leq p \in P\}$ and therefore $f^\sigma(x) = \bigwedge\{f(p) : x \leq p \in P\}$. Notice also that the set $\{f(p) : x \leq p \in P\}$ is down-directed so that f^σ sends closed elements to closed elements. Furthermore this description of f^σ for closed elements together with the fact that f is order preserving easily yields the fact that f^σ extends f. Finally, it also easily implies that f^σ is order preserving on $K(P^\sigma)$ and that $f^\sigma(u) = \bigvee\{f^\sigma(x) : u \geq x \in K(P^\sigma)\}$. This last fact in turn easily implies that f^σ is order preserving on all of P^σ. By duality all the corresponding statements hold for f^π. In terms of the claims about the relationship between f^σ and f^π, notice that for $x \in K(P^\sigma)$

$$\begin{aligned} f^\pi(x) &= \bigwedge\{f^\pi(y) : x \leq y \in O(P^\sigma)\} \\ &\leq \bigwedge\{f^\pi(p) : x \leq p \in P\} \\ &= \bigwedge\{f(p) : x \leq p \in P\} = f^\sigma(x). \end{aligned}$$

Also, for each $y \in O(P^\sigma)$ with $x \leq y$, by compactness, there exists $p \in P$ with $x \leq p \leq y$. Thus

$$\bigwedge\{f^\pi(y) : x \leq y \in O(P^\sigma)\} \geq \bigwedge\{f^\pi(p) : x \leq p \in P\}$$

and we conclude that $f^\pi(x) = f^\sigma(x)$. By duality $f^\sigma(y) = f^\pi(y)$ for $y \in O(P^\sigma)$ as well. Finally we see that for $u \in P^\sigma$

$$\begin{aligned} f^\sigma(u) &= \bigvee\{f^\sigma(x) : u \geq x \in K(P^\sigma)\} \\ &= \bigvee\{f^\pi(x) : u \geq x \in K(P^\sigma)\} \\ &\leq \bigwedge\{f^\pi(y) : u \leq y \in O(P^\sigma)\} = f^\pi(u). \end{aligned}$$

□

Since we consider two, generally, different ways of extending maps, the canonical extension of an MPE is not uniquely determined. As in the lattice and distributive lattice settings, whether we want to extend a particular additional operation using the σ- or the π-extension depends on the properties of the particular operation to be extended. Here, as in [8] we make a general definition of β-canonical extensions.

Definition 3.5. Let $(P, (f_i)_{i \in I})$ be an MPE with additional operations indexed by a set I. Let β be a map from I to the set $\{\sigma, \pi\}$. Then the β-canonical extension of $(P, (f_i)_{i \in I})$ is the MPE $(P^\sigma, (f_i^{\beta(i)})_{i \in I})$.

In order to obtain relational semantics from a class of algebras it is necessary that the additional operations are such that they can be encoded by relations on the dual. This will be the case when the extensions of the maps are (possibly up to some turning upside down of coordinates) complete operators or complete dual operators. Here we show that this happens for residuated maps.

First we recall some facts about residuation of maps. Let P, Q and R be partial orders and let $f : P \times Q \to R$, $g : P \times R \to Q$ and $h : R \times Q \to P$. The map g is called the *right residual* of f provided for every $p \in P$, $q \in Q$ and $r \in R$,

$$f(p, q) \leq r \text{ if and only if } q \leq g(p, r).$$

h is called the *left residual* of f provided for every $p \in P$, $q \in Q$ and $r \in R$,

$$f(p, q) \leq r \text{ if and only if } p \leq h(r, q).$$

Further, if g and h are the right and left residuals of f, respectively, then

1. $f : P \times Q \to R$, $g : P^\partial \times R \to Q$ and $h : R \times Q^\partial \to P$ are order preserving.

2. If P, Q and R are complete lattices, then $f : P \times Q \to R$ preserves arbitrary joins in each coordinate, and $g : P^\partial \times R \to Q$ and $h : R \times Q^\partial \to P$ preserve arbitrary meets in each coordinate.

Proposition 3.6. *Let P, Q and R be partial orders and let $f : P \times Q \to R$ be order preserving. Then for $g : P \times R \to Q$ and $h : R \times Q \to P$,*

1. *if g is the right residual of f, then g^π is the right residual of f^σ.*

2. *If h is the left residual of f, then h^π is the left residual of f^σ.*

Proof. To prove statement 1, we have to show that for every $u \in P^\sigma$, $v \in Q^\sigma$, and $w \in R^\sigma$,

$$f^\sigma(u, v) \leq w \text{ if and only if } v \leq g^\pi(u, w).$$

We first show that if $s \in K(P^\sigma)$, $t \in K(Q^\sigma)$, $y \in O(R^\sigma)$, and $f^\sigma(s, t) \leq y$, then $t \leq g^\pi(s, y)$. We have $f^\sigma(s, t) = \bigwedge\{f(p, q) : s \leq p \in P, t \leq q \in Q\} \leq m$. Since s and t are closed, so is $f^\sigma(s, t)$. In fact $\bigwedge\{f(p, q) : s \leq p \in P, t \leq q \in Q\}$ is a down-directed family in $P \times Q$ whose infimum is $f^\sigma(s, t)$. Also, y is open and thus by compactness, $f(p, q) \leq m$ for some $p \in P$ and $q \in Q$ with $s \leq p$ and $t \leq q$. Let $f(p, q) = r \in R$. As $f(p, q) \leq r$ and g is the right residual of f, $q \leq g(p, r)$. Now since $p \leq s$

and $r = f(p,q) \leq m$, $g(p,r) \in \{g(p',r') : s \leq p' \in P, m \geq r' \in R\}$ and thus $t \leq q \leq g(p,r) \leq \bigvee\{g(p',r') : s \leq p' \in P, m \geq r' \in R\} = g^\pi(s,y)$ as desired. For the general case, assume that $f^\sigma(u,v) \leq w$. We need to show that $v \leq g^\pi(u,w) = \bigwedge\{g^\pi(s,y) : u \geq s \in K(P^\sigma), w \leq y \in O(R^\sigma)\}$, i.e. that $v \leq g^\pi(s,y)$ for every $s \in K(P^\sigma)$ and $y \in O(R^\sigma)$, with $s \leq u$ and $w \leq y$. By denseness, $v = \bigvee\{t \in K(Q^\sigma) : t \leq v\}$ and it is enough to show that $t \leq g^\pi(s,y)$ for every $s \in K(P^\sigma))$, $t \in K(Q^\sigma)$, and $y \in O(R^\sigma)$ with $s \leq u$, $t \leq v$, and $w \leq y$. By assumption, $\bigvee\{f^\sigma(s,t) : u \geq s \in K(P^\sigma), v \geq t \in K(Q^\sigma)\} = f^\sigma(u,v) \leq w$, so $f^\sigma(s,t) \leq y$ for every $s \in K(P^\sigma), t \in K(Q^\sigma)$, and $y \in O(R^\sigma)$ such that $s \leq u$, $t \leq v$ and $w \leq y$. We have already seen that this implies $t \leq g^\pi(s,y)$.

For the other direction, assume $v \leq g^\pi(u,w)$. Then $t \leq g^\pi(s,y)$ whenever $s \in K(P^\sigma), t \in K(Q^\sigma), y \in O(R^\sigma), s \leq u, t \leq v$, and $w \leq y$. Since $g^\pi(s,y) = \bigvee\{g(p,r) : s \leq p \in P \text{ and } y \geq r \in R\}$, by compactness, there are $p \in P, r \in R$, with $s \leq p, y \geq r$ and $t \leq g(p,r)$. Using the fact that g is the right residual of f, we get $f(p, g(p,r)) \leq r$, and thus $f^\sigma(s,t) \leq f(p, g(p,r)) \leq r \leq y$. It follows that $f^\sigma(u,v) \leq w$ as desired. Statement 2 is obtained by interchanging the role of the coordinates of f. $\square$

Corollary 3.7. *Let P, Q, and R be partial orders and $f : P \times Q \to R$ a map with right and left residuals $g : P \times R \to Q$, and $h : R \times Q \to P$, respectively. Then*

1. *f^σ is a complete operator;*
2. *$g^\pi : (P^\sigma)^\partial \times R^\sigma \to Q^\sigma$ and $h^\pi : R^\sigma \times (Q^\sigma)^\partial \to P^\sigma$ are complete dual operators.*

4 Basic Discrete duality

Here we develop a duality between the categories of what we have named perfect lattices and what we will call perfect posets.

Definition 4.1. A lattice C is called *perfect* provided $J^\infty(C)$ is join-dense in C and $M^\infty(C)$ is meet-dense in C. A homomorphism of perfect lattices is a complete lattice homomorphism.

Given a perfect lattice C, we define the dual of C to be $\underline{C} = J^\infty(C) \cup M^\infty(C)$.

Example 4.2. Given a complete and atomic Boolean algebra B, we usually define the discrete dual to be B_+, the set of atoms of B, which of course is the same as the set of all completely join irreducible elements of B. The set of completely join irreducible elements is sufficient as long as we stay within the realm of DL^+s. But here we must keep both the completely join and the completely meet irreducible elements to be able to reconstruct the lattice. Notice that $\underline{B}$ may be obtained from B_+ by taking the disjoint union $B_+ \uplus B_+$ and defining an order by $x \leq y$ if and only if either x and y are in the same copy of B_+ and $x = y$, or x is in the first copy of B_+, y is in the second copy of B_+, and $x \neq y$.

Definition 4.3. A partial order Z is *perfect* provided $J^\infty(Z)$ is join-dense in Z, $M^\infty(Z)$ is meet-dense in Z, and $Z = J^\infty(Z) \cup M^\infty(Z)$.

Remark 4.4. The morphisms of this category are *not* the order preserving maps or anything like that. The remaining work of this section discovers what they are. While it is not the most obvious and simple thing, the remarkable fact here is that they are *something* that is first order definable (also in the non-surjective case).

We first finish working out the duality on objects. Given a perfect poset Z, we define the dual of Z to be the Dedekind-MacNeille completion $\overline{Z}$ of Z.

Example 4.5. Given a complete and atomic Boolean algebra, notice that the Dedekind-MacNeille completion of $\underline{B}$ is isomorphic to the complex algebra (namely the power set) of B_+.

We will need the following possibly not so well-known but very important property of the Dedekind-MacNeille completion.

Proposition 4.6. *For every poset P, $J^\infty(\overline{P}) = J^\infty(P)$ and $M^\infty(\overline{P}) = M^\infty(P)$.*

Proof. Let $x \in J^\infty(\overline{P})$. Then, as $x = \bigvee\{p : x \geq p \in P\}$ by join-denseness of P, $x \in P$. If $x = \bigvee_P A$ for some $A \subseteq P \subseteq \overline{P}$, then, as the Dedekind-MacNeille completion preserves all existing joins, $x = \bigvee_{\overline{P}} A$, hence, by complete join irreducibility of x in $\overline{P}$, $x \in A$. So $J^\infty(\overline{P}) \subseteq J^\infty(P)$. Now let $x \in J^\infty(P)$ and assume $x = \bigvee A$ for some $A \subseteq \overline{P}$. Now using the join-denseness of P in $\overline{P}$, we get $x = \bigvee X$, where $X = \{p \in P : p \leq a \text{ for some } a \in A\}$, and since x is completely join irreducible in P, it follows that $x = x'$ for some $x' \in X$. But then, by the definition of X, there is $a \in A$ with $x' \leq a$. Finally, since $a \leq x$ it follows that $x = a \in A$, and x is also completely join irreducible in $\overline{P}$. The statement for completely meet irreducible elements follows by order duality. $\square$

Proposition 4.7. *For every perfect lattice C,*

1. $\overline{(\underline{C})} \cong C$;

2. $J^{\infty}(\underline{C}) = J^{\infty}(C)$ *and* $M^{\infty}(\underline{C}) = M^{\infty}(C)$;

3. $\underline{C}$ *is a perfect poset.*

Proof. The statement 1 follows from the abstract characterization of the Dedekind-MacNeille completion of a poset as the complete lattice in which the poset is both join- and meet-dense: By the definition of a perfect lattice clearly $\underline{C} = J^{\infty}(C) \cup M^{\infty}(C)$ is both join- and meet-dense in C. By 1, the statement 2 is exactly the content of Proposition 4.6. The statement 3 now clearly follows by 2 and the definition of $\underline{C}$. □

Proposition 4.8. *The following are equivalent for every poset P:*

1. P *is a perfect poset;*

2. $P = J^{\infty}(P) \cup M^{\infty}(P)$ *and* $\overline{P}$ *is a perfect lattice.*

Proof. If P is a perfect poset, then $P = J^{\infty}(P) \cup M^{\infty}(P)$ by definition. Also, P is join-dense in $\overline{P}$, and $J^{\infty}(P)$ is join-dense in P, and Dedekind-MacNeille completion preserves existing joins, so $J^{\infty}(P)$ is join-dense in $\overline{P}$. Finally, by Proposition 4.6, it follows that $J^{\infty}(P) = J^{\infty}(\overline{P})$. The fact that $M^{\infty}(\overline{P})$ is meet-dense in $\overline{P}$ follows by order duality. For the converse, assume that $P = J^{\infty}(P) \cup M^{\infty}(P)$ and that $\overline{P}$ is a perfect lattice. By Proposition 4.7, $J^{\infty}(P) = J^{\infty}(\overline{P})$, and since $\overline{P}$ is assumed to be a perfect lattice this set is join-dense in $\overline{P}$. But then it is certainly also join-dense in the smaller set P. By order duality we conclude that $M^{\infty}(P)$ is meet-dense in P. We have thus shown that 2 implies 1. □

Corollary 4.9. *For every perfect poset Z, $\overline{Z}$ is a perfect lattice and $\underline{(\overline{Z})} = Z$.*

Now that we have defined the dual of objects we turn to maps. We will want to apply this discrete duality not only to perfect lattices but actually to perfect lattices with additional monotone operations that are either complete operators or complete dual operators. Thus, eventually at least, we need to find duals not only for homomorphisms but also for complete operators and complete dual operators. We will do this, for the binary case, in the next section. However, here we already treat the unary case as a complete homomorphism may be seen as a complete operator and a complete dual operator with the additional equation that says the two are equal.

Definition 4.10. Let $f : C \to D$ be a completely join preserving map between perfect lattices. Define $R_f \subseteq J^\infty(C) \times M^\infty(D)$ as follows:

$$xR_f\, n \quad \text{if and only if} \quad f(x) \leq n.$$

Remark 4.11. Notice that this definition is a departure from the way R_f is defined in the distributive and Boolean settings. There, R_f is a binary relation from $J^\infty(D)$ to $J^\infty(C)$ given by $y \leq f(x)$. This definition is also available to us here and would also work. In fact, it is an easy 'toggle' between these two and it may very well be that one in general wants to keep both available. However, for the work we will do in this paper, the above definition makes the exposition slightly smoother, and it is all we need to name explicitely.

In the distributive and Boolean settings, we have a tight relationship between $J^\infty(C)$ and $M^\infty(C)$ as they are isomorphic. This means that all the structure can be moved to just one of these sets, as it traditionally is. In the general setting that we treat here, $J^\infty(C)$ and $M^\infty(C)$ are no longer isomorphic in general, and both sets must be kept in play. Accordingly, we will need some machinery to toggle back and forth between $J^\infty(C)$ and $M^\infty(C)$.

Definition 4.12. Given a poset P in which $J^\infty(P)$ is join-dense and $M^\infty(P)$ is meet-dense, we define

$$\begin{aligned}
()^u : \mathcal{P}(J^\infty(P)) &\longrightarrow \mathcal{P}(M^\infty(P)) \\
A &\longmapsto \{m : a \in A \text{ implies } a \leq m\}. \\
()^l : \mathcal{P}(M^\infty(P)) &\longrightarrow \mathcal{P}(J^\infty(P)) \\
B &\longmapsto \{x : b \in B \text{ implies } x \leq b\}.
\end{aligned}$$

Remark 4.13. It is well known that these maps form a Galois connection. It is easy to see that the Dedekind-MacNeille completion of P, which is equal to the Dedekind-MacNeille completion of $J^\infty(P) \cup M^\infty(P)$, is isomorphic to the lattice of the Galois closed sets of this Galois connection. This is of course the restriction of the upper/lower Galois connection on P, which gives the standard construction of the Dedekind-MacNeille completion. The above construction is sufficient exactly because $J^\infty(P)$ is join-dense in P and $M^\infty(P)$ is meet-dense in P. Thus for a perfect poset Z this Galois connection gives $\overline{Z}$, and for a perfect lattice C this Galois connection gives

$\underline{C} = C$. We only prove the following two facts about this Galois connection, which will be used repeatedly in what follows.

Lemma 4.14. *If C is a perfect lattice, then for every $X \subseteq J^\infty(C)$ and every $Y \subseteq M^\infty(C)$,*

1. $\bigwedge X^u = \bigvee X$.
2. $\bigvee Y^l = \bigwedge Y$.

Proof. We just prove 1. Let $\bigvee X = x_0$. As C is a perfect lattice, $x_0 = \bigwedge\{m \in M^\infty(C) : m \geq x_0\}$, so

$$\begin{aligned}\bigvee X &= \bigwedge\{m \in M^\infty(C) : m \geq x_0\}\\ &= \bigwedge\{m \in M^\infty(C) : x \in X \text{ implies } m \geq x\}\\ &= \bigwedge X^u.\end{aligned}$$

□

Proposition 4.15. *For every completely join preserving map $f : C \to D$ between perfect lattices, every $x \in J^\infty(C)$ and every $n \in M^\infty(D)$,*

*R*1. $(R_f[x])^{lu} = R_f[x]$.

*R*2. $(R_f^{-1}[n])^{ul} = R_f^{-1}[n]$.

Proof. In order to prove 1, as $(\ )^{lu}$ is a closure operator, we just need to show that $(R_f[x])^{lu} \subseteq R_f[x]$. Let $n \in (R_f[x])^{lu}$. We have to show that $f(x) \leq n$. As $n \in (R_f[x])^{lu}$, then $n \geq \bigvee R[x]^l = \bigwedge R_f[x]$, so it is enough to show that $f(x) = \bigwedge R_f[x]$. As D is perfect, then $f(x) = \bigwedge\{n \in M^\infty(D) : f(x) \leq n\} = \bigwedge R_f[x]$. The proof of 2 is similar. □

Definition 4.16. Let Y and Z be perfect posets, and let $R \subseteq J^\infty(Y) \times M^\infty(Z)$. We say that R is an *(unary) operator relation* provided for every $x \in J^\infty(Y)$ and every $n \in M^\infty(Z)$, the following conditions are satisfied:

*R*1. $(R[x])^{lu} = R[x]$.

*R*2. $(R^{-1}[n])^{ul} = R^{-1}[n]$.

Given an operator relation $R \subseteq J^\infty(Y) \times M^\infty(Z)$, define $f_R : \overline{Y} \to \overline{Z}$ by setting:

1. For every $x \in J^\infty(Y)$, $f_R(x) = \bigwedge R[x]$;

2. For every $u \in \overline{Y}$, $f_R(u) = \bigvee\{f_R(x) : u \geq x \in J^\infty(Y)\}$.

Lemma 4.17. *Let Y and Z be perfect posets and $R \subseteq J^\infty(Y) \times M^\infty(Z)$ an operator relation. For every $u \in \overline{Y}$ we have*

$$\begin{aligned} f_R(u) &= \bigwedge(\bigcap\{R[x] : u \geq x \in J^\infty(Y)\}) \\ &= \bigwedge\{m \in M^\infty(Z) : u \geq x \in J^\infty(Y) \textit{ implies } xRm\}. \end{aligned}$$

Proof. For every $u \in \overline{Y}$,

$$\begin{aligned} f_R(u) &= \bigvee\{f_R(x) : u \geq x \in J^\infty(Y)\} \\ &= \bigvee\{\bigwedge R[x] : u \geq x \in J^\infty(Y)\} \\ &= \bigvee\{\bigwedge(R[x])^{lu} : u \geq x \in J^\infty(Y)\} \\ &= \bigvee\{\bigvee(R[x])^{l} : u \geq x \in J^\infty(Y)\} \\ &= \bigvee(\bigcup\{(R[x])^{l} : u \geq x \in J^\infty(Y)\}) \\ &= \bigvee(\bigcap\{R[x] : u \geq x \in J^\infty(Y)\})^{l} \\ &= \bigwedge(\bigcap\{R[x] : u \geq x \in J^\infty(Y)\}). \end{aligned}$$

□

Proposition 4.18. *Given Y and Z perfect posets and an operator relation $R \subseteq J^\infty(Y) \times M^\infty(Z)$, we have:*

1. $\leq \circ R \circ \leq \; = \; R$.
2. $f_R(x) \leq n$ *if and only if* xRn.
3. f_R *is order preserving.*
4. f_R *preserves arbitrary joins.*

Proof. To prove 1 assume that $x \leq x'Rn$, and let us show that xRn. By $R2$, it is enough to show that $x \in (R^{-1}[n])^{ul}$. As $x'Rn$, $x' \in R^{-1}[n]$, so $x' \leq m$ for every $m \in (R^{-1}[n])^u$. Hence $x \leq x' \leq m$ for every $m \in (R^{-1}[x])^u$, and so $x \in (R^{-1}[n])^{ul}$. This shows that $\leq \circ R \subseteq R$. Now assume that $xRn' \leq n$, and let us show that xRn. By $R1$, it is enough to show that $n \in (R[x])^{lu}$. As xRn', $n' \in R[x]$, so $y \leq n'$ for every $y \in (R[x])^l$. Hence

$y \le n' \le n$ for every $y \in (R[x])^l$, and so $n \in (R[x])^{lu}$. This shows that $\le \circ R \circ \le \subseteq R$. The other containment is trivial as the order is reflexive. For 2 first notice that if xRn, then $n \in R[x]$ and hence $f_R(x) = \bigwedge R[x] \le n$. For the converse, assume that $f_R(x) \le n$. By $R1$, it is enough to show that $n \in (R[x])^{lu}$. For every $y \in (R[x])^l$, $y \le \bigwedge R[x] = f_R(x) \le n$, hence $n \in (R[x])^{lu}$. To prove 3 it is enough to show that f_R is order preserving on $J^\infty(Y)$. Let $x, x' \in J^\infty(Y)$ with $x \le x'$. If we show that $R[x'] \subseteq R[x]$ then $f_R(x) = \bigwedge R[x] \le \bigwedge R[x'] = f_R(x')$ and we are done. But if $x'Rn$ then $x \le x'Rn$ and so by 1 xRn. To prove 4 let $u \in \overline{Y}$ and suppose $u = \bigvee A$. Since f_R is order preserving we have $f_R(u) \ge \bigvee f_R(A)$. Thus we need to show that $f_R(u) \le \bigvee f_R(A)$. To this end first notice that as $\overline{Y}$ is join-generated by $J^\infty(Y)$ we can suppose that $A \subseteq J^\infty(Y)$. But then $f_R(a) = \bigwedge R[a]$ and then, as in the proof of Lemma 4.17, we get $\bigvee f_R[A] = \bigwedge(\bigcap\{R[a] : a \in A\})$. Finally since each $a \in A$ satisfies $u \ge a \in J^\infty(Y)$, we have $\bigwedge(\bigcap\{R[a] : a \in A\}) \ge \bigwedge(\bigcap\{R[x] : u \ge x \in J^\infty(Y)\})$ and thus $\bigvee f_R[A] \ge f_R(u)$ as desired. □

Proposition 4.19. *For every completely join preserving map $f : C \to D$ between perfect lattices, $f_{R_f} = f$.*

Proof. For every $x \in J^\infty(C) = J^\infty(\underline{C})$, $f_{R_f}(x) = \bigwedge\{n \in M^\infty(D) : xR_f n\} = \bigwedge\{n \in M^\infty(D) : f(x) \le n\} = f(x)$ and then the result follows as both functions are completely join preserving. □

Proposition 4.20. *For any two perfect posets Y and Z and operator relation $R \subseteq J^\infty(Y) \times M^\infty(Z)$, $R_{f_R} = R$.*

Proof. For $x \in J^\infty(Y)$ and $n \in M^\infty(Z)$, we have $xR_{f_R}n$ if and only if $f_R(x) \le n$ if and only if xRn. □

The above takes care of dualizing unary completely join preserving maps. By order duality we then also are able to dualize completely meet preserving maps: If $g : C \to D$ is a completely meet preserving map between perfect lattices, then $g^\partial : C^\partial \to D^\partial$ is a completely join preserving map between perfect lattices. The associated relation $S_g \subseteq M^\infty(C) \times J^\infty(D)$ is defined by setting for every $m \in M^\infty(C)$ and every $y \in J^\infty(D)$

$$mS_g\, y \quad \text{if and only if} \quad g(m) \ge y,$$

and the dual relations, which we will refer to as *(unary) dual operator relations*, are characterized by the following properties:

$S1$. $(S_g[m])^{ul} = S_g[m]$.

$S2$. $(S_g^{-1}[y])^{lu} = S_g^{-1}[y]$.

Notice that $S1$ is $R2$ for R^{-1}, where R is a relation corresponding to a completely join preserving map, and similarly for $S2$ and $R1$.

Now that we have found duals for completely join preserving as well as completely meet preserving maps we are ready to dualize homomorphisms. As mentioned above, we will do this by finding the dual characterization of an equational theory where we have one complete operator f, one complete dual operator g, and one equation $f = g$ – or actually rather two inequalities: $f \leq g$ and $g \leq f$. Since we have already described the relations dual to f and g, what we still need to do is to find first order correspondents of the two inequalities $f \leq g$ and $g \leq f$. Notice that, using modal notation, these are of the form $\Diamond \leq \Box$ and $\Box \leq \Diamond$, respectively. The first is the most basic type of Sahlqvist equation, namely the strictly positive kind, and the second is the more involved kind. Nevertheless, finding the first order duals in this setting is not much more involved than in Sahlqvist theory for classical modal logic based on Boolean algebras. Thus this certainly indicates the basic building blocks of a Sahlqvist-type correspondence theory for the partially ordered setting. Working this out further is the subject of current work by the two last authors and H. Priestley.

Proposition 4.21. *Let $f, g : C \to D$ be maps between perfect lattices with f completely join preserving and g completely meet preserving. Furthermore, let R be the operator relation corresponding to f, and let S be the dual operator relation corresponding to g. Then the following are equivalent:*

1. *For all $u \in C$ $f(u) \leq g(u)$;*
2. *For all $x \in J^\infty(C)$ and $m \in M^\infty(C)$, if $x \leq m$ then $f(x) \leq g(m)$;*
3. *For all $x \in J^\infty(C)$ and $m \in M^\infty(C)$, if $x \leq m$ then $(R[x])^l \subseteq S[m]$;*
4. *For all $x \in J^\infty(C)$ and $m \in M^\infty(C)$, if $x \leq m$ then $(S[m])^u \subseteq R[x]$.*

Proof. That 1 implies 2 is clear. On the other hand if 2 holds and $u \in C$, then

$$\begin{aligned} f(u) &= \bigvee\{f(x) : u \geq x \in J^\infty(C)\} \\ &\leq \bigwedge\{g(m) : u \leq m \in M^\infty(C)\} = g(u) \end{aligned}$$

and thus 1 holds. To show that 2 is equivalent to 3, notice that $f(x) = \bigwedge R[x] = \bigvee(R[x])^l$ and $g(m) = \bigvee S[m] = \bigvee(S[m])^{ul}$. Finally, the equivalence of 3 and 4 follows simply from the fact that $S[m]$ and $R[x]$ are closed under ul and lu, respectively. $\Box$

The following lemma is the essential content of the Sahlqvist mechanism for equations like $\Box \leq \Diamond$. It is just the fact that completely join preserving and completely meet preserving maps are residuated combined with the observation that the relation for the residual is the converse of the relation for the original maps.

Lemma 4.22. *Let $f : C \to D$ be a completely join preserving map between perfect lattices, and let $n \in M^\infty(D)$. Then*

$$\textit{For all } u \in C \quad f(u) \leq n \textit{ if and only if } u \leq u_n$$

where $u_n = \bigvee R^{-1}[n]$. Dually, let $g : C \to D$ be a completely meet preserving map between perfect lattices, and let $y \in J^\infty(D)$. Then

$$\textit{For all } u \in C \quad g(u) \geq y \textit{ if and only if } u \geq v_y$$

where $v_y = \bigwedge S^{-1}[y]$.

Proof. This is because f is completely join preserving and g is completely meet preserving. □

Proposition 4.23. *Let $f, g : C \to D$ be maps between perfect lattices with f completely join preserving and g completely meet preserving. Then the following are equivalent:*

1. *For all $u \in C$ $g(u) \leq f(u)$;*
2. *For all $n \in M^\infty(D)$ $g(u_n) \leq f(u_n)$;*
3. *For all $y \in J^\infty(D)$ and $n \in M^\infty(D)$, if $v_y \leq u_n$ then $y \leq f(u_n)$;*
4. *For all $y \in J^\infty(D)$ and $n \in M^\infty(D)$,*

$$\textit{if } (S^{-1}[y])^l \subseteq R^{-1}[n] \quad \textit{then} \quad \bigcap\{R[x] : x \in R^{-1}[n]\} \subseteq y\uparrow;$$

5. *For all $y \in J^\infty(D)$ and $n \in M^\infty(D)$, if $v_y \leq u_n$ then $g(v_y) \leq n$;*
6. *For all $y \in J^\infty(D)$ and $n \in M^\infty(D)$,*

$$\textit{if } (S^{-1}[y])^l \subseteq R^{-1}[n] \quad \textit{then} \quad \bigcap\{S[m] : m \in S^{-1}[y]\} \subseteq n\downarrow.$$

Proof. The fact that 1 implies 2 is clear. To see that 2 implies 1, let $u \in C$ and $n \in M^\infty(D)$ with $n \geq f(u)$. Then $u \leq u_n$ and thus we have $g(u) \leq g(u_n) \leq f(u_n) \leq n$. Since this holds for every $n \in M^\infty(D)$ $g(u) \leq f(u)$ and 1 holds. The fact that 2 is equivalent to 3 follows from the fact that $g(u_n) \leq f(u_n)$ holds if and only if for all $y \in J^\infty(D)$ $y \leq g(u_n)$ implies $y \leq f(u_n)$, and from the fact that $y \leq g(u_n)$ holds if and only if $v_y \leq u_n$. The statement 4 is equivalent to 3 as it just uses the definitions of the various entities in 3 in terms of R and S. Finally 5 and 6 follow by order duality from 3 and 4, respectively. □

Definition 4.24. Given perfect posets Y and Z, we define a *morphism* from Z to Y to be a pair (R, S) of binary relations $R \subseteq J^\infty(Y) \times M^\infty(Z)$ and $S \subseteq M^\infty(Y) \times J^\infty(Z)$ satisfying $R1$ and $R2$, and $S1$ and $S2$, respectively, as well as the equivalent conditions of both Proposition 4.21 and Proposition 4.23.

Remark 4.25. While it is not clear at this point how to best understand these morphisms, the main point is that *all* morphisms, not just surjective ones, have first-order definable duals.

One alternative option which may be better than 4 and 6 in Proposition 4.23 is:
4. $((R^{-1}[n])^u)^S \subseteq ((R^{-1}[n])^R)^l$;
6. $((S^{-1}[y])^S)^u \subseteq ((S^{-1}[y])^l)^R$
where $A^R = \{n : \text{for all } x \text{ if } x \in A \text{ then } xRn\}$. Note that in this notation $A^u = A^{\leq}$.

For surjective homomorphisms we get a somewhat simpler description of the dual, thus explaining why these were easier to discover.

Lemma 4.26. *Let $h : C \to D$ be a complete homomorphism between perfect lattices that is surjective, then*

1. *For all $n \in M^\infty(D)$ we have $u_n \in M^\infty(C)$;*
2. *For all $y \in J^\infty(D)$ we have $v_y \in J^\infty(C)$.*

Proof. To prove 1, recall that u_n is the largest element of C that is mapped below n by h. Also, by surjectivity, we must have $h(u_n) = n$. Now we get $n = h(u_n) = \bigwedge\{h(m) : u_n \geq m \in M^\infty(C)\}$ since h is also completely meet preserving. But then, by complete meet irreducibility of n, we get $n = h(m)$ for some m with $u_n \geq m \in M^\infty(C)$. But then, by maximality of u_n with respect to $h(\_\_) \leq n$ we have that $u_n = m \in M^\infty(C)$. The proof of 2 is (order) dual. □

Thus for $h : C \to D$ a surjective complete homomorphism between perfect lattices, define

$$r_h : M^\infty(D) \to M^\infty(C)$$
$$n \mapsto u_n = \bigvee\{x : h(x) \le n\}$$

and

$$s_h : J^\infty(D) \to J^\infty(C)$$
$$y \mapsto v_y = \bigwedge\{m : h(m) \ge y\}.$$

Proposition 4.27. *Let $h : C \to D$ be a surjective complete homomorphism between perfect lattices, then*

1. *For all $n, n' \in M^\infty(D)$ $n \le n'$ if and only if $r_h(n) \le r_h(n')$;*
2. *For all $y, y' \in J^\infty(D)$ $y \le y'$ if and only if $s_h(y) \le s_h(y')$;*
3. *For all $y \in J^\infty(D)$ and $n \in M^\infty(D)$ $y \le n$ if and only if $s_h(y) \le r_h(n)$;*
4. *For all $x \in J^\infty(C)$ $(r_h^{-1}(x\uparrow))^l = s_h^{-1}(x\downarrow)$ and $(s_h^{-1}(x\downarrow))^u = r_h^{-1}(x\uparrow)$;*
5. *For all $m \in M^\infty(C)$ $(r_h^{-1}(m\uparrow))^l = s_h^{-1}(m\downarrow)$ and $(s_h^{-1}(m\downarrow))^u = r_h^{-1}(m\uparrow)$.*

Proof. For 1, if $n \le n'$ then clearly $r_h(n) \le r_h(n')$. For the converse, since h is onto, we have $n = h(r_h(n)) \le h(r_h(n')) = n'$. The statement 2 is dual to 1. For 3, if $y \le n$, then $h(s_h(y)) = y \le n$ so $s_h(y) \le r_h(n)$. Conversely, if $s_h(y) \le r_h(n)$, then $y = h(s_h(y)) \le h(r_h(n)) = n$. Statements 4 and 5 follow from the fact that for any $u \in C$ $r_h^{-1}(u\uparrow) = \{n : h(u) \le n\}$ and $s_h^{-1}(u\downarrow) = \{y : h(u) \ge y\}$. $\square$

Proposition 4.28. *Let C and D be perfect lattices. For a pair of maps $(r : M^\infty(D) \to M^\infty(C), s : J^\infty(D) \to J^\infty(C))$ satisfying the five properties in Proposition 4.27, the map*

$$h = h_{(r,s)} : C \to D$$

where $h(u) = \bigwedge\{n : u \le r_h(n)\} = \bigvee\{y : u \ge s_h(y)\}$ is a surjective complete homomorphism with $r_h = r$ and $s_h = s$.

Proof. Define $R \subseteq J^\infty(C) \times M^\infty(D)$ by xRn if and only if $x \leq r_h(n)$ and $S \subseteq M^\infty(C) \times J^\infty(D)$ by mRy if and only if $m \geq s_h(y)$. Then we have for $u \in C$

$$\begin{aligned} f_R(u) &= \bigwedge\{n : \text{For all } x \text{ if } x \leq u \text{ then } xRn\} \\ &= \bigwedge\{n : \text{For all } x \text{ if } x \leq u \text{ then } x \leq r(n)\} \\ &= \bigwedge\{n : u \leq r(n)\} \end{aligned}$$

and dually

$$g_S(u) = \bigvee\{y : u \geq s(y)\}.$$

Thus we will have that h is a well-defined complete homomorphism if we can show that (R, S) is a morphism of perfect posets. The fact that h is surjective and that $r_h = r$ and $s_h = s$ is easy to check.

Let $n \in M^\infty(D)$, then $R^{-1}[n] = \{r(n)\}^l$ and thus $(R^{-1}[n])^{ul} = R^{-1}[n]$. Dually we have $(S^{-1}[y])^{lu} = S^{-1}[y]$. For $x \in J^\infty(C)$,

$$\begin{aligned} (R[x])^{lu} &= (\{n : x \leq r(n)\})^{lu} \\ &= (r^{-1}(x\uparrow))^{lu} \\ &= r^{-1}(x\uparrow) \\ &= R[x] \end{aligned}$$

by property 4 in Proposition 4.27, and dually $(S[m])^{ul} = S[m]$ by property 5 in Proposition 4.27. Thus f_R is completely join preserving and g_S is completely meet preserving. We now check that R and S give the same function. Let $x \in J^\infty(C)$ and $m \in M^\infty(C)$ with $x \leq m$ then

$$\begin{aligned} (R[x])^l &= s^{-1}(x\downarrow) \\ &= \{y : s(y) \leq x\} \\ &\subseteq \{y : s(y) \leq x\} = S[m]. \end{aligned}$$

Finally we show that for all $n \in M^\infty(D)$ we have $g_S(u_n) \leq f_R(u_n)$. Notice that

$$\begin{aligned} f_R(u_n) &= f_R(r(n)) \\ &= \bigwedge\{n' : r(n) \leq r(n')\} \\ &= \bigwedge\{n' : n \leq n'\} = n, \end{aligned}$$

so we just need to show that $g_s(u_n) \leq n$. We have

$$\begin{aligned} g_S(u_n) &= g_S(r(n)) \\ &= \bigvee\{y : s(y) \leq r(n)\} \\ &= \bigvee\{y : y \leq n\} = n. \end{aligned}$$

This completes the proof. $\square$

Thus one may take the duals of surjective perfect lattice morphisms to be pairs of maps $(r : M^\infty(Z) \to M^\infty(Y), s : J^\infty(Z) \to J^\infty(Y))$, where Y and Z are perfect posets, satisfying the five properties in Proposition 4.27.

5 Discrete duals of binary operators and dual operators

We now return to the task of developing duals of complete operators and complete dual operators.

Lemma 5.1. *Let C_1, C_2 and D be perfect lattices, $f : C_1 \times C_2 \to D$ be completely join preserving in each coordinate, $g : C_1^\partial \times C_2 \to D$ and $h : C_1 \times C_2^\partial \to D$ be completely meet preserving in each coordinate. Then for every $u_i \in C_i$, $i = 1, 2$,*

1. $f(u_1, u_2) = \bigvee\{f(x_1, x_2) : u_i \geq x_i \in J^\infty(C_i)\}$.
2. $g(u_1, u_2) = \bigwedge\{g(x_1, m_2) : u_1 \geq x_1 \in J^\infty(C_1)$ *and* $u_2 \leq m_2 \in M^\infty(C_2)\}$.
3. $h(u_1, u_2) = \bigwedge\{h(m_1, x_2) : u_2 \geq x_2 \in J^\infty(C_2)$ *and* $u_1 \leq m_1 \in M^\infty(C_1)\}$.

Proof. To prove 1, let $u_i \in C_i$, $i = 1, 2$. As C_i is perfect, $u_i = \bigvee A_i$, where $A_i = \{x_i \in J^\infty(C_i) : x_i \leq u_i\}$. As f preserves arbitrary joins in each coordinate, we get that

$$\begin{aligned} f(u_1, u_2) &= f(\bigvee A_1, u_2) \\ &= \bigvee\{f(x_1, u_2) : x_1 \in A_1\} \\ &= \bigvee\{\bigvee\{f(x_1, x_2) : x_2 \in A_2\} : x_1 \in A_1\} \\ &= \bigvee\{f(x_1, x_2) : x_i \in A_i, i = 1, 2\}. \end{aligned}$$

To prove 2, let $u_i \in C_i$, $i = 1, 2$. As C_i is perfect, $u_1 = \bigvee A_1$, where $A_1 = \{x_1 \in J^\infty(C_1) : x_1 \leq u_1\}$, and $u_2 = \bigwedge B_2$, where $B_2 = \{m_2 \in M^\infty(C_2) : m_2 \geq u_2\}$. As g preserves arbitrary meetss in each coordinate, we get that

$$\begin{aligned}
g(u_1, u_2) &= g(\textstyle\bigvee A_1, u_2) \\
&= \textstyle\bigwedge\{g(x_1, u_2) : x_1 \in A_1\} \\
&= \textstyle\bigwedge\{\bigwedge\{f(x_1, m_2) : m_2 \in B_2\} : x_1 \in A_1\} \\
&= \textstyle\bigwedge\{f(x_1, m_2) : x_1 \in A_1 \text{ and } m_2 \in B_2\}.
\end{aligned}$$

The proof of 3 is similar to the proof of statement 2. □

Remark 5.2. Notice that the result and the arguments given in the above lemma and its proof depend on the fact that $f(u_1, u_2) = 0$ as soon as at least one of u_1 and u_2 is 0 (and corresponding statements may be made for g and h). This will indeed be the case for all the basic operations considered in this paper, but it is of course a little restrictive. A duality theory can easily be developed without this restriction, however then 0 must be added to the set of completely join irreducibles and 1 must be added to the set of completely meet irreducibles. For work of this kind in the DL setting see [12].

Let C_1, C_2 and D be perfect lattices, and let $f : C_1 \times C_2 \to D$ be completely join preserving in each coordinate. Let us define $R_f \subseteq J^\infty(C_1) \times J^\infty(C_2) \times M^\infty(D)$ as follows:

$$R_f(x_1, x_2, n) \quad \text{if and only if} \quad f(x_1, x_2) \leq n.$$

Proposition 5.3. *For every $f : C_1 \times C_2 \to D$ completely join preserving in each coordinate, evey $x_i \in J^\infty(C_i)$, $i = 1, 2$, and every $n \in M^\infty(D)$,*

*R*1. $(R_f(x_1, x_2, \_)^{lu} = R_f(x_1, x_2, \_)$.

*R*2$_1$. $R_f(\_, x_2, n)^{ul} = R_f(\_, x_2, n)$.

*R*2$_2$. $R_f(x_1, \_, n)^{ul} = R_f(x_1, \_, n)$.

Proof. To prove 1 let $n \in (R_f(x_1, x_2, \_))^{lu}$. We have to show that $f(x_1, x_2) \leq n$. As $n \in (R_f(x_1, x_2, \_))^{lu}$, then $n \geq \bigvee(R_f(x_1, x_2, \_))^{l} = \bigwedge R_f(x_1, x_2, \_)$

by lemma 4.14, so it is enough to show that $f(x_1, x_2) = \bigwedge R_f(x_1, x_2, \_)$. Using the fact that D is a perfect lattice, we get the desired equality:

$$\begin{aligned} f(x_1, x_2) &= \bigwedge\{n \in M^\infty(D) : f(x_1, x_2) \leq n\} \\ &= \bigwedge\{n \in M^\infty(D) : R_f(x_1, x_2, n)\} \\ &= \bigwedge R_f(x_1, x_2, \_). \end{aligned}$$

To prove 2 let $x_2 \in (R_f(x_1, \_, n))^{ul}$, then $x_2 \leq \bigwedge(R_f(x_1, \_, n))^u$. By lemma 4.14, $\bigwedge(R_f(x_1, \_, n))^u = \bigvee R_f(x_1, \_, n)$. As f is completely join preserving, f is order preserving, and so

$$\begin{aligned} f(x_1, x_2) &\leq f(x_1, \bigvee R_f(x_1, \_, n)) \\ &= \bigvee\{f(x_1, x') : x' \in R_f(x_1, \_, n)\} \\ &= \bigvee\{f(x_1, x') : f(x_1, x') \leq n\} \leq n. \end{aligned}$$

The proof of 3 is like the proof of statement 2. $\square$

Definition 5.4. Let Y_1, Y_2, and Z be perfect posets, and let $R \subseteq J^\infty(Y_1) \times J^\infty(Y_2) \times M^\infty(Z)$. We call R a *(binary) operator relation* provided, for every $x_i \in J^\infty(Y_i)$, $i = 1, 2$ and for every $n \in M^\infty(Z)$

$R1$. $(R(x_1, x_2, \_)^{lu} = R(x_1, x_2, \_)$.

$R2_1$. $R(\_, x_2, n)^{ul} = R(\_, x_2, n)$.

$R2_2$. $R(x_1, \_, n)^{ul} = R(x_1, \_, n)$.

Given such a binary operator relation, define $f_R : \overline{Y}_1 \times \overline{Y}_2 \to \overline{Z}$ by setting:

1. For $x_i \in J^\infty(\overline{Y}_i)$, $i = 1, 2$, $f_R(x_1, x_2) = \bigwedge R(x_1, x_2, \_)$.
2. For $u_i \in \overline{Y}_i$, $i = 1, 2$,

$$f(u_1, u_2) = \bigvee\{f(x_1, x_2) : u_i \geq x_i \in J^\infty(\overline{Y}_i), i = 1, 2\}$$

.

Proposition 5.5. *Let Y_1, Y_2, and Z be perfect posets and $R \subseteq J^\infty(Y_1) \times J^\infty(Y_2) \times M^\infty(Z)$ be an operator relation, then*

1. *$R(\_, x_2, n)$ and $R(x_1, \_, n)$ are downsets, and $R(x_1, x_2, \_)$ is an upset.*

2. $f_R(x_1, x_2) \leq n$ *if and only if* $R(x_1, x_2, n)$.

3. f_R *is order preserving.*

4. f_R *is completely join preserving in each coordinate.*

Proof. To prove 1 we show that $R(\_, x_2, n)$ is a downset: Assume $x_1 \leq x'$ and $R(x', x_2, n)$. We show that $x_1 \in (R(\_, x_2, n))^{ul}$. As $R(x', x_2, n)$, $x' \in R(\_, x_2, n)$, so $x' \leq m$ for every $m \in (R(\_, x_2, n))^u$. So $x_1 \leq x' \leq m$ for every $m \in (R(\_, x_2, n))^u$ and consequently $x_1 \in (R(\_, x_2, n))^{ul}$. We also show that $R(x_1, x_2, \_)$ is an upset: Assume that $n' \leq n$ and that $n' \in R(x_1, x_2, \_)$. We show that $n \in (R(x_1, x_2, \_))^{lu}$. As $n' \in R(x_1, x_2, \_)$, $y \leq n'$ for every $y \in (R(x_1, x_2, \_))^l$. Hence $y \leq n' \leq n$ for every $y \in (R(x_1, x_2, \_))^l$ and $n \in (R(x_1, x_2, \_))^{lu}$. The proof of the last statement is similar. To prove 2, first notice that if $R(x_1, x_2, n)$, then $n \in R(x_1, x_2, \_)$ and thus $f_R(x_1, x_2) \leq n$. Conversely, if $f_R(x_1, x_2) \leq n$, then $y \leq n$ for each $y \in (R(x_1, x_2, \_))^l$. That is, $n \in (R(x_1, x_2, \_))^{lu} = R(x_1, x_2, \_)$ as desired. To prove 3, it is enough to show that f_R is order preserving on $J^\infty(Y_1) \times J^\infty(Y_2)$. To this end, let $x_i, x_i' \in J^\infty(Y_i)$ with $x_i \leq x_i'$. Let $n \in R(x_1', x_2', \_)$, then $x_1 \leq x_1' \in R(\_, x_2', n)$, and so by 1 $R(x_1, x_2', n)$, and so $x_2 \leq x_2' \in R(x_1, \_, n)$ and we get $R(x_1, x_2, n)$ by 1. It now follows that

$$f_R(x_1, x_2) = \bigwedge R(x_1, x_2, \_) \leq \bigwedge R(x_1', x_2', \_) = f_R(x_1', x_2').$$

To prove 4 we show that f_R is completely join preserving in the first coordinate: Let $u_i \in \overline{Y}_i$ and assume that $u_1 = \bigvee A$. That $f_R(u_1, u_2) \geq \bigvee \{f_R(a, u_2) : a \in A\}$ follows from 3. For the other direction we may assume that $A \subseteq J^\infty(Y_1)$ since $J^\infty(Y_1)$ is join-dense in $\overline{Y}_1$. Also, since $\overline{Z}$ is meet-generated by $M^\infty(Z)$, it is enough to show that if $n \in M^\infty(Z)$ and $n \geq \bigvee \{f_R(a, u_2) : a \in A\}$ then $n \geq f_R(u_1, u_2) = \bigvee \{f_R(x_1, x_2) : x_i \leq u_i\}$, i.e. $f_R(x_1, x_2) \leq n$ whenever $x_i \leq u_i$, for $i = 1, 2$. But $n \geq \bigvee \{f_R(a, u_2) : a \in A\}$ implies that $n \geq \bigvee \{f_R(a, x_2) : a \in A\}$ and this means that $R(a, x_2, n)$ for each $a \in A$. Now let $m \in (R(\_, x_2, n))^u$, then $m \geq a$ for each $a \in A$, and thus $x_1 \leq u_1 = \bigvee A \leq m$. So $x_1 \leq m$ for each $m \in (R(\_, x_2, n))^u$. That is, $x_1 \in (R(\_, x_2, n))^{ul} = R(\_, x_2, n)$ and $R(x_1, x_2, n)$ or $f_R(x_1, x_2) \leq n$ as desired. □

Proposition 5.6. *Let* C_1, C_2, D *be perfect lattices, and* $f : C_1 \times C_2 \to D$ *a map that is completely join preserving in each coordinate, then* $f_{R_f} = f$.

Proof. For $x_i \in J^\infty(C_i)$ for $i = 1, 2$,

$$\begin{aligned} f_{R_f}(x_1, x_2) &= \bigwedge\{n \in M^\infty(Z) : R_f(x_1, x_2, n)\} \\ &= \bigwedge\{n \in M^\infty(Z) : f(x_1, x_2) \leq n\} f(x_1, x_2). \end{aligned}$$

So $f_{R_f} = f$ on $J^\infty(C_1) \times J^\infty(C_2)$, and for $(u_1, u_2) \in C_1 \times C_2$ we get

$$\begin{aligned} f_{R_f}(u_1, u_2) &= \bigvee\{f_{R_f}(x_1, x_2) : u_i \geq x_i \in J^\infty(C_i)\} \\ &= \bigvee\{f(x_1, x_2) : u_i \geq x_i \in J^\infty(C_i)\} \\ &= \bigvee\{\bigvee\{f(x_1, x_2) : u_1 \geq x_1 \in J^\infty(C_1)\} : u_2 \geq x_2 \in J^\infty(Y_2)\} \\ &= \bigvee\{f(u_1, x_2) : u_2 \geq x_2 \in J^\infty(C_2)\} \\ &= f(u_1, u_2). \end{aligned}$$

□

Proposition 5.7. *Let Y_1, Y_2, and Z be perfect posets, and let $R \subseteq J^\infty(Y_1) \times J^\infty(Y_2) \times M^\infty(Z)$ be a operator relation, then $R_{f_R} = R$.*

Proof. We have $R_{f_R}(x_1, x_2, n)$ if and only if $f_R(x_1, x_2) \leq n$ if and only if $R(x_1, x_2, n)$. □

The above treatment also takes care of binary maps $g : C_1^\partial \times C_2 \to D$ and $h : C_1 \times C_2^\partial \to D$ that are completely meet preserving in each coordinate: If $g : C_1^\partial \times C_2 \to D$ is completely meet preserving in each coordinate, then $g^\partial : C_1 \times C_2^\partial \to D^\partial$ is a completely join preserving map. Given $x_1 \in J^\infty(C_1)$, $m_2 \in M^\infty(C_2)$, and $x \in J^\infty(D)$, the relation associated with g $S_g \subseteq J^\infty(C_1) \times M^\infty(C_2) \times J^\infty(D)$ is defined by

$$S_g(x_1, m_2, x) \quad \text{if and only if} \quad g(x_1, m_2) \geq x,$$

and these dual relations are characterized by the following properties:

$S1$. $(S_g(x_1, m_2, \_))^{ul} = S_g(x_1, m_2, \_)$.

$S2_1^\partial$. $(S_g(\_, m_2, x))^{ul} = S_g(\_, m_2, x)$.

$S2_2$. $(S_g(x_1, \_, x))^{lu} = S_g(x_1, \_, x)$.

Notice that if $R \subseteq J^\infty(Y_1) \times J^\infty(Y_2) \times M^\infty(Z)$ is an operator relation, then the relation $S \subseteq J^\infty(Y_1) \times M^\infty(Z) \times J^\infty(Y_2)$ obtained by simply moving around the order of the coordinates: $R(x_1, x_2, n)$ if and only if $S(x_1, n, x_2)$ is a relation of the type just described, that is, it satisfies $S1$, $S2_1^\partial$, and $S2_2^\partial$. As the next proposition shows this is no coincidence:

Proposition 5.8. *Let C be a perfect lattice, $f : C \times C \to C$ a complete operator, and $g : C^\partial \times C \to C$ a complete dual operator. The following two statement are equivalent:*

1. *g is the right residual of f;*
2. *For all $x, y \in J^\infty(C)$ and for all $m \in M^\infty(C)$ we have*

$$R_f(x, y, m) \text{ if and only if } S_g(x, m, y)$$

Proof. Suppose g is the right residual of f, and let $x, y \in J^\infty(C)$ and $m \in M^\infty(C)$, then $R_f(x, y, m)$ if and only if $f(x, y) \leq m$ if and only if $y \leq g(x, m)$ if and only if $S_g(x, m, y)$. Conversely assume 2, and let $u, v, w \in C$. If $f(u, v) \leq w$, then for each $x, y \in J^\infty(C)$ with $x \leq u, y \leq v$ we have $f(x, y) \leq w$. Consequently, for each $m \in M^\infty(C)$ with $m \geq w$, we have $f(x, y) \leq m$, that is, $R_f(x, y, m)$. But then, by assumption, $S_g(x, m, y)$ holds, that is, $g(x, m) \geq y$. It now follows that

$$\begin{aligned} v &= \bigvee \{y : v \geq y \in J^\infty(C)\} \\ &\leq \bigvee \{g(x, m) : u \geq x \in J^\infty(C), w \leq m \in M^\infty(C)\} = g(u, w). \end{aligned}$$

Similarly $v \leq g(u, w)$ implies $f(u, v) \leq w$. □

Analogously one can define the relations T_h dual to complete dual operators of the type of h, and one can prove corresponding statements about these.

6 Applications to substructural logic

We will now obtain relational semantics for the fragment given by implication and fusion for each of the following logics: Lambeck calculus, linear logic, relevance logic, BCK logic, and intuitionistic logic. Our set-up is as in [5], and we refer to that paper for details.

As explained there, each of these logics correspond to classes of ordered algebras. In all cases the appropriate algebras are MPEs of the form $(P, ;, \to$

$, \leftarrow)$ where $; , \rightarrow, \leftarrow$ are binary operations. The operation ; is known as *fusion*, and the two arrows are known as *implication*, and it is assumed that the fusion operation is residuated with $\rightarrow$ and $\leftarrow$ as its right and left residual, respectively. We call these algebras residuated algebras:

Definition 6.1. A *residuated algebra* is an MPE (P, f, g, h) where P is a partially ordered set, f, g, and h are binary operations on P, and g and h are the right and left residual of f, respectively.

Each of the logics we will consider corresponds to a subclass of these algebras given by some combination of the following inequations:

$$\begin{aligned} (\alpha;\beta);\gamma &\approx \alpha;(\beta;\gamma) && (\textbf{associative}) \\ \alpha;\beta &\approx \beta;\alpha && (\textbf{commutative}) \\ \alpha &\precsim \alpha;\alpha && (\textbf{square-increasing}) \\ \alpha;\beta &\precsim \beta && (\textbf{right-lower-bounded}) \end{aligned}$$

The Lambeck calculus corresponds to the class of **associative** residuated algebras, the fragment of linear logic to **associative** and **commutative** residuated algebras, the fragment of relevance logic to **square-increasing associative** and **commutative** residuated algebras, and the fragment of BCK logic corresponds to **right-lower-bounded associative** and **commutative** residuated algebras. Finally, the fragment of intuitionistic logic corresponds to the class of **associative** and **commutative** residuated algebras that satisfy both **square-increasing** and **right-lower-bounded**.

Here we will first show that each of these equations are canonical for residuated algebras, that is, if any one of them is satisfied in a residuated algebra, then it is also satisfied in the canonical extension of that residuated algebra. This of course implies that the class of algebras corresponding to each of these logics is generated by its perfect members, where perfect residuated algebras are defined by:

Definition 6.2. A *perfect residuated algebra* is an MPE (C, f, g, h) where C is a perfect lattice, f, g, and h are binary operations on C, and g and h are the right and left residual of f, respectively. In particular f is a complete operator on C, and $g : C^\partial \times C \rightarrow C$ and $h : C \times C^\partial \rightarrow C$ are complete dual operators.

Thus we are in a situation where the lattices obtain as canonical extensions fall under the discrete duality as described in the previous two sections, we may define:

Definition 6.3. Given a perfect residuated algebra, $\mathbf{C} = (C, f, g, h)$, its discrete dual is the structure $\underline{\mathbf{C}} = (\underline{C}, R)$, where $R \subseteq J^{\infty}(C) \times J^{\infty}(C) \times M^{\infty}(C)$ is given by

$$\begin{aligned} R(x, y, m) &\text{ if and only if } f(x, y) \leq m \\ &\text{ if and only if } y \leq g(x, m) \\ &\text{ if and only if } x \leq h(m, y) \end{aligned}$$

Proposition 6.4. *Let* $\mathbf{C} = (C, f, g, h)$ *be a perfect residuated algebra, and* $\underline{\mathbf{C}} = (\underline{C}, R)$ *its dual structure. Then* $\underline{C}$ *is a perfect poset, and* R *satisfies* $R1$ *and* $R2$*.*

Proof. This follows readily from the results in Sections 4 and 5. □

Definition 6.5. Let $\mathbf{X} = (X, R)$ be a structure where X is a perfect poset, and $R \subseteq J^{\infty}(X) \times J^{\infty}(X) \times M^{\infty}(X)$ satisfies $R1$ and $R2$. We will call such a structure a *Kripke structure* for now. Define relations S and T by

$$\begin{aligned} R(x, y, m) &\text{ if and only if } S(x, m, y) \\ &\text{ if and only if } T(m, y, x). \end{aligned}$$

Then the dual of $\mathbf{X}$ is the residuated algebra $\overline{\mathbf{X}} = (\overline{X}, f_R, g_S, h_T)$.

Proposition 6.6. *For any perfect residuated algebra* $\mathbf{C}$, $\overline{(\underline{\mathbf{C}})} = \mathbf{C}$*, and for any Kripke structure* $\mathbf{X} = (X, R)$*, we have* $\underline{(\overline{\mathbf{X}})} = \mathbf{X}$*.*

Proof. Again this follows from the results in sections 4 and 5, and we leave the details to the reader. □

Secondly, after showing that each of the equations **associative**, **commutative**, **square-increasing**, and **right-lower-bounded** are canonical for residuated algebras, we show that each of them has a first order correspondent, that is, it holds in a perfect residuated algebra if and only if its first order correspondent holds on the dual structure of that perfect residuated algebra.

The consequence of these two facts then is that the class of dual structures satisfying the appropriate first order correspondents provides a complete relational semantics for the given logic.

We start by proving the canonicity of the equations:

Proposition 6.7. *Let* $f : P \times P \to P$ *be such that* f^{σ} *preserves arbitrary joins in each coordinate. If* f *satisfies* ***associative****, then so does* f^{σ}*.*

Proof. Let $t : P \times P \times P \to P$ be defined by

$$t(p_1, p_2, p_3) = f(p_1, f(p_2, p_3)) = f(f(p_1, p_2), p_3).$$

It is enough to show that for all $u_1, u_2, u_3 \in P^\sigma$,

$$f^\sigma(u_1, f^\sigma(u_2, u_3)) = t^\sigma(u_1, u_2, u_3) = f^\sigma(f^\sigma(u_1, u_2), u_3).$$

We first show it for $x_1, x_2, x_3 \in K(P^\sigma)$:

$$\begin{array}{rcl} t^\sigma(x_1, x_2, x_3) & = & \bigwedge\{t(p_1, p_2, p_3) : x_i \leq p_i \in P, i = 1, 2, 3\} \\ & = & \bigwedge\{f(p_1, f(p_2, p_3)) : x_i \leq p_i \in P, i = 1, 2, 3\} \\ & \geq & \bigwedge\{f(p_1, q) : x_1 \leq p_1 \in P, f^\sigma(x_2, x_3) \leq q \in P\}. \end{array}$$

The reverse inequality $t^\sigma(x_1, x_2, x_3) \leq \bigwedge\{f(p_1, q) : x_1 \leq p_1 \in P, f^\sigma(x_2, x_3) \leq q \in P\}$ follows from the fact that $\{f(p_2, p_3) : x_1 \leq p_1, i = 1, 2\}$ is down-directed in the set of elements from P that are above $f^\sigma(x_2, x_3)$. As for the general case, since f^σ preserves arbitrary joins in each coordinate, we get:

$$\begin{array}{rcl} t^\sigma(u_1, u_2, u_3) & = & \bigvee\{t^\sigma(x_1, x_2, x_3) : u_i \geq x_i \in K(P^\sigma), i = 1, 2, 3\} \\ & = & \bigvee\{f^\sigma(x_1, f^\sigma(x_2, x_3)) : u_i \geq x_i \in K(P^\sigma), i = 1, 2, 3\} \\ & = & \bigvee\{f^\sigma(\bigvee\{x_1 : u_1 \geq x_1 \in K(P^\sigma)\}, f^\sigma(x_2, x_3)) : u_i \geq x_i \in K(P^\sigma), i = 2, 3\} \\ & = & f^\sigma(u_1, \bigvee\{f^\sigma(x_2, x_3)) : u_i \geq x_i \in K(P^\sigma), i = 2, 3\}) \\ & = & f^\sigma(u_1, \bigvee\{f^\sigma(u_2, x_3)) : u_3 \geq x_3 \in K(P^\sigma)\}) \\ & = & f^\sigma(u_1, f^\sigma(u_2, u_3)) \end{array}$$

Similarly, one can show that $t^\sigma(u_1, u_2, u_3) = f^\sigma(f^\sigma(u_1, u_2), u_3)$. □

Proposition 6.8. *Let $f : P \times P \to Q$ be order preserving. If f satisfies **commutative**, then so does f^σ.*

Proof. Let $x_1, x_2 \in K(P^\sigma)$,

$$\begin{array}{rcl} f^\sigma(x_1, x_2) & = & \bigwedge\{f(p_1, p_2) : x_i \leq p_i \in P, i = 1, 2\} \\ & = & \bigwedge\{f(p_2, p_1) : x_i \leq p_i \in P, i = 1, 2\} \\ & = & f^\sigma(x_2, x_1). \end{array}$$

So for any $u_1, u_2 \in P^\sigma$,

$$\begin{array}{rcl} f^\sigma(u_1, u_2) & = & \bigvee\{f^\sigma(x_1, x_2) : u_i \geq x_i \in K(P^\sigma), i = 1, 2\} \\ & = & \bigvee\{f^\sigma(x_2, x_1) : u_i \geq x_i \in K(P^\sigma), i = 1, 2\} \\ & = & f^\sigma(u_2, u_1). \end{array}$$

□

Proposition 6.9. *Let $f : P \times P \to P$ be order preserving. If f satisfies **square-increasing**, then so does f^σ.*

Proof. Let $x \in K(P^\sigma)$ then

$$\begin{aligned}
x &= \bigwedge\{p : x \leq p \in P\} \\
&\leq \bigwedge\{f(p,p) : x \leq p \in P\} \\
&\leq \bigwedge\{f(p,q) : x \leq p, q \in P\} = f^\sigma(x,x)
\end{aligned}$$

The second inequality holds because the set $\{(p,p) : x \leq p \in P\}$ is filtering in $\{(p,q) : x \leq p, q \in P\}$, and this is because the projections of the second set is a filter, namely the filter of P corresponding to the closed element x. Now for $u \in P^\sigma$

$$\begin{aligned}
u &= \bigvee\{x : u \geq x \in K(P^\sigma)\} \\
&\leq \bigvee\{f^\sigma(x,x) : u \geq x \in K(P^\sigma)\} \\
&\leq \bigvee\{f(x,y) : u \geq x, y \in K(P^\sigma)\} = f^\sigma(u,u).
\end{aligned}$$

Here the second $\leq$ simply holds because $\{f(x,y) : u \geq x, y \in K(P^\sigma)\}$ contains $\{f^\sigma(x,x) : u \geq x \in K(P^\sigma)\}$. □

Proposition 6.10. *Let $f : P \times Q \to Q$ be order preserving. If f satisfies* ***right-lower-bounded****, then so does f^σ.*

Proof. Suppose $f(p,q) \leq q$ for every $p \in P$, and $q \in Q$, then for every $x \in K(P^\sigma)$, $y \in K(Q^\sigma)$

$$\begin{aligned}
f^\sigma(x,y) &= \bigwedge\{f(p,q) : x \leq p \in P, y \leq q \in Q\} \\
&\leq \bigwedge\{q : y \leq q \in Q\} = y
\end{aligned}$$

And then for $u \in P^\sigma$ and $v \in Q^\sigma$,

$$\begin{aligned}
f^\sigma(u,v) &= \bigvee\{f^\sigma(x,y) : u \geq x \in K(P^\sigma), v \geq y \in K(Q^\sigma)\} \\
&\leq \bigvee\{y : v \geq y \in K(Q^\sigma)\} = v
\end{aligned}$$

□

This completes the canonicity proofs. Now we turn to correspondence.

Proposition 6.11. *Let C be a perfect lattice and $f : C \times C \to C$ a complete operator. Then the following statements are equivalent:*

1. *For all $u_1, u_2, u_3 \in C$, $f(u_1, f(u_2, u_3)) = f(f(u_1, u_2), u_3)$.*
2. *For all $x_1, x_2, x_3 \in J^\infty(C)$, $f(x_1, f(x_2, x_3)) = f(f(x_1, x_2), x_3)$.*

Proof. Clearly 1 implies 2. The converse follows from the fact that for $u_1, u_2, u_3 \in C$

$$f(u_1, f(u_2, u_3)) = \bigvee\{f(x_1, f(x_2, x_3)) : u_i \geq x_i \in J^\infty(C), i = 1, 2, 3\}$$

and

$$f(f(u_1, u_2), u_3)) = \bigvee\{f(f(x_1, x_2), x_3)) : u_i \geq x_i \in J^\infty(C), i = 1, 2, 3\}.$$

This is obtained by using the fact that f preserves joins in each coordinate repeatedly. □

The point is that the second statement in the above proposition only refers to elements of the dual structure. These kind of statements easily translate to first order statements on the dual. Here we do the translation in an algorithmic way not worrying about getting the simplest possible statements.

Proposition 6.12. *Let $f : C \times C \to C$ be a complete operator on a perfect lattice. Let $x_i \in J^\infty(C)$, $i = 1, 2, 3$ and $m \in M^\infty(C)$.*

1. *The following are equivalent:*
   (a) $f(x_1, f(x_2, x_3)) \leq m$.
   (b) $\forall x_2'([\forall m'(R_f(x_2, x_3, m') \to x_2' \leq m')] \to R_f(x_1, x_2', m))$.
2. *The following are equivalent:*
   (a) $f(f(x_1, x_2), x_3) \leq m$.
   (b) $\forall x_1'([\forall m'(R_f(x_1, x_2, m') \to x_1' \leq m')] \to R_f(x_1', x_3, m))$.

Proof. We just prove the first equivalence, and we resort to formal first order statements in doing so. We assume that any element named x with any super- or subscript comes from $J^\infty(C)$ whereas any element named m with any superscript comes from $M^\infty(C)$.

$$\begin{aligned} &f(x_1, f(x_2, x_3)) \leq m \\ \Leftrightarrow &\forall x_2'(x_2' \leq f(x_2, x_3) \Rightarrow R_f(x_1, x_2', m)) \\ \Leftrightarrow &\forall x_2'(\forall m'[R_f(x_2, x_3, m') \Rightarrow x_2' \leq m'] \Rightarrow R_f(x_1, x_2', m)) \end{aligned}$$

□

Definition 6.13. Let Φ_a denote the first order statement:

$$\begin{aligned}
&\forall x_1, x_2, x_3 \quad \forall m \\
&([\forall x_2'(\forall m'[R(x_2, x_3, m') \Rightarrow x_2' \leq m'] \Rightarrow R(x_1, x_2', m))] \\
&\qquad\qquad \Leftrightarrow [\forall x_1'([\forall m''(R(x_1, x_2, m'') \rightarrow x_1' \leq m'')] \rightarrow R(x_1', x_3, m))])
\end{aligned}$$

When interpreting this statement in a structure we assume the variables named by x's range over completely join irreducibles and the variables named by m's range over completely meet irreducibles.

Corollary 6.14. *Let (X, R) be a Kripke structure, then the following statements are equivalent:*

1. *(X, R) satisfies Φ_a;*
2. *$\overline{(X, R)}$ satisfies **associative**.*

Proposition 6.15. *Let C be a perfect lattice, and $f : C \times C \to D$ be a complete operator. Then the following statements are equivalent:*

1. *For all $u_1, u_2 \in C$, $f(u_1, u_2) = f(u_2, u_1)$.*
2. *For all $x_1, x_2 \in J^\infty(C)$, $f(x_1, x_2) = f(x_2, x_1)$.*

Proof. Similar to the proof of Proposition 6.11. □

Definition 6.16. Let Φ_c denote the first order statement:

$$\forall x_1, x_2 \; \forall m \;\; (R(x_1, x_2, m) \Leftrightarrow R(x_2, x_1, m))$$

When interpreting this statement in a structure we assume the variables named by x's range over completely join irreducibles and the variable named by m ranges over completely meet irreducibles.

Corollary 6.17. *Let (X, R) be a Kripke structure, then the following statements are equivalent:*

1. *(X, R) satisfies Φ_c;*
2. *$\overline{(X, R)}$ satisfies **commutative**.*

Proposition 6.18. *Let C be a perfect lattice, and $f : C \times C \to C$ be a complete operator. Then the following statements are equivalent:*

1. *For all $u \in C$, $u \leq f(u, u)$.*

2. *For all* $x \in J^\infty(C)$, $x \leq f(x,x)$.

Proof. Similar to Prop 6.11. □

Definition 6.19. Let Φ_{si} denote the first order statement:

$$\forall x \; \forall m \;\; (R(x,x,m) \Rightarrow x \leq m)$$

When interpreting this statement in a structure we assume the variable named by x ranges over completely join irreducibles and the variable named by m ranges over completely meet irreducibles.

Corollary 6.20. *Let* (X,R) *be a Kripke structure, then the following statements are equivalent:*

1. (X,R) *satisfies* Φ_{si}*;*

2. $\overline{(X,R)}$ *satisfies* ***square-increasing****.*

Proposition 6.21. *Let* C *be a perfect lattice, and* $f : C \times C \to C$ *be a complete operator. Then the following statements are equivalent:*

1. *For all* $u_1, u_2 \in C$, $f(u_1,u_2) \leq u_1$.

2. *For all* $x_1, x_2 \in J^\infty(C)$, $f(x_1,x_2) \leq x_1$.

Proof. Similar to the proof of Proposition 6.11. □

Definition 6.22. Let Φ_{rlb} denote the first order statement:

$$\forall x_1, x_2 \; \forall m \;\; (x_1 \leq m \Rightarrow R(x_1,x_2,m))$$

When interpreting this statement in a structure we assume the variables named by x's range over completely join irreducibles and the variable named by m ranges over completely meet irreducibles.

Corollary 6.23. *Let* (X,R) *be a Kripke structure, then the following statements are equivalent:*

1. (X,R) *satisfies* Φ_{rlb}*;*

2. $\overline{(X,R)}$ *satisfies* ***right-lower-bounded****.*

We conclude:

Theorem 6.24. *The class of Kripke structures satisfying* Φ_a *is a complete semantics for the Lambeck calculus.*

Theorem 6.25. *The class of Kripke structures satisfying Φ_a and Φ_c is a complete semantics for the fragment of linear logic given by implication and fusion.*

Theorem 6.26. *The class of Kripke structures satisfying Φ_a, Φ_c, and Φ_{si} is a complete semantics for the fragment of relevance logic given by implication and fusion.*

Theorem 6.27. *The class of Kripke structures satisfying Φ_a, Φ_c, and Φ_{rlb} is a complete semantics for the fragment of linear logic given by implication and fusion.*

Theorem 6.28. *The class of Kripke structures satisfying Φ_a, Φ_c, Φ_{si}, and Φ_{rlb} is a complete semantics for the fragment of intuitionistic logic given by implication and fusion.*

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