CANONICAL EXTENSIONS FOR CONGRUENTIAL LOGICS WITH THE DEDUCTION THEOREM

MAI GEHRKE RAMON JANSANA ALESSANDRA PALMIGIANO

Abstract. We introduce a new and general notion of canonical extension for algebras in the algebraic counterpart $\mathsf{Alg}\mathcal{S}$ of any finitary and congruential logic $\mathcal{S}$. This definition is logic-based rather than purely order-theoretic and is in general different from the one given e.g. in [3], but it agrees with it whenever the algebras in $\mathsf{Alg}\mathcal{S}$ are based on lattices. As a case study on logics purely based on implication, we prove that the varieties of Hilbert and Tarski algebras are canonical in this new sense.

1. Introduction

Abstract Algebraic Logic (AAL) is a general framework for studying the connections between algebra and logic. In particular between logics, taken as consequence relations, and their associated classes of algebras. The basic set-up implies that the appropriate algebras are at least quasiordered and, for logics in the important class of congruential logics[1], the algebras are ordered. Canonical extension is a general tool for ordered algebras which allows for the smooth development of representation theory and duality, also at the limits of availability of such tools. Since representation theory and duality are central and powerful tools for the treatment of algebras pertinent to logic such as modal algebras, Heyting algebras, MV-algebras[2], and the algebraic counterparts of substructural logics, and since canonical extension has been particularly useful in several of these settings [10, 11, 12, 3], it is natural to explore whether canonical extension can be developed as a *logical* construct within AAL rather than just as a purely order theoretic construct. This is exactly what this paper does.

We now give a short, non-technical, account of the gist of our results and an outline of the paper before introducing the machinery necessary to talk more precisely about our work. Central in the theory of canonical extension is a choice of filters and ideals, of which the canonical extension is then the least completion, see [13], and our paper [14] on a parametric treatment of such completions with respect to varying families of filters and ideals of a poset. Central in AAL is the notion of logical filter that is, in general, different from the purely order-theoretic notion of

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[1]Congruential logics are referred to as strongly selfextensional in [4] and fully selfextensional in [15, 16].

[2]All of them are the algebraic counterpart of some congruential logic.

filter as a down-directed upset. In addition to the notion of logical filter, we need a notion of logical ideal in order to be able to give a logic-inspired notion of canonical extension. Our first contribution is giving such a notion and showing that the logical notions of filter and ideal agree with the order theoretic ones used in canonical extension for a wide and distinguished class of logics. Specifically, congruential logics with the properties of conjunction (PC) and disjunction, in a weak (PWD) or a strong form (PD), have algebras that are lattices (or distributive lattices in the strong case) and in this setting the logical and order-theoretic notions of filters and ideals agree. This is an encouraging preliminary result.

Of fundamental importance in logic is of course implication, and implication, without necessarily having conjunctions - or at least without having disjunctions - is an important test case for theories pertinent to logic. Thus it is not surprising that both AAL and canonical extension have already been tested in this setting. Canonical extension has been successfully applied to obtain the first fully uniform and modular treatment of relational semantics for the basic hierarchy of substructural logics [3] and in AAL, logics with the property of deduction-detachment (PDD) have been extensively studied (cf. [4] and [16]). A case in point is that of Hilbert logic, that is, the implication fragment of intuitionistic logic. This is a very well behaved logic from the point of view of AAL and its associated algebras are subalgebras of the implication reducts of Heyting algebras. Thus it would be desirable that a *logically* determined notion of canonical extension would preserve this property. However, canonical extension, as defined in [3], fails badly: the canonical extension of a Hilbert algebra is not a Heyting algebra in general; in fact, it isn't even necessarily a Hilbert algebra. Our second and main purpose in this paper is to understand this mismatch between AAL and canonical extension which occurs once we leave the lattice setting.

We give an AAL inspired notion of *logic-based canonical extension*, i.e. based on the logical filters and our associated notion of logical ideal. We show that the classes of Hilbert and Tarski algebras are canonical with respect to this logic-based canonical extension and that the logic-based canonical extension of a Hilbert algebra is a (complete) Heyting algebra. In addition, we reconcile logic-based canonical extensions with the purely order theoretic canonical extensions given in [3] by showing that, for any finitary congruential logic with PDD $\mathcal{S}$ and any algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, the logic-based canonical extension of $\boldsymbol{A}$ is equal to the order canonical extension of the meet semi-lattice of the *finitely generated* logical $\mathcal{S}$-filters of $\boldsymbol{A}$.

This paper is organized as follows. In Section 2 we expound the necessary preliminaries on basic notions of AAL, in particular on congruential logics, recall some properties characterizing the behaviour of conjunction, disjunction and implication w.r.t. the entailment relation of a logic $\mathcal{S}$ and discuss some of their effects on the algebras of $\mathsf{Alg}\mathcal{S}$. Moreover, we introduce the notion of *logical ideal* induced by $\mathcal{S}$ on the algebras of the corresponding similarity type. In Section 3 we recall the concepts and results of [14] that we will need in this paper. Section 4 is the central one, where we introduce the notion of logic canonical extension for the algebras $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, for every finitary congruential logic $\mathcal{S}$. It essentially consists in taking the canonical extension, as defined in [3], of the meet semi-lattice of the *finitely generated* logical $\mathcal{S}$-filters of $\boldsymbol{A}$ in $\mathsf{Alg}\mathcal{S}$. In Sections 5 and 6 we show that Hilbert algebras and Tarski algebras are canonical w.r.t. the notion of canonical extensions introduced in Section 4.

2. Congruential logics and logical ideals

2.1. **General concepts.** In this section we are going to introduce the basic concepts of Abstract Algebraic Logic that we will use in the paper, as well as the new notion of logical ideal. For a general view of AAL the reader is addressed to [5] and the references therein.

Consequence operations and their duals. Given a set A, a *consequence operation* (or closure operator) on A is a map $C : \mathcal{P}(A) \to \mathcal{P}(A)$ such that for every $X, Y \subseteq A$: (1) $X \subseteq C(X)$, (2) if $X \subseteq Y$, then $C(X) \subseteq C(Y)$ and (3) $C(C(X)) = C(X)$. C is *finitary* if in addition satisfies (4) $C(X) = \bigcup\{C(Z) : Z \subseteq X, Z \text{ finite}\}$.

Given a consequence operation C on A, a set $X \subseteq A$ is C-*closed* if $C(X) = X$. The set of all C-closed subsets of A is a *closure system* on A, i.e. it contains A and it is closed under intersections of arbitrary non-empty families. The family of C-closed subsets of A will be denoted by $\mathcal{C}_C$. If C is finitary, then $\mathcal{C}_C$ is an algebraic closure system, that is, it is closed under unions of up-directed families. It is well-known that a closure system $\mathcal{C}$ on a set A defines a consequence operation $C_{\mathcal{C}}$ on A by setting $C_{\mathcal{C}}(X) = \bigcap\{Y \in \mathcal{C} : X \subseteq Y\}$ for every $X \subseteq A$. The $C_{\mathcal{C}}$-closed sets are exactly the elements of $\mathcal{C}$. Moreover, $\mathcal{C}$ is algebraic if and only if $C_{\mathcal{C}}$ is finitary.

The *dual consequence operation* of C is the map $C^d : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by

$$C^d(X) = \{a \in A : \bigcap_{b \in Y} C(b) \subseteq C(a) \text{ for some finite } Y \subseteq X\}$$

for every $X \subseteq A$. So $a \in C^d(\emptyset)$ if and only if $A = \bigcap_{b \in \emptyset} C(b) \subseteq C(a)$, therefore $C^d(\emptyset) = \{a \in A : C(a) = A\}$.

Other straightforward consequences of the definition of C^d are that C^d is a finitary consequence operation on A and for every $a, b \in A$

$$a \in C(b) \quad \text{iff} \quad b \in C^d(a).$$

The specialization quasi-order of a consequence operation. For every consequence operation C on A, the *specialization quasi-order* of C is the binary relation $\leq_C^A$ on A defined by

$$a \leq_C^A b \quad \text{iff} \quad C(b) \subseteq C(a).$$

This means that

$$a \leq_C^A b \quad \text{iff} \quad \forall X \in \mathcal{C}_C (a \in X \Rightarrow b \in X),$$

which justifies its name. For every $a, b \in A$

$$a \leq_C^A b \quad \text{iff} \quad b \leq_{C^d}^A a,$$

so the specialization quasi-order of C^d is the converse quasi-order of $\leq_C^A$.

Logics. Let $\mathcal{L}$ be a propositional language (i.e. a set of connectives, that we will also regard as a set of function symbols) and let $\mathbf{Fm}_{\mathcal{L}}$ denote the algebra of formulas (or term algebra) of $\mathcal{L}$ over a denumerable set V of variables, i.e. the absolutely free $\mathcal{L}$-algebra over V. A *logic* (or deductive system) of type $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathbf{Fm}_{\mathcal{L}}, \vdash_{\mathcal{S}} \rangle$ where the *consequence* or *entailment* relation $\vdash_{\mathcal{S}}$ is a relation between subsets of the carrier $Fm_{\mathcal{L}}$ of $\mathbf{Fm}_{\mathcal{L}}$ and elements of $Fm_{\mathcal{L}}$ such that the operator $C_{\vdash_{\mathcal{S}}} : \mathcal{P}(Fm_{\mathcal{L}}) \to \mathcal{P}(Fm_{\mathcal{L}})$ defined by

$$\varphi \in C_{\vdash_{\mathcal{S}}}(\Gamma) \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \varphi$$

is a consequence operation with the property of *invariance under substitutions*; this means that for every substitution σ (i.e. for every $\sigma \in \mathrm{Aut}(\mathbf{Fm}_{\mathcal{L}})$) and for every $\Gamma \subseteq Fm_{\mathcal{L}}$,

$$\sigma[C_{\vdash_{\mathcal{S}}}(\Gamma)] \subseteq C_{\vdash_{\mathcal{S}}}(\sigma[\Gamma]).$$

A logic is *finitary* if the consequence operation $C_{\vdash_{\mathcal{S}}}$ is finitary. When necessary we will refer by $\mathcal{L}_{\mathcal{S}}$ to the propositional language of a logic $\mathcal{S}$.

The *interderivability relation* of a logic $\mathcal{S}$ is the relation $\equiv_{\mathcal{S}}$ defined by

$$\varphi \equiv_{\mathcal{S}} \psi \quad \text{iff} \quad \varphi \vdash_{\mathcal{S}} \psi \text{ and } \psi \vdash_{\mathcal{S}} \varphi.$$

$\mathcal{S}$ satisfies the *congruence property* if $\equiv_{\mathcal{S}}$ is a congruence of $\mathbf{Fm}_{\mathcal{L}}$[3].

Logical filters. Let $\mathcal{S}$ be a logic of type $\mathcal{L}$ and $\boldsymbol{A}$ an $\mathcal{L}$-algebra (from now on, we will drop reference to the type $\mathcal{L}$, and when we refer to an algebra or class of algebras in relation with $\mathcal{S}$, we will always assume that the algebra and the algebras in the class are of the type $\mathcal{L}$).

A subset $F \subseteq A$ is an $\mathcal{S}$-*filter* of $\boldsymbol{A}$ if for every $\Gamma \cup \{\varphi\} \subseteq Fm$ and every $h \in \mathrm{Hom}(\mathbf{Fm}_{\mathcal{L}}, \boldsymbol{A})$,

$$\text{if } \Gamma \vdash_{\mathcal{S}} \varphi \text{ and } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F.$$

The collection $\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$ of the $\mathcal{S}$-filters of $\boldsymbol{A}$ is a closure system. And $\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$ is an algebraic closure system if $\mathcal{S}$ is finitary. The consequence operation associated with $\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$ is denoted by $C_{\mathcal{S}}^{\boldsymbol{A}}$. Thus, for every $X \subseteq A$, $C_{\mathcal{S}}^{\boldsymbol{A}}(X)$ is the $\mathcal{S}$-filter of $\boldsymbol{A}$ generated by X. If $\mathcal{S}$ is finitary, then $C_{\mathcal{S}}^{\boldsymbol{A}}$ is finitary for every algebra $\boldsymbol{A}$.

An $\mathcal{S}$-filter F of $\boldsymbol{A}$ is *finitely generated* if $F = C_{\mathcal{S}}^{\boldsymbol{A}}(X)$ for some finite $X \subseteq A$. $\mathrm{Fi}_{\mathcal{S}}^{\omega}(\boldsymbol{A})$ denotes the collection of the finitely generated $\mathcal{S}$-filters of $\boldsymbol{A}$.

On the algebra of formulas $\mathbf{Fm}$, $C_{\mathcal{S}}^{\mathbf{Fm}}$ coincides with $C_{\vdash_{\mathcal{S}}}$ and the $C_{\mathcal{S}}^{\mathbf{Fm}}$-closed sets are the $\mathcal{S}$-*theories*; they are exactly the sets of formulas that are closed under the relation $\vdash_{\mathcal{S}}$.

The $\mathcal{S}$-specialization quasi-order. For every finitary logic $\mathcal{S}$ and every algebra $\boldsymbol{A}$, the $\mathcal{S}$-*specialization quasi-order* of $\boldsymbol{A}$, denoted by $\leq_{\mathcal{S}}^{\boldsymbol{A}}$, is the specialization quasi-order associated with $C_{\mathcal{S}}^{\boldsymbol{A}}$. Thus, for every $a, b \in A$,

$$a \leq_{\mathcal{S}}^{\boldsymbol{A}} b \;\text{ iff }\; C_{\mathcal{S}}^{\boldsymbol{A}}(b) \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(a) \;\text{ iff }\; a \vdash_{C_{\mathcal{S}}^{\boldsymbol{A}}} b$$

and

$$a \leq_{\mathcal{S}}^{\boldsymbol{A}} b \;\text{ iff }\; (C_{\mathcal{S}}^{\boldsymbol{A}})^d(a) \subseteq (C_{\mathcal{S}}^{\boldsymbol{A}})^d(b) \;\text{ iff }\; b \vdash_{(C_{\mathcal{S}}^{\boldsymbol{A}})^d} a.$$

Clearly, every $\mathcal{S}$-filter is an up-set w.r.t. $\leq_{\mathcal{S}}^{\boldsymbol{A}}$. Let $\geq_{\mathcal{S}}^{\boldsymbol{A}}$ denote the converse relation of $\leq_{\mathcal{S}}^{\boldsymbol{A}}$. Then the equivalence relation $\equiv_{\mathcal{S}}^{\boldsymbol{A}}$ associated with $\leq_{\mathcal{S}}^{\boldsymbol{A}}$ is $\leq_{\mathcal{S}}^{\boldsymbol{A}} \cap \geq_{\mathcal{S}}^{\boldsymbol{A}}$. Thus, for every $a, b \in A$,

$$a \equiv_{\mathcal{S}}^{\boldsymbol{A}} b \quad \text{iff} \quad C_{\mathcal{S}}^{\boldsymbol{A}}(a) = C_{\mathcal{S}}^{\boldsymbol{A}}(b).$$

The relation $\equiv_{\mathcal{S}}^{\boldsymbol{A}}$ is not in general a congruence for every $\boldsymbol{A}$, even if $\mathcal{S}$ satisfies the congruence property.

[3]Logics with the congruence property are also known as selfextensional logics.

Logical ideals. As we remarked early on in the introduction, in order to give an account of canonical extensions within AAL, we need to introduce a logic-based notion of *ideal*. Just like the $\mathcal{S}$-filters, the logical ideals should be defined purely in terms of the consequence relation of $\mathcal{S}$. Moreover, they should reduce to the familiar notion of lattice ideal whenever $\mathcal{S}$ has enough metalogical properties. The following definition satisfies both requirements (see also Proposition 2.8 below).

Let $\mathcal{S}$ be a finitary logic and $\boldsymbol{A}$ an algebra of its type. An $\mathcal{S}$-*ideal* of $\boldsymbol{A}$ is a closed set of the dual consequence operation $(C_{\mathcal{S}}^{\boldsymbol{A}})^d$ of $C_{\mathcal{S}}^{\boldsymbol{A}}$, i.e. it is a $(C_{\mathcal{S}}^{\boldsymbol{A}})^d$-closed set. The closure system of the $\mathcal{S}$-ideals of $\boldsymbol{A}$ will be denoted by $\mathrm{Id}_{\mathcal{S}}\boldsymbol{A}$. By the definition of $(C_{\mathcal{S}}^{\boldsymbol{A}})^d$, $\mathrm{Id}_{\mathcal{S}}\boldsymbol{A}$ is always an algebraic closure system.

The canonical algebraic counterpart of a logic. One of the main conceptual achievements of AAL is the identification of the canonical algebraic counterpart $\mathsf{Alg}\mathcal{S}$ of every logic $\mathcal{S}$ (see [5]). $\mathsf{Alg}\mathcal{S}$ can be defined in several equivalent ways: the definition we present here is the most convenient for the purposes of this paper. For every logic $\mathcal{S}$, $\mathsf{Alg}\mathcal{S}$ is the class of those algebras $\boldsymbol{A}$ such that the identity relation Δ_A is the only congruence of $\boldsymbol{A}$ that is included in $\equiv_{\mathcal{S}}^{\boldsymbol{A}}$. That is,

$$\mathsf{Alg}\mathcal{S} := \{\boldsymbol{A} : \forall \theta \in \mathrm{Co}\boldsymbol{A}(\text{if } \theta \subseteq \equiv_{\mathcal{S}}^{\boldsymbol{A}} \text{ then } \theta = \Delta_A)\}.$$

2.2. Congruential logics.

Definition 2.1. A logic $\mathcal{S}$ is *congruential* if for every algebra $\boldsymbol{A}$, $\equiv_{\mathcal{S}}^{\boldsymbol{A}}$ is a congruence of $\boldsymbol{A}$

Of course, if $\mathcal{S}$ is congruential, then $\mathcal{S}$ has the congruence property; but the converse is not true (cf. [2]).

If a logic $\mathcal{S}$ is congruential, $\mathsf{Alg}\mathcal{S}$ can be characterized in a simpler way: indeed, since $\equiv_{\mathcal{S}}^{\boldsymbol{A}}$ is a congruence for every algebra $\boldsymbol{A}$, then

$$\mathsf{Alg}\mathcal{S} = \{\boldsymbol{A} : \equiv_{\mathcal{S}}^{\boldsymbol{A}} = \Delta_A\}.$$

Then, recalling that $\equiv_{\mathcal{S}}^{\boldsymbol{A}}$ was defined as $\leq_{\mathcal{S}}^{\boldsymbol{A}} \cap \geq_{\mathcal{S}}^{\boldsymbol{A}}$, we get that if $\mathcal{S}$ is congruential, then $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ if and only if $\leq_{\mathcal{S}}^{\boldsymbol{A}}$ is a partial order. In fact this condition characterizes congruentiality:

Theorem 2.2. *A logic $\mathcal{S}$ is congruential if and only if for every algebra $\boldsymbol{A}$, $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ iff $\langle A, \leq_{\mathcal{S}}^{\boldsymbol{A}}\rangle$ is a poset.*

This innocuous-looking fact identifies congruential logics as the largest class of logics to which the theory of canonical extensions can be applied.

Definition 2.3. For every congruential logic $\mathcal{S}$ and every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ the poset $\langle A, \leq_{\mathcal{S}}^{\boldsymbol{A}}\rangle$ is the $\mathcal{S}$-*poset* of $\boldsymbol{A}$.

Note that if $\mathcal{S}$ is congruential, then for every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ and every $a \in A$, $C_{\mathcal{S}}^{\boldsymbol{A}}(a)$ is the principal upset $\uparrow a$ relative to $\leq_{\mathcal{S}}^{\boldsymbol{A}}$ and $(C_{\mathcal{S}}^{\boldsymbol{A}})^d(a)$ is the principal down-set $\downarrow a$; so $\{\uparrow a : a \in A\} \subseteq \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$ and $\{\downarrow a : a \in A\} \subseteq \mathrm{Id}_{\mathcal{S}}\boldsymbol{A}$.

2.3. **Consequence relations and logical connectives.** So far the treatment has been uniform in every algebraic similarity type $\mathcal{L}$, but conjunction, disjunction and implication will play a prominent role in what follows. Therefore in this section we are going to present the well-known properties characterizing these connectives in terms of their behaviour w.r.t. the entailment relation of a logical system $\mathcal{S}$, and discuss their effects on the algebras of $\mathsf{Alg}\mathcal{S}$, especially when $\mathcal{S}$ is congruential.

For the sake of greater generality, we will not assume that either connective mentioned above is primitive in the language, but only that it can be defined from the connectives in $\mathcal{L}_\mathcal{S}$:

(1) $\mathcal{S}$ satisfies the *property of conjunction* (PC-$\wedge$) *relative to the term* $t_1(x, y)$ that we rewrite as $x \wedge y$, if for all formulas φ and ψ, (a) $\varphi \wedge \psi \vdash_\mathcal{S} \varphi$, (b) $\varphi \wedge \psi \vdash_\mathcal{S} \psi$ and (c) $\varphi, \psi \vdash_\mathcal{S} \varphi \wedge \psi$.
(2) $\mathcal{S}$ satisfies the *property of weak disjunction* (PWD-$\vee$) *relative to the term* $t_2(x, y)$ that we rewrite as $x \vee y$, if for all formulas φ, ψ and δ: (a) $\varphi \vdash_\mathcal{S} \varphi \vee \psi$, $\varphi \vdash_\mathcal{S} \psi \vee \varphi$ and (b) if $\varphi \vdash_\mathcal{S} \delta$ and $\psi \vdash_\mathcal{S} \delta$, then $\varphi \vee \psi \vdash_\mathcal{S} \delta$. If the following stronger condition holds: (b') for every set of formulas Γ, if $\Gamma, \varphi \vdash_\mathcal{S} \delta$ and $\Gamma, \psi \vdash_\mathcal{S} \delta$, then $\Gamma, \varphi \vee \psi \vdash_\mathcal{S} \delta$, then $\mathcal{S}$ satisfies the *property of disjunction* (PD-$\vee$) *relative to* $t_2(x, y)$.
(3) $\mathcal{S}$ satisfies the *property of deduction* (PDe-$\rightarrow$) *relative to a term* $t_3(x, y)$ that we rewrite as $x \rightarrow y$, if for every set of formulas $\Gamma \cup \{\varphi, \psi\}$, if $\Gamma, \varphi \vdash_\mathcal{S} \psi$, then $\Gamma \vdash_\mathcal{S} \varphi \rightarrow \psi$. $\mathcal{S}$ satisfies the *property of detachment* (PDt-$\rightarrow$) if for every set of formulas $\Gamma \cup \{\varphi, \psi\}$, if $\Gamma \vdash_\mathcal{S} \varphi \rightarrow \psi$, then $\Gamma, \varphi \vdash_\mathcal{S} \psi$. If both (PDe-$\rightarrow$) and (PDt-$\rightarrow$) hold for $\mathcal{S}$, then $\mathcal{S}$ satisfies the *property of deduction-detachment* (PDD-$\rightarrow$) relative to $x \rightarrow y$.

In the remainder, we will assume that the terms relative to which the various properties hold are fixed, and drop reference to them.

Proposition 2.4. *If $\mathcal{S}$ is finitary and satisfies (PWD) and (PDD), then $\mathcal{S}$ satisfies (PD).*

Proof. To prove that $\mathcal{S}$ satisfies (PD), it is enough to see that if $\Gamma, \varphi \vdash_\mathcal{S} \delta$ and $\Gamma, \psi \vdash_\mathcal{S} \delta$, then $\Gamma, \varphi \vee \psi \vdash_\mathcal{S} \delta$. If $\Gamma, \varphi \vdash_\mathcal{S} \delta$ and $\Gamma, \psi \vdash_\mathcal{S} \delta$, then, since $\mathcal{S}$ is finitary, we can assume that $\{\psi_1, \ldots, \psi_n\}, \varphi \vdash_\mathcal{S} \delta$ and $\{\psi_1, \ldots, \psi_n\}, \psi \vdash_\mathcal{S} \delta$ for some $\psi_1, \ldots, \psi_n \in \Gamma$. Then by (PDD) we obtain $\varphi \vdash_\mathcal{S} \psi_1 \rightarrow (\ldots \rightarrow (\psi_n \rightarrow \delta) \ldots)$ and $\psi \vdash_\mathcal{S} \psi_1 \rightarrow (\ldots \rightarrow (\psi_n \rightarrow \delta) \ldots)$. So by (PWD), $\varphi \vee \psi \vdash_\mathcal{S} \psi_1 \rightarrow (\ldots \rightarrow (\psi_n \rightarrow \delta) \ldots)$. Hence by (PDD), $\{\psi_1, \ldots, \psi_n\}, \varphi \vee \psi \vdash_\mathcal{S} \delta$. Therefore, $\Gamma, \varphi \vee \psi \vdash_\mathcal{S} \delta$. □

It is well known that if $\mathcal{S}$ satisfies (PC) and (PD), the distributive laws for the corresponding $\wedge$ and $\vee$ hold:

$$\varphi \wedge (\psi \vee \delta) \dashv\vdash_\mathcal{S} (\varphi \wedge \psi) \vee (\varphi \wedge \delta) \text{ and } \varphi \vee (\psi \wedge \delta) \dashv\vdash_\mathcal{S} (\varphi \vee \psi) \wedge (\varphi \vee \delta). \tag{2.1}$$

The properties introduced so far can be stated using the consequence operation $C_{\vdash_\mathcal{S}}$ associated with $\vdash_\mathcal{S}$:

(1) $\mathcal{S}$ satisfies (PC) iff $C_{\vdash_\mathcal{S}}(\varphi \wedge \psi) = C_{\vdash_\mathcal{S}}(\varphi, \psi)$ for all formulas φ, ψ.
(2) $\mathcal{S}$ satisfies (PWD) iff $C_{\vdash_\mathcal{S}}(\varphi \vee \psi) = C_{\vdash_\mathcal{S}}(\varphi) \cap C_{\vdash_\mathcal{S}}(\psi)$ for all formulas φ, ψ.
(3) $\mathcal{S}$ satisfies (PD) iff for every set of formulas $\Gamma \cup \{\varphi, \psi\}$, $C_{\vdash_\mathcal{S}}(\Gamma, \varphi \vee \psi) = C_{\vdash_\mathcal{S}}(\Gamma, \varphi) \cap C_{\vdash_\mathcal{S}}(\Gamma, \psi)$.
(4) $\mathcal{S}$ satisfies (PDD) iff for every set of formulas $\Gamma \cup \{\varphi, \psi\}$, $\psi \in C_{\vdash_\mathcal{S}}(\Gamma, \varphi)$ iff $\varphi \rightarrow \psi \in C_{\vdash_\mathcal{S}}(\Gamma)$.

This is useful because we can then extend these properties to closure operators on arbitrary algebras: for every algebra $\boldsymbol{A}$ and every closure operator C on A,

(1) C satisfies (PC) if $C(a \wedge^{\boldsymbol{A}} b) = C(a, b)$ for every $a, b \in A$,
(2) C satisfies (PWD) if $C(a \vee^{\boldsymbol{A}} b) = C(a) \cap C(b)$ for every $a, b \in A$,
(3) C satisfies (PD) if $C(X, a \vee^{\boldsymbol{A}} b) = C(X, a) \cap C(X, b)$ for every $a, b \in A$ and every $X \subseteq A$,

(4) C satisfies (PDD) if $b \in C(X, a)$ iff $a \to^{\boldsymbol{A}} b \in C(X)$, for every $X \subseteq A$ and every $a, b \in A$.

Let Φ be any of the properties introduced at the beginning of this section and let $\mathcal{S}$ be a logic satisfying Φ. Φ *transfers to every algebra* if for every algebra $\boldsymbol{A}$ the closure operator $C^{\boldsymbol{A}}_{\mathcal{S}}$ satisfies Φ relative to the same term for which $\mathcal{S}$ satisfies Φ. For example, a logic $\mathcal{S}$ satisfying (PDD) transfers (PDD) to every algebra if, for every algebra $\boldsymbol{A}$, $C^{\boldsymbol{A}}_{\mathcal{S}}$ satisfies (PDD), that is, if for every algebra $\boldsymbol{A}$, every $X \subseteq A$, and every $a, b \in A$, $b \in C^{\boldsymbol{A}}_{\mathcal{S}}(X, a)$ iff $a \to^{\boldsymbol{A}} b \in C^{\boldsymbol{A}}_{\mathcal{S}}(X)$.

If $\mathcal{S}$ satisfies (PC), (PD) or (PDD), then the property transfers to every algebra. Proving this for (PC) is easy. Proofs that the other two properties transfer to every algebra can be found in [4], cf. Thm. 2.48 and Thm. 2.52.

As we already mentioned, not every logic satisfying the congruence property is congruential. But if either (PC) or (PDD) holds for $\mathcal{S}$, the congruence property is enough for $\mathcal{S}$ to be congruential. These facts were first proved in [4] (see also [15, 16] for simpler proofs). Moreover, if $\mathcal{S}$ satisfies either (PC) or (PDD), and if in addition $\mathcal{S}$ satisfies (PWD), then this property transfers to every algebra of the corresponding similarity type.

Proposition 2.5. *For every congruential logic $\mathcal{S}$ satisfying (PC) and (PWD) and every algebra $\boldsymbol{A}$, $C^{\boldsymbol{A}}_{\mathcal{S}}$ satisfies (PWD).*

Proof. To show that $\mathcal{S}$ transfers (PWD) to every algebra, let $\boldsymbol{A}$ be an algebra and $a, b \in A$. Since $p \vdash_{\mathcal{S}} p \vee q$ and $q \vdash_{\mathcal{S}} p \vee q$, then $C^{\boldsymbol{A}}_{\mathcal{S}}(a \vee b) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(a) \cap C^{\boldsymbol{A}}_{\mathcal{S}}(b)$. Conversely, if $c \in C^{\boldsymbol{A}}_{\mathcal{S}}(a) \cap C^{\boldsymbol{A}}_{\mathcal{S}}(b)$, since (PC) transfers, we get $C^{\boldsymbol{A}}_{\mathcal{S}}(a \wedge c) = C^{\boldsymbol{A}}_{\mathcal{S}}(a)$ and $C^{\boldsymbol{A}}_{\mathcal{S}}(b \wedge c) = C^{\boldsymbol{A}}_{\mathcal{S}}(b)$, i.e. $a \wedge c \equiv^{\boldsymbol{A}}_{\mathcal{S}} a$ and $b \wedge c \equiv^{\boldsymbol{A}}_{\mathcal{S}} b$. Since by assumption $\equiv^{\boldsymbol{A}}_{\mathcal{S}}$ is a congruence, this implies that $C^{\boldsymbol{A}}_{\mathcal{S}}(a \vee b) = C^{\boldsymbol{A}}_{\mathcal{S}}((a \wedge c) \vee (b \wedge c))$. Now notice that $(p \wedge r) \vee (q \wedge r) \vdash_{\mathcal{S}} r$, because by assumption $\mathcal{S}$ satisfies (PC) and (PWD). Therefore, since $C^{\boldsymbol{A}}_{\mathcal{S}}((a \wedge c) \vee (b \wedge c))$ is an $\mathcal{S}$-filter, then $c \in C^{\boldsymbol{A}}_{\mathcal{S}}((a \wedge c) \vee (b \wedge c))$. Hence $c \in C^{\boldsymbol{A}}_{\mathcal{S}}(a \vee b)$. □

Proposition 2.6. *For every finitary logic $\mathcal{S}$ satisfying (PDD) and (PWD) and every algebra $\boldsymbol{A}$, $C^{\boldsymbol{A}}_{\mathcal{S}}$ satisfies (PD).*

Proof. By Proposition 2.4, $\mathcal{S}$ satisfies (PD). But (PD) transfers to every algebra (cf. [4], Thm. 2.48), which implies the statement. □

The fact that a congruential logic $\mathcal{S}$ satisfies (PC) has important consequences for the structure of the algebras in $\mathsf{Alg}\mathcal{S}$ and the shape of their $\mathcal{S}$-filters.

In order to avoid unnecessary complications in stating the results we assume in the remainder of the section that $\mathcal{S}$ has theorems, namely there is at least one formula φ such that $\vdash_{\mathcal{S}} \varphi$. This holds for every $\mathcal{S}$ with (PDD) and implies that the $\mathcal{S}$-filters are non-empty.

Proposition 2.7. *If $\mathcal{S}$ is congruential and satisfies (PC), then for every algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $\langle A, \wedge^{\boldsymbol{A}} \rangle$ is a meet semi-lattice, the semi-lattice order is $\leq^{\boldsymbol{A}}_{\mathcal{S}}$, and the semi-lattice filters are the $\mathcal{S}$-filters of $\boldsymbol{A}$.*

Proof. Proof that $\langle A, \wedge^{\boldsymbol{A}} \rangle$ is a meet semi-lattice and its semi-lattice filters are the non-empty $\mathcal{S}$-filters of $\boldsymbol{A}$ can be found in [15]. To see that the semi-lattice order $\leq$ is $\leq^{\boldsymbol{A}}_{\mathcal{S}}$, simply note that for every $a, b \in A$, $a \leq^{\boldsymbol{A}}_{\mathcal{S}} b$ iff $C^{\boldsymbol{A}}_{\mathcal{S}}(b) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(a)$ iff $C^{\boldsymbol{A}}_{\mathcal{S}}(a \wedge b) = C^{\boldsymbol{A}}_{\mathcal{S}}(b)$ iff $a \wedge b = b$ iff $a \leq b$. □

If $\mathcal{S}$ in addition satisfies (PWD) then also the logical and order-theoretic notions of ideals can be identified:

Proposition 2.8. *For every finitary congruential logic $\mathcal{S}$ satisfying (PC) and (PWD) and every algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}} \rangle$ is a lattice with the following properties:*

1. *the lattice order $\leq$ is $\leq_{\mathcal{S}}^{\boldsymbol{A}}$,*
2. *the lattice filters are the $\mathcal{S}$-filters of $\boldsymbol{A}$,*
3. *the lattice ideals are the non-emty $\mathcal{S}$-ideals of $\boldsymbol{A}$.*

If in addition $\mathcal{S}$ satisfies (PD), then $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}} \rangle$ is distributive.

Proof. The fact that $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}} \rangle$ is a lattice easily follows from the fact that $C_{\mathcal{S}}^{\boldsymbol{A}}$ satisfies (PC) and (PWD) and that for every $a, b \in A$, $C_{\mathcal{S}}^{\boldsymbol{A}}(a) = C_{\mathcal{S}}^{\boldsymbol{A}}(b)$ iff $a = b$. (1) and (2) follow from Proposition 2.7. As for (3), if J is an $\mathcal{S}$-ideal of $\boldsymbol{A}$, then J is a down-set: if $a \leq_{\mathcal{S}}^{\boldsymbol{A}} b \in J$, then since $C_{\mathcal{S}}^{\boldsymbol{A}}(b) \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(a)$, so $a \in J$. Moreover, if $a, b \in J$, since by (PWD) $C_{\mathcal{S}}^{\boldsymbol{A}}(a) \cap C_{\mathcal{S}}^{\boldsymbol{A}}(b) = C_{\mathcal{S}}^{\boldsymbol{A}}(a \vee b)$, $a \vee b \in J$. This shows that J is a lattice ideal. Conversely, if I is a lattice ideal, $a_1, \ldots, a_n \in I$ and $C_{\mathcal{S}}^{\boldsymbol{A}}(a_1) \cap \ldots \cap C_{\mathcal{S}}^{\boldsymbol{A}}(a_n) \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(b)$, then by (PWD), $C_{\mathcal{S}}^{\boldsymbol{A}}(a_1) \cap \ldots \cap C_{\mathcal{S}}^{\boldsymbol{A}}(a_n) = C_{\mathcal{S}}^{\boldsymbol{A}}(a_1 \vee \ldots \vee a_n)$. Hence, $b \leq_{\mathcal{S}}^{\boldsymbol{A}} a_1 \vee \ldots \vee a_n$. Since I is a lattice ideal, $a_1 \vee \ldots \vee a_n \in I$ and therefore $b \in I$. Therefore I is an $\mathcal{S}$-ideal. If in addition $\mathcal{S}$ satisfies (PD), then, using the fact that both (PC) and (PD) transfer to every algebra and that $\equiv_{\boldsymbol{A}}^{\mathcal{S}} = \Delta_{\boldsymbol{A}}$, it is easy to show that $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}} \rangle$ is distributive. □

The considerations above imply that in the setting of congruential logics $\mathcal{S}$ satisfying (PC) and (PWD) the theory of canonical extensions for lattice expansions presented in [9] can be applied directly to the algebras in $\mathsf{Alg}\mathcal{S}$, provided that the operations on these algebras have the suitable monotonicity properties. Moreover, for congruential logics satisfying (PC) and (PD), the theory of canonical extensions for distributive lattice expansions [8] applies.

2.4. Congruential logics satisfying PDD.

Congruential logics satisfying (PDD) have been studied in [4, 16] from the perspective of AAL. In this subsection we are going to report the facts that are relevant for this paper.

Let $\mathcal{L} = \{\rightarrow\}$. The least finitary congruential $\mathcal{L}$-logic $\mathcal{S}$ that satisfies (PDD) is the $\rightarrow$-fragment of intuitionistic logic, and its algebraic counterpart $\mathsf{Alg}\mathcal{S}$ is the variety of Hilbert algebras.

A *Hilbert algebra* (cf. [19], *positive implication algebra*) is an algebra $\boldsymbol{A} = \langle A, \rightarrow \rangle$ that satisfies the following equations:

- H1. $x \rightarrow x \approx y \rightarrow y$
- H2. $(x \rightarrow x) \rightarrow x \approx x$
- H3. $x \rightarrow (y \rightarrow z) \approx (x \rightarrow y) \rightarrow (x \rightarrow z)$
- H4. $(x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow y) \approx (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow x)$.

The variety of Hilbert algebras can be obtained as the class of $\rightarrow$-subalgebras of the $\rightarrow$-reducts of Heyting algebras.

Results in [16] imply that a finitary and congruential logic $\mathcal{S}$ satisfies (PDD) relative to a definable binary term $x \rightarrow y$ if and only if $\mathsf{Alg}\mathcal{S}$ is a subvariety of the variety of $\mathcal{L}_{\mathcal{S}}$-algebras axiomatized by the equations H1-H4.

Rather than working in the most general setting, in this paper we restrict our attention to finitary congruential logics satisfying (PDD). The results we obtain can be easily extended to finitary congruential logics satisfying (PC) and (PDD) and also to finitary congruential logics satisfying (PD) and (PDD).

3. Preliminaries on Δ_1-completions and canonical extensions of posets

Let $P = \langle P, \leq \rangle$ be a poset. A subset $X \subseteq P$ is an *up-set* if for every $x \in X$ and every $y \in P$, if $x \leq y$ then $y \in X$. *Down-sets* are defined order-dually. For every $x \in P$, the least down-set (resp. up-set) to which x belongs is denoted by $\downarrow x$ ($\uparrow x$). A subset $X \subseteq P$ is *down-directed* if for every $x, y \in X$ there exists some $z \in X$ such that $z \leq x, y$. *Up-directed* subsets are defined order-dually. A *poset-filter* of a poset $\langle P, \leq \rangle$ is a non-empty down-directed up-set and a *poset-ideal* is a non-empty up-directed down-set. In [3] poset-filters and poset-ideals of a poset are called filters and ideals respectively.

A *completion* of a poset P is a pair $\langle \mathbb{C}, e \rangle$ such that $\mathbb{C}$ is a complete lattice and e is an embedding of P into $\mathbb{C}$. We will suppress the embedding e and identify P with its image under e. If $\mathbb{C}$ is a completion of P, the joins of sets of elements of P are called the *open* elements of $\mathbb{C}$ (relative to P) and the meets of sets of elements of P are called the *closed* elements of $\mathbb{C}$ (relative to P). The set of closed elements is denoted by $K(\mathbb{C})$ and the set of open elements by $O(\mathbb{C})$. A Δ_1-*completion* of P ([14]) is a completion $\mathbb{C}$ in which $K(\mathbb{C})$ is join-dense and $O(\mathbb{C})$ meet-dense, that is, it is a completion $\mathbb{C}$ each element of which can be obtained as a join of closed elements and as a meet of open elements.

If P is a lattice, the canonical extension of P introduced in [9] is the unique (up to isomorphism fixing P) Δ_1-completion $\mathbb{C}$ such that for every filter F and every ideal I of P, if $\bigwedge_{\mathbb{C}} F \leq \bigvee_{\mathbb{C}} I$, then $F \cap I \neq \emptyset$. In [3] a notion of *canonical extension* for posets is introduced as the unique (up to isomorphism fixing P) Δ_1-completion $\mathbb{C}$ of P such that the following two properties hold:

(1) for every poset-filter F of P and every poset-ideal I of P, if $\bigwedge_{\mathbb{C}} F \leq \bigvee_{\mathbb{C}} I$, then $F \cap I \neq \emptyset$,
(2) every element of $\mathbb{C}$ is a join of the meets of the elements of some family of poset-filters of P and a meet of the joins of the elements of some family of poset-ideals of P.

In [14], special Δ_1-completions of a poset P are parametrically defined in the choice of a collection $\mathcal{F}$ of up-sets of P such that $\{\uparrow x : x \in P\} \subseteq \mathcal{F}$ and a collection $\mathcal{I}$ of down-sets of P such that $\{\downarrow x : x \in P\} \subseteq \mathcal{I}$. This parametric definition encompasses the canonical extensions defined in [3]. In the remainder of this section, we will briefly expound the relevant concepts and results of [14] about these Δ_1-completions.

Let P be a poset, $\mathcal{F}$ be a family of up-sets of P and $\mathcal{I}$ be a family of down-sets of P such that $\{\uparrow x : x \in P\} \subseteq \mathcal{F}$ and $\{\downarrow x : x \in P\} \subseteq \mathcal{I}$. If $\mathbb{C}$ is a completion of P, let

$$K^{\mathcal{F}}(\mathbb{C}) = \{a \in \mathbb{C} : a = \bigwedge_{\mathbb{C}} F \text{ for some } F \in \mathcal{F}\}$$

$$O^{\mathcal{I}}(\mathbb{C}) = \{a \in \mathbb{C} : a = \bigvee_{\mathbb{C}} I \text{ for some } I \in \mathcal{I}\}.$$

The elements of $K^{\mathcal{F}}(\mathbb{C})$ (resp. $O^{\mathcal{I}}(\mathbb{C})$) are the $\mathcal{F}$-*closed* ($\mathcal{I}$-*open*) elements of $\mathbb{C}$. Every $\mathcal{F}$-closed element is closed and every $\mathcal{I}$-open element is open.

A completion $\mathbb{C}$ of P is $(\mathcal{F},\mathcal{I})$-*compact* if for every $F \in \mathcal{F}$ and $I \in \mathcal{I}$, if $\bigwedge_{\mathbb{C}} F \leq_{\mathbb{C}} \bigvee_{\mathbb{C}} I$, then $F \cap I \neq \emptyset$. A completion $\mathbb{C}$ of P is $(\mathcal{F},\mathcal{I})$-*dense* if $K^{\mathcal{F}}(\mathbb{C})$ is join-dense in $\mathbb{C}$ and $O^{\mathcal{I}}(\mathbb{C})$ is meet-dense in $\mathbb{C}$. Thus, every $(\mathcal{F},\mathcal{I})$-dense completion is in particular a Δ_1-completion.

An $(\mathcal{F},\mathcal{I})$-compact and $(\mathcal{F},\mathcal{I})$-dense completion of P is an $(\mathcal{F},\mathcal{I})$-*completion* of P. Thus, every $(\mathcal{F},\mathcal{I})$-completion is a Δ_1-completion. For every poset P and every $\mathcal{F},\mathcal{I}$ as above, an $(\mathcal{F},\mathcal{I})$-completion of P exists and it is unique up to an isomorphism that fixes P (cf. [14]).

Let us now describe the main steps of the proof of existence given in [14]: First we consider the polarity $(\mathcal{F},\mathcal{I},R)$, where $R \subseteq \mathcal{F} \times \mathcal{I}$ is the relation defined by

$$FRI \quad \text{iff} \quad F \cap I \neq \emptyset.$$

Then we associate the following quasi-ordered set $Int(\mathcal{F},\mathcal{I},R)$ to the polarity: The domain of $Int(\mathcal{F},\mathcal{I},R)$ is the disjoint union $\mathcal{F} \uplus \mathcal{I}$ of $\mathcal{F}$ and $\mathcal{I}$ and the quasi-order is defined by setting, for every $F, G \in \mathcal{F}$ and every $I, J \in \mathcal{I}$ (we assume, using the same notation, that $\mathcal{F}$ and $\mathcal{I}$ are disjoint copies of $\mathcal{F}$ and $\mathcal{I}$),

(1) $F \leq^* G$ iff $G \subseteq F$,
(2) $I \leq^* J$ iff $I \subseteq J$,
(3) $F \leq^* I$ iff $F \cap I \neq \emptyset$,
(4) $I \leq^* F$ iff for every $p \in F$ and every $q \in I$, $q \leq p$.

Then we consider the quotient $\mathcal{F} \oplus_{\mathbb{P}} \mathcal{I}$ of $Int(\mathcal{F},\mathcal{I},R)$ by the equivalence relation $\equiv\ =\ \leq^* \cap \geq^*$ and denote the quotient partial order by $\leq$. The elements of the quotient are denoted by $[F]$ for $F \in \mathcal{F}$ and by $[I]$ for $I \in \mathcal{I}$. The only non-singleton $\equiv$-classes are of the form $\{\uparrow p, \downarrow p\}$ for every $p \in P$. Let $[p] = [\uparrow p] = [\downarrow p]$.

Finally, the $(\mathcal{F},\mathcal{I})$-completion of P is the MacNeille completion of the poset $\mathcal{F} \oplus_{\mathbb{P}} \mathcal{I}$.

4. $\mathcal{S}$-Canonical extensions for finitary congruential logics

Let $\mathcal{S}$ be a finitary congruential logic. Recall that because of congruentiality, for every algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ it is possible to define the $\mathcal{S}$-poset of $\boldsymbol{A}$ as $\langle A, \leq_{\mathcal{S}}^{\boldsymbol{A}} \rangle$.

From the logical point of view we take in this paper, the definition of "canonical extension" for $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ ought to be in principle based on taking the $(\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}, \mathrm{Id}_{\mathcal{S}}\boldsymbol{A})$-completion of $\langle A, \leq_{\mathcal{S}}^{\boldsymbol{A}} \rangle$. On the other hand, for this definition to be independent of the algebraic signature of $\boldsymbol{A}$, the consequence relation of $\mathcal{S}$ should be represented purely in terms of the order-theoretic properties of the $\mathcal{S}$-poset (and so every operation/logical connective in the algebraic signature should be extended to the completion like it is done for the modal operators in the Boolean case). When $\mathcal{S}$ satisfies (PC), the $\mathcal{S}$-poset is in particular a meet-semilattice, and the hypothesis of $\mathcal{S}$ being finitary makes it possible to encode the consequence relation of $\mathcal{S}$ purely in terms of the partial order $\leq_{\mathcal{S}}^{\boldsymbol{A}}$, because any consequence only depends on a finite number of premises, which in turn can be encoded by their finite meet. But if the $\mathcal{S}$-poset is not a meet-semilattice, the consequence relation cannot anymore be encoded purely in terms of $\leq_{\mathcal{S}}^{\boldsymbol{A}}$, because the finite subsets cannot be replaced by their infima, since they may not exist.

The solution we propose here to solve this tension is based on the fact that the poset $\langle \mathrm{Fi}_{\mathcal{S}}^{\omega}\boldsymbol{A}, \supseteq \rangle$ of the finitely generated $\mathcal{S}$-filters of $\boldsymbol{A}$ is a meet-semilattice. So the idea is to define the canonical extension of $\boldsymbol{A}$ not by taking some $(\mathcal{F},\mathcal{I})$-completion of the $\mathcal{S}$-poset $\langle A, \leq_{\mathcal{S}}^{\boldsymbol{A}}\rangle$ but of the meet-semilattice $\mathrm{Fi}^{\omega}_{\mathcal{S}}\boldsymbol{A}$. We will see that the order-theoretic properties of $\mathrm{Fi}^{\omega}_{\mathcal{S}}\boldsymbol{A}$ are well suited to encode the consequence relation of $\mathcal{S}$: indeed, the poset-filters of $\mathrm{Fi}^{\omega}_{\mathcal{S}}\boldsymbol{A}$ are in one-to-one correspondence with the $\mathcal{S}$-filters of $\boldsymbol{A}$. An analogous correspondence holds between certain $\mathcal{S}$-ideals of $\boldsymbol{A}$ and certain poset-ideals of $\mathrm{Fi}^{\omega}_{\mathcal{S}}\boldsymbol{A}$.

We will define the $\mathcal{S}$-canonical extension $\boldsymbol{A}^{\mathcal{S}}$ of $\boldsymbol{A}$ as the canonical extension, in the sense of [3], of $\mathrm{Fi}^{\omega}_{\mathcal{S}}\boldsymbol{A}$; that is, as its $(\mathcal{F},\mathcal{I})$-completion, where $\mathcal{F}$ is the collection of its poset-filters (non-empty down-directed up-sets) and $\mathcal{I}$ the collection of its poset-ideals (non-empty up-directed down-sets). The most important result of this section is Proposition 4.20: in the special case in which $\boldsymbol{A}^{\mathcal{S}}$ satisfies the $(\vee, \bigwedge)$-distributive law, $\boldsymbol{A}^{\mathcal{S}}$ is isomorphic to the $(\mathcal{F},\mathcal{I})$-completion of $\langle A, \leq_{\mathcal{S}}^{\boldsymbol{A}}\rangle$ such that $\mathcal{F}$ is the set of the $\mathcal{S}$-filters of $\boldsymbol{A}$ and $\mathcal{I}$ is the set of the *non-empty up-directed* $\mathcal{S}$-ideals.

4.1. **The meet-semilattice $\mathrm{Fi}^{\omega}_{\mathcal{S}}\boldsymbol{A}$.** Let $\mathcal{P}_{\omega}(X)$ denote, as usual, the set of finite subsets of X. $Y \subseteq_{\omega} X$ will mean that Y is a finite subset of X.

Let $\mathcal{S}$ be a finitary congruential logic and $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, which we assume fixed throughout the section. In what follows we will give an alternative presentation of $\mathrm{Fi}^{\omega}_{\mathcal{S}}\boldsymbol{A}$ that is based on identifying every finitely generated $\mathcal{S}$-filter with the equivalence class of the finite sets that generate it. Let us define the relation $\leq_{\mathcal{S}}$ on $\mathcal{P}_{\omega}(A)$ as follows:

$$X \leq_{\mathcal{S}} Y \quad \text{iff} \quad Y \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(X).$$

This relation is a quasi-order. Its associated equivalence relation $\sim_{\mathcal{S}}$ identifies two finite subsets X, Y of A if $X \leq_{\mathcal{S}} Y$ and $Y \leq_{\mathcal{S}} X$, that is, if $C^{\boldsymbol{A}}_{\mathcal{S}}(X) = C^{\boldsymbol{A}}_{\mathcal{S}}(Y)$. The equivalence class of $X \subseteq_{\omega} A$ will be denoted by $\overline{X}$. Thus, for $X, Y \subseteq_{\omega} A$

$$\overline{X} = \overline{Y} \quad \text{iff} \quad C^{\boldsymbol{A}}_{\mathcal{S}}(X) = C^{\boldsymbol{A}}_{\mathcal{S}}(Y).$$

Let $\mathcal{P}_{\omega}(A)/{\sim_{\mathcal{S}}}$ be the quotient of $\mathcal{P}_{\omega}(A)$ by $\sim_{\mathcal{S}}$. The partial order induced on $\mathcal{P}_{\omega}(A)/{\sim_{\mathcal{S}}}$ by $\leq_{\mathcal{S}}$ will be also denoted by $\leq_{\mathcal{S}}$. Note that for every $X, Y \in \mathcal{P}_{\omega}(A)$,

$$\overline{X} \leq_{\mathcal{S}} \overline{Y} \quad \text{iff} \quad C^{\boldsymbol{A}}_{\mathcal{S}}(Y) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(X).$$

Hence, for every $X \in \mathcal{P}_{\omega}(A)$, $\overline{X} \leq_{\mathcal{S}} \overline{\emptyset}$.

Lemma 4.1. *For every $\overline{X}, \overline{Y} \in \mathcal{P}_{\omega}(A)/{\sim_{\mathcal{S}}}$, the meet of $\overline{X}, \overline{Y}$ w.r.t. $\leq_{\mathcal{S}}$ exists and*

$$\overline{X} \wedge \overline{Y} = \overline{X \cup Y}.$$

Proof. Since $C^{\boldsymbol{A}}_{\mathcal{S}}(X), C^{\boldsymbol{A}}_{\mathcal{S}}(Y) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(X \cup Y)$, we have $\overline{X \cup Y} \leq_{\mathcal{S}} \overline{X}, \overline{Y}$. Conversely, suppose that $\overline{Z} \leq_{\mathcal{S}} \overline{X}, \overline{Y}$. Then $C^{\boldsymbol{A}}_{\mathcal{S}}(X), C^{\boldsymbol{A}}_{\mathcal{S}}(Y) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Z)$; therefore $C^{\boldsymbol{A}}_{\mathcal{S}}(X \cup Y) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Z)$. Hence, $\overline{Z} \leq \overline{X \cup Y}$, which shows that $\overline{X \cup Y}$ is the meet of $\overline{X}$ and $\overline{Y}$. $\square$

Proposition 4.2. *The poset $\langle \mathcal{P}_{\omega}(A)/{\sim_{\mathcal{S}}}, \leq_{\mathcal{S}}\rangle$ is a topped meet-semilattice.*

We denote by $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ the poset $\langle \mathcal{P}_{\omega}(A)/{\sim_{\mathcal{S}}}, \leq_{\mathcal{S}}\rangle$ and we refer to it as the *meet $\mathcal{S}$-semi-lattice of $\boldsymbol{A}$*.

The poset $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is in fact isomorphic to the poset $\langle \mathrm{Fi}^{\omega}_{\mathcal{S}}\boldsymbol{A}, \supseteq\rangle$. We will rather work with $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ than with $\langle \mathrm{Fi}^{\omega}_{\mathcal{S}}\boldsymbol{A}, \supseteq\rangle$ because the results we present in this paper are mainly proved using sets of generators.

Let $j : A \to \mathcal{P}_{\omega}(A)/{\sim_{\mathcal{S}}}$ be the map defined by

$$j(a) = \overline{\{a\}}.$$

For simplicity we will abuse of notation and write $\overline{a}$ for $\overline{\{a\}}$.

Proposition 4.3. *The map j is an order embedding from $\langle A, \leq_{\mathcal{S}}^{\boldsymbol{A}}\rangle$ into $L_{\mathcal{S}}^{\wedge}(\boldsymbol{A})$ and $L_{\mathcal{S}}^{\wedge}(\boldsymbol{A})$ is meet-generated by $j[A]$.*

Proof. For every $a, b \in A$, $a \leq_{\mathcal{S}}^{\boldsymbol{A}} b$ iff $C_{\mathcal{S}}^{\boldsymbol{A}}(b) \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(a)$ iff $\overline{a} \leq_{\mathcal{S}} \overline{b}$, which shows that j is an order embedding. Let $\overline{X} \in \mathcal{P}_\omega(A)/{\sim_{\mathcal{S}}}$. Since $X = \bigcup_{a\in X}\{a\}$, then by Proposition 4.1 $\overline{X} = \bigwedge\{\overline{a} : a \in X\} = \bigwedge\{j(a) : a \in X\}$. □

Remark 4.4. If $\mathcal{S}$ satisfies (PC), then every finitely generated $\mathcal{S}$-filter of every $\mathcal{L}$-algebra $\boldsymbol{A}$ is generated by a single element. Therefore, all the elements of $\mathcal{P}_\omega(A)/{\sim_{\mathcal{S}}}$ are of the form $\overline{a}$. In this case j is an isomorphism between $\langle A, \leq_{\mathcal{S}}^{\boldsymbol{A}}\rangle$ and $L_{\mathcal{S}}^{\wedge}(\boldsymbol{A})$.

4.2. **$\mathcal{S}$-filters of $\boldsymbol{A}$ and filters of $L_{\mathcal{S}}^{\wedge}(\boldsymbol{A})$.** We are now going to show that the collection $\mathcal{F}$ of the poset-filters of $L_{\mathcal{S}}^{\wedge}(\boldsymbol{A})$, ordered by inclusion, is order-isomorphic to $\langle \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}, \subseteq\rangle$.

For every $F \in \mathcal{F}$, let $F^* = \bigcup\{C_{\mathcal{S}}^{\boldsymbol{A}}(X) : \overline{X} \in F\}$. Clearly, if $F_1, F_2 \in \mathcal{F}$ and $F_1 \subseteq F_2$, then $F_1^* \subseteq F_2^*$.

Lemma 4.5. *For every $F \in \mathcal{F}$, F^* is an $\mathcal{S}$-filter of $\boldsymbol{A}$.*

Proof. It is enough to show that $C_{\mathcal{S}}^{\boldsymbol{A}}(F^*) \subseteq F^*$. Suppose $a \in C_{\mathcal{S}}^{\boldsymbol{A}}(F^*)$. Because $\mathcal{S}$ is finitary, $a \in C_{\mathcal{S}}^{\boldsymbol{A}}(X)$ for some $X \subseteq_\omega F^* = \bigcup\{C_{\mathcal{S}}^{\boldsymbol{A}}(Y) : \overline{Y} \in F\}$. Then for every $b \in X$, $b \in C_{\mathcal{S}}^{\boldsymbol{A}}(Y_b)$ for some some $Y_b \subseteq_\omega A$ such that $\overline{Y_b} \in F$. Since F is down-directed, there exists some $Y \subseteq_\omega A$ such that $\overline{Y} \in F$ and $\overline{Y} \leq_{\mathcal{S}} \overline{Y_b}$ for every $b \in X$. Then $C_{\mathcal{S}}^{\boldsymbol{A}}(Y_b) \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(Y)$ for every $b \in X$, so $X \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(Y)$ and hence $a \in C_{\mathcal{S}}^{\boldsymbol{A}}(Y)$. Therefore, $a \in F^*$. □

Let G be an $\mathcal{S}$-filter of $\boldsymbol{A}$. Consider the set

$$\overline{G} = \{\overline{X} : X \subseteq_\omega G\}$$

and notice that since G is an $\mathcal{S}$-filter, for every $Y \subseteq_\omega A$,

$$\overline{Y} \in \overline{G} \quad \text{iff} \quad Y \subseteq G.$$

Lemma 4.6. *If G is an $\mathcal{S}$-filter of $\boldsymbol{A}$, then $\overline{G} \in \mathcal{F}$.*

Proof. Suppose that $\overline{X} \leq_{\mathcal{S}} \overline{Y}$ and $\overline{X} \in \overline{G}$. Thus $Y \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(X) \subseteq G$. Hence $\overline{Y} \in \overline{G}$, which shows that $\overline{G}$ is an up-set. Now suppose that $\overline{X}, \overline{Y} \in \overline{G}$. Then $X, Y \subseteq G$, so $X \cup Y \subseteq_\omega G$. Therefore $\overline{X \cup Y} \in \overline{G}$. Now, since $C_{\mathcal{S}}^{\boldsymbol{A}}(X), C_{\mathcal{S}}^{\boldsymbol{A}}(Y) \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(X \cup Y)$, then $\overline{X \cup Y} \leq_{\mathcal{S}} \overline{X}$ and $\overline{X \cup Y} \leq_{\mathcal{S}} \overline{Y}$, which shows that $\overline{G}$ is down-directed. Finally, since $\overline{\emptyset} \in \overline{G}$, $\overline{G}$ is non-empty. □

Lemma 4.7. *If $F \in \mathcal{F}$, then $\overline{F^*} = F$.*

Proof. If $\overline{X} \in F$, then $C_{\mathcal{S}}^{\boldsymbol{A}}(X) \in \{C_{\mathcal{S}}^{\boldsymbol{A}}(X) : \overline{X} \in F\}$, so $X \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(X) \subseteq \bigcup\{C_{\mathcal{S}}^{\boldsymbol{A}}(X) : \overline{X} \in F\}$, which shows that $\overline{X} \in \overline{F^*}$. If $\overline{X} \in \overline{F^*}$, then $X \subseteq_\omega F^*$. So for each $b \in X$, $b \in C_{\mathcal{S}}^{\boldsymbol{A}}(Y_b)$ for some Y_b such that $\overline{Y_b} \in F$. Since F is down-directed, there exists some $Y \subseteq_\omega A$ such that $\overline{Y} \in F$ and $\overline{Y} \leq_{\mathcal{S}} \overline{Y_b}$, for every $b \in X$. Then $C_{\mathcal{S}}^{\boldsymbol{A}}(Y_b) \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(Y)$ for every $b \in X$. So, $X \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(Y)$, i.e. $\overline{Y} \leq_{\mathcal{S}} \overline{X}$. Since F is an up-set, this shows that $\overline{X} \in F$. □

Lemma 4.8. *If G is an $\mathcal{S}$-filter, then $(\overline{G})^* = G$.*

Proof. By definition, and because $\mathcal{S}$ is finitary, $(\overline{G})^* = \bigcup\{C_{\mathcal{S}}^{\boldsymbol{A}}(Y) : \overline{Y} \in \overline{G}\} = \bigcup\{C_{\mathcal{S}}^{\boldsymbol{A}}(Y) : Y \subseteq_\omega G\} = G$. □

The two lemmas above show that the maps $(.)^* : \mathcal{F} \to \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$ and $\overline{(.)} : \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A} \to \mathcal{F}$ are order isomorphisms (when both sets are ordered by inclusion), and are inverse to one another.

4.3. **Poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$.** In this section, we will turn to the relationship between $\mathcal{S}$-ideals of $\boldsymbol{A}$ and poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$. While in general the analogous correspondence cannot be established as for the filters, there is a strict correspondence between interesting subclasses on both sides.

Recall that a poset-ideal I of a meet semi-lattice $\langle P, \wedge\rangle$ is *prime* if it is proper and for every $a, b \in P$, if $a \wedge b \in I$, then $a \in I$ or $b \in I$.

The following characterization of the prime poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ will be useful in understanding how poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ are related to $\mathcal{S}$-ideals of $\boldsymbol{A}$.

Lemma 4.9. *A poset-ideal I of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is prime iff for every $\overline{X} \in I$ there exists some $a \in X$ such that $\overline{a} \in I$.*

Proof. For the right to left direction, let I be a poset-ideal of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ such that for every $\overline{X} \in I$ there exists some $a \in X$ such that $\overline{a} \in I$. Then $\overline{\emptyset} \notin I$, so I is proper. Suppose that $\overline{X \cup Y} = \overline{X} \wedge \overline{Y} \in I$. By the assumption on I, there exists some $a \in X \cup Y$ such that $\overline{a} \in I$. Then $\overline{X} \leq_{\mathcal{S}} \overline{a}$ or $\overline{Y} \leq_{\mathcal{S}} \overline{a}$. Since I is a down-set, this implies that $\overline{X} \in I$ or $\overline{Y} \in I$.

Conversely, let I be prime and let $\overline{X} \in I$. Since I is proper, $X \neq \emptyset$. It is easy to see by induction on the cardinality of X that there exists some $a \in X$ such that $\overline{a} \in I$. □

There is a strict correspondence between the non-empty $\mathcal{S}$-ideals of $\boldsymbol{A}$ which are up-directed w.r.t. $\leq^{\boldsymbol{A}}_{\mathcal{S}}$ and the poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ satisfying a property that we are going to introduce below and which is satisfied by the prime poset-ideals. The correspondence we establish allows us to introduce a notion of prime $\mathcal{S}$-ideal which will be very useful in what follows.

A poset-ideal I of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is an $\boldsymbol{A}$*-ideal* if for every $\overline{X} \in I$ there exists some $a \in A$ such that $\overline{X} \leq_{\mathcal{S}} \overline{a}$ and $\overline{a} \in I$. Notice that, by Lemma 4.9, every prime poset-ideal of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is an $\boldsymbol{A}$-ideal.

For every poset-ideal I of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$, let us define

$$I^{\star} = \{a \in A : \overline{a} \in I\}.$$

The map $(\cdot)^{\star}$ is clearly monotone: if $I_1 \subseteq I_2$, then $I_1^{\star} \subseteq I_2^{\star}$.

Proposition 4.10. *If I is a poset-ideal of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$, then $I^{\star}$ is an $\mathcal{S}$-ideal of $\boldsymbol{A}$. If, in addition, I is an $\boldsymbol{A}$-ideal, then $I^{\star}$ is up-directed (w.r.t. $\leq^{\boldsymbol{A}}_{\mathcal{S}}$).*

Proof. As for the first part, it is enough to show that $(C^{\boldsymbol{A}}_{\mathcal{S}})^d(I^{\star}) \subseteq I^{\star}$: Let $b \in A$ and $a_0, \ldots, a_n \in I^{\star}$ such that $C^{\boldsymbol{A}}_{\mathcal{S}}(a_0) \cap \ldots C^{\boldsymbol{A}}_{\mathcal{S}}(a_n) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(b)$. Since I is up-directed and $\overline{a_0}, \ldots, \overline{a_n} \in I$, there exists some $\overline{X} \in I$ such that $\overline{a_i} \leq_{\mathcal{S}} \overline{X}$ for every $i \leq n$. Then $X \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(a_0) \cap \ldots C^{\boldsymbol{A}}_{\mathcal{S}}(a_n) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(b)$. Therefore $\overline{b} \leq_{\mathcal{S}} \overline{X}$. This implies that $\overline{b} \in I$, and so $b \in I^{\star}$. If $C^{\boldsymbol{A}}_{\mathcal{S}}(b) = A$, since I is non-empty, it follows that there exists $\overline{X} \in I$ with $X \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(b)$ and therefore, as before, we have $b \in I^{\star}$.

Let I be an $\boldsymbol{A}$-ideal and let $a, b \in I^{\star}$. Then $\overline{a}, \overline{b} \in I$. Since I is up-directed, $\overline{a}, \overline{b} \leq_{\mathcal{S}} \overline{X}$ for some $\overline{X} \in I$. Since I is an $\boldsymbol{A}$-ideal, there exists some $c \in A$ such that $\overline{X} \leq_{\mathcal{S}} \overline{c}$ and $\overline{c} \in I$. Hence $c \in I^{\star}$, and $a, b \leq^{\boldsymbol{A}}_{\mathcal{S}} c$. □

For every $\mathcal{S}$-ideal J of $\boldsymbol{A}$, let us define

$$\overline{J} = \{\overline{X} \in L^{\wedge}_{\mathcal{S}}(\boldsymbol{A}) : C^{\boldsymbol{A}}_{\mathcal{S}}(X) \cap J \neq \emptyset\}.$$

Note that $\overline{J} = {\downarrow}\{\overline{a} : a \in J\}$ and that the map $\overline{(\cdot)}$ is monotone: if $J_1 \subseteq J_2$, then $\overline{J_1} \subseteq \overline{J_2}$.

Proposition 4.11. *For every $J \in \mathrm{Id}_{\mathcal{S}}\boldsymbol{A}$, if J is non-empty and up-directed w.r.t. $\leq^{\boldsymbol{A}}_{\mathcal{S}}$, then $\overline{J}$ is an $\boldsymbol{A}$-ideal of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$.*

Proof. Let J be a non-empty up-directed $\mathcal{S}$-ideal of $\boldsymbol{A}$. Then it follows straightforwardly from the definition that $\overline{J}$ is a non-empty down-set. To show that it is up-directed, let $\overline{X}, \overline{Y} \in \overline{J}$. Then let $a, b \in J$ such that $\overline{X} \leq_{\mathcal{S}} \overline{a}$ and $\overline{Y} \leq_{\mathcal{S}} \overline{b}$. Since J is up-directed, $a, b \leq^{\boldsymbol{A}}_{\mathcal{S}} c$ for some $c \in J$. Then $\overline{a}, \overline{b} \leq_{\mathcal{S}} \overline{c}$. Therefore, $\overline{X}, \overline{Y} \leq_{\mathcal{S}} \overline{c} \in \overline{J}$. Finally, from the definition of $\overline{J}$ it follows that it is an $\boldsymbol{A}$-ideal. □

Proposition 4.12. *If J is a non-empty $\mathcal{S}$-ideal of $\boldsymbol{A}$, then $(\overline{J})^{\star} = J$.*

Proof. Since $\mathcal{S}$-ideals are down-sets w.r.t. $\leq^{\boldsymbol{A}}_{\mathcal{S}}$, $(\overline{J})^{\star} = \{a \in \boldsymbol{A} : \overline{a} \in \overline{J}\} = \{a \in \boldsymbol{A} : C^{\boldsymbol{A}}_{\mathcal{S}}(a) \cap J \neq \emptyset\} = \{a \in \boldsymbol{A} : {\uparrow}a \cap J \neq \emptyset\} = \{a \in \boldsymbol{A} : a \in J\} = J$. □

Proposition 4.13. *For every $\boldsymbol{A}$-ideal I of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$, $\overline{I^{\star}} = I$.*

Proof. By assumption, if $\overline{X} \in I$, then $\overline{X} \leq_{\mathcal{S}} \overline{a}$ for some $\overline{a} \in I$. So $a \in C^{\boldsymbol{A}}_{\mathcal{S}}(X) \cap I^{\star} \neq \emptyset$, hence $\overline{X} \in \overline{I^{\star}}$. Conversely, if $\overline{X} \in \overline{I^{\star}}$ then there exists some $a \in C^{\boldsymbol{A}}_{\mathcal{S}}(X) \cap I^{\star}$. Hence $\overline{X} \leq_{\mathcal{S}} \overline{a} \in I$ and so $\overline{X} \in I$. □

The two propositions above imply that:

Proposition 4.14. *The maps $\overline{(\cdot)}$ and $(\cdot)^{\star}$ establish order isomorphisms between the non-empty up-directed $\mathcal{S}$-ideals of an algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ and the $\boldsymbol{A}$-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$, both collections being ordered by inclusion.*

Note that since every prime poset-ideal of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is an $\boldsymbol{A}$-ideal, its corresponding $\mathcal{S}$-ideal is up-directed.

The previous considerations naturally lead to the following notion:

Definition 4.15. An $\mathcal{S}$-ideal J of $\boldsymbol{A}$ is *prime* if $\overline{J}$ is a prime poset-ideal of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$. Equivalently, J is prime if J is non-empty, up-directed and $X \cap J \neq \emptyset$ for every $X \subseteq_{\omega} A$ such that $C^{\boldsymbol{A}}_{\mathcal{S}}(X) \cap J \neq \emptyset$.

Proposition 4.16. *The maps $\overline{(\cdot)}$ and $(\cdot)^{\star}$ establish order isomorphisms between the prime $\mathcal{S}$-ideals of an algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ and the prime poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$, both collections being ordered by inclusion.*

4.4. **The $\mathcal{S}$-canonical extension of $\boldsymbol{A}$.** Let $\mathcal{S}$ be a finitary congruential logic and $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$. The theory of canonical extensions for posets developed in [3] can now be applied to the meet semi-lattice $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$. That is, we can define the canonical extension of $\boldsymbol{A}$ as the canonical extension $(L^{\wedge}_{\mathcal{S}}(\boldsymbol{A}))^{\sigma}$ of the poset $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ as it is defined in [3].

Definition 4.17. The *$\mathcal{S}$-canonical extension of* $\boldsymbol{A}$ is the $(\mathcal{F}, \mathcal{I})$-completion of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ where $\mathcal{F}$ is the family of poset-filters and $\mathcal{I}$ the family of poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$. The $\mathcal{S}$-canonical extension of $\boldsymbol{A}$ will be denoted by $\boldsymbol{A}^{\mathcal{S}}$.

Let $m : L^{\wedge}_{\mathcal{S}}(\boldsymbol{A}) \to (L^{\wedge}_{\mathcal{S}}(\boldsymbol{A}))^{\sigma}$ be the canonical embedding, and let

$$k := (m \circ j) : A \to (L^{\wedge}_{\mathcal{S}}(\boldsymbol{A}))^{\sigma}$$

be the order embedding obtained by composing the map m with the embedding $j : \langle A, \leq^{\boldsymbol{A}}_{\mathcal{S}}\rangle \to L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ defined above Proposition 4.3.

The correspondences between $\mathcal{S}$-filters of $\boldsymbol{A}$ and poset-filters of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ and between non-empty up-directed $\mathcal{S}$-ideals of $\boldsymbol{A}$ and $\boldsymbol{A}$-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ are at the basis of the following facts:

Lemma 4.18.
1. *For every $\mathcal{S}$-filter G of $\boldsymbol{A}$, $\bigwedge k[G] = \bigwedge m[\overline{G}]$,*
2. *For every non-empty up-directed $\mathcal{S}$-ideal J of $\boldsymbol{A}$, $\bigvee k[J] = \bigvee m[\overline{J}]$.*

Proof. (1) Let G be an $\mathcal{S}$-filter of $\boldsymbol{A}$. Notice that for every $X \subseteq_{\omega} G$, $\overline{X} = \bigwedge\{j(a) : a \in X\}$, moreover the canonical embedding m preserves all finite meets. This implies that $m(\overline{X}) = \bigwedge k[X]$, and so

$$\bigwedge m[\overline{G}] = \bigwedge\{m(\overline{X}) : X \subseteq_{\omega} G\} = \bigwedge\{\bigwedge k[X] : X \subseteq_{\omega} G\} = \bigwedge k[G].$$

(2) Let J be a non-empty up-directed $\mathcal{S}$-ideal of $\boldsymbol{A}$. Since $j[J] \subseteq \overline{J}$, then $k[J] \subseteq m[\overline{J}]$ and so $\bigvee k[J] \leq \bigvee m[\overline{J}]$. As for the converse inequality, since J is non-empty and up-directed, then $\overline{J}$ is an $\boldsymbol{A}$-ideal, hence for every $\overline{X} \in \overline{J}$ there exists some $\overline{a_{\overline{X}}} \in \overline{J}$ such that $\overline{X} \leq_{\mathcal{S}} \overline{a_{\overline{X}}}$, which implies that $\bigvee m[\overline{J}] = \bigvee m[\{\overline{a_{X}} : \overline{X} \in \overline{J}\}]$. But $\overline{a_{\overline{X}}} \in \overline{J}$ implies $a_{\overline{X}} \in J$. Thus $k[\{a_{\overline{X}} : \overline{X} \in \overline{J}\}] \subseteq k[J]$ and so

$$\bigvee m[\overline{J}] = \bigvee m[\{\overline{a_X} : \overline{X} \in \overline{J}\}] = \bigvee k[\{a_{\overline{X}} : \overline{X} \in \overline{J}\}] \leq \bigvee k[J].$$

□

Let us finish this subsection by showing now that the prime poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ and the completely meet-prime elements of $\boldsymbol{A}^{\mathcal{S}}$ exactly correspond:

Proposition 4.19.
1. *For every poset-ideal I of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$, I is prime iff $\bigvee m[I]$ is completely meet-prime in $\boldsymbol{A}^{\mathcal{S}}$.*
2. *If $c \in A^{\mathcal{S}}$ is completely meet-prime, then $c = \bigvee m[I]$ for some prime poset-ideal I of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$*

Proof. 1. For simplicity let us suppress the embedding m. As for the 'if' direction, by Lemma 4.9, in order to show that I is prime, it is enough to show that for every $\overline{X} \in I$ there exists some $a \in X$ such that $\overline{a} \in I$. If $\overline{X} \in I$, then $\bigwedge_{a\in X} \overline{a} = \overline{X} \leq \bigvee I$. Since $\bigvee I$ is completely meet-prime, $\overline{a} \leq \bigvee I$ for some $a \in X$. Hence $\overline{a} \in I$ by compactness.
As for the converse implication, let I be a prime poset-ideal of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$. Since every element of $\boldsymbol{A}^{\mathcal{S}}$ is a meet of joins of poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$, to show that $\bigvee I$ is completely meet-prime, it is enough to show that if $\{I_s : s \in S\} \subseteq \mathrm{Id}(L^{\wedge}_{\mathcal{S}}(\boldsymbol{A}))$ and $\bigwedge_{s\in S} \bigvee I_s \leq \bigvee I$, then $\bigvee I_s \leq \bigvee I$ for some $s \in S$. Suppose for contradiction that $\bigvee I_s \not\leq \bigvee I$ for every $s \in S$. Then for every $s \in S$ there exists some $\overline{X_s} \in I_s$ such that $\overline{X_s} \notin I$. $\overline{X_s} \in I_s$ implies that $\overline{X_s} \leq \bigvee I_s$ and so

$$\bigwedge_{s\in S} \overline{X_s} \leq \bigwedge_{s\in S} \bigvee I_s \leq \bigvee I. \tag{4.1}$$

Since I is prime, if $\overline{X_s} \notin I$ for every $s \in S$ then $\bigwedge_{s\in S'} \overline{X_s} \notin I$ for every $S' \subseteq_\omega S$. This implies that

$$\bigwedge_{s\in S} \overline{X_s} \not\leq \bigvee I : \tag{4.2}$$

indeed if $\bigwedge_{s\in S} \overline{X_s} \leq \bigvee I$, then by the compactness of $\boldsymbol{A}^{\mathcal{S}}$ we would get that $\bigwedge_{s\in S'} \overline{X_s} \leq \bigvee I$ (i.e. $\bigwedge_{s\in S'} \overline{X_s} \in I$) for some $S' \subseteq_\omega S$. Now (4.1) and (4.2) contradict one another.

2. If $c \in A^{\mathcal{S}}$ is completely meet-prime, then $c \in M^\infty(\boldsymbol{A}^{\mathcal{S}}) \subseteq O^{\mathcal{I}}(\boldsymbol{A}^{\mathcal{S}})$ (cf. [14]), hence $c = \bigvee I$ for some poset-ideal I of $L^\wedge_{\mathcal{S}}(\boldsymbol{A})$. Then by the 'if' direction of the first item of this proposition, I is prime. □

4.5. **$\boldsymbol{A}^{\mathcal{S}}$ satisfying the $(\vee, \bigwedge)$-distributive law.** We will now work under the additional hypothesis that $\boldsymbol{A}^{\mathcal{S}}$ satisfies the $(\vee, \bigwedge)$-distributive law

$$p \vee \bigwedge S = \bigwedge_{s\in S} p \vee s$$

because, as we will see, this situation applies to the setting of congruential logics satisfying (PDD). The most important result of this section is that, under this additional hypothesis, $\boldsymbol{A}^{\mathcal{S}}$ coincides (up to an isomorphism fixing $\boldsymbol{A}$) with the $(\mathcal{F}, \mathcal{I})$-completion of $\boldsymbol{A}$, $\mathcal{F}$ being the collection of $\mathcal{S}$-filters of $\boldsymbol{A}$ and $\mathcal{I}$ being the collection of the non-empty up-directed $\mathcal{S}$-ideals of $\boldsymbol{A}$.

Recall that if a complete lattice satisfies the $(\vee, \bigwedge)$-distributive law, then every completely meet-irreducible element is completely meet-prime and if it satisfies the $(\wedge, \bigvee)$-distributive law, then every completely join-irreducible element is completely join-prime.

We are now ready to show the main result of this section.

Theorem 4.20. *If $\boldsymbol{A}^{\mathcal{S}}$ satisfies the $(\vee, \bigwedge)$-distributive law, then $\boldsymbol{A}^{\mathcal{S}}$ is the $(\mathcal{F}, \mathcal{I})$-completion of $\boldsymbol{A}$, for the collection $\mathcal{F}$ of $\mathcal{S}$-filters and the collection $\mathcal{I}$ of the non-empty up-directed $\mathcal{S}$-ideals of $\boldsymbol{A}$.*

Proof. Let $A^{\mathcal{S}}$ be the domain of $\boldsymbol{A}^{\mathcal{S}}$. By definition, if $a \in A^{\mathcal{S}}$ then

$$a = \bigvee \{ \bigwedge m[F] \mid F \in \mathcal{X} \}$$

for some collection $\mathcal{X}$ of poset-filters of $L^\wedge_{\mathcal{S}}(\boldsymbol{A})$. By Lemmas 4.7 and 4.18 (1), we get that for every $F \in \mathcal{X}$,

$$\bigwedge m[F] = \bigwedge m[\overline{F^*}] = \bigwedge k[F^*]$$

and because of Lemma 4.5 we conclude that every element of $\boldsymbol{A}^{\mathcal{S}}$ is a join of meets of $\mathcal{S}$-filters. Similarly, every $a \in A^{\mathcal{S}}$ is a meet of joins of non-empty up-directed $\mathcal{S}$-ideals: indeed, $a = \bigwedge M$ for some subset M of completely meet-irreducible elements of $A^{\mathcal{S}}$ (cf. [3]); since by assumption $\boldsymbol{A}^{\mathcal{S}}$ satisfies the $(\vee, \bigwedge)$-distributive law, every $c \in M$ is completely-meet prime, therefore, by Proposition 4.19 (2), $c = \bigvee m[I]$ for some prime poset-ideal I of $L^\wedge_{\mathcal{S}}(\boldsymbol{A})$. Since I, being prime, is an $\boldsymbol{A}$-ideal of $L^\wedge_{\mathcal{S}}(\boldsymbol{A})$, then by Proposition 4.10 $I^\star$ is a non-empty up-directed $\mathcal{S}$-ideal of $\boldsymbol{A}$ and by Lemmas 4.13 and 4.18 (2), we get

$$c = \bigvee m[I] = \bigvee m[\overline{I^\star}] = \bigvee k[I^\star].$$

Therefore every element of $\boldsymbol{A}^{\mathcal{S}}$ is a a meet of joins of non-empty up-directed $\mathcal{S}$-ideals.

Let us show that $\boldsymbol{A}^{\mathcal{S}}$ is $(\mathcal{F}, \mathcal{I})$-compact. Let $G \in \mathcal{F}$ and $J \in \mathcal{I}$ such that $\bigwedge k[G] \leq \bigvee k[J]$. Since by Lemma 4.18 $\bigwedge k[G] = \bigwedge m[\overline{G}]$ and $\bigvee k[J] = \bigvee m[\overline{J}]$, then $\bigwedge m[\overline{G}] \leq \bigvee m[\overline{J}]$. Then the compactness of $\boldsymbol{A}^{\mathcal{S}}$ w.r.t. the poset-filters and poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ implies that there exists some $\overline{X} \in \overline{G} \cap \overline{J}$. Since $\overline{J}$ is an $\boldsymbol{A}$-ideal, there exists some $a \in A$ such that $\overline{X} \leq_{\mathcal{S}} \overline{a}$ and $\overline{a} \in \overline{J}$. Then $a \in G \cap J \neq \emptyset$. □

In order to be able to apply Proposition 4.20 we will need the following result.

Proposition 4.21. *If $\boldsymbol{A}^{\mathcal{S}}$ satisfies the $(\wedge, \bigvee)$-distributive law, then $\boldsymbol{A}^{\mathcal{S}}$ is a completely distributive lattice.*

Proof. Because $\boldsymbol{A}^{\mathcal{S}}$ is the canonical extension of the poset $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$, $\boldsymbol{A}^{\mathcal{S}}$ is join generated by its completely join-irreducible elements, and so it is join generated by its completely join-prime elements. Therefore $\boldsymbol{A}^{\mathcal{S}}$ is a completely distributive lattice (cf. Thm. 16 in Ch. XII.4 of [1]). □

5. The canonicity of Hilbert algebras

In the previous section we introduced the $\mathcal{S}$-canonical extension $\boldsymbol{A}^{\mathcal{S}}$ of $\boldsymbol{A}$; this construction extends the known lattice-based settings of canonical extensions and uniformly applies to every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ for every finitary and congruential logic $\mathcal{S}$. We also showed that, if $\boldsymbol{A}^{\mathcal{S}}$ satisfies the $(\vee, \bigwedge)$-distributive law, then $\boldsymbol{A}^{\mathcal{S}}$ is the $(\mathcal{F}, \mathcal{I})$-completion of $\boldsymbol{A}$ corresponding to the choice of the collections $\mathcal{F}$ of $\mathcal{S}$-filters and $\mathcal{I}$ of non-empty up-directed $\mathcal{S}$-ideals of $\boldsymbol{A}$.

In this section we are going to show that, if our notion of canonical extension is based on this construction, the variety of Hilbert algebras is indeed canonical. This will be a consequence of a canonicity result that we show for every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $\mathcal{S}$ being any finitary and congruential logic satisfying (PDD) relative to a definable binary term $x \to y$. Our proof strategy goes as follows: using a lemma of Köhler-Pigozzi's [18], we will show that for every finitary congruential logic $\mathcal{S}$ satisfying (PDD) and every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, the meet operation of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is residuated. Then, by applying Proposition 3.6 in [3], the σ-extension of the meet in $\boldsymbol{A}^{\mathcal{S}}$ is also residuated and its residuum is the π-extension $\to^{\pi}$ of the residuum of the meet in $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$. Therefore $\langle A^{\mathcal{S}}, \bigwedge, \bigvee, \to^{\pi}, \bot, \top\rangle$ is a complete Heyting algebra. In particular this implies both that $\langle A^{\mathcal{S}}, \to^{\pi}\rangle$ is a Hilbert algebra and that $\boldsymbol{A}^{\mathcal{S}}$ satisfies the $(\wedge, \bigvee)$-distributive law, hence by Proposition 4.21 it satisfies the $(\vee, \bigwedge)$-distributive law, and so by Proposition 4.20, $\boldsymbol{A}^{\mathcal{S}}$ is the $(\mathcal{F}, \mathcal{I})$-completion of $\boldsymbol{A}$ corresponding to the choice of the collections $\mathcal{F}$ of $\mathcal{S}$-filters and $\mathcal{I}$ of non-empty up-directed $\mathcal{S}$-ideals of $\boldsymbol{A}$.

5.1. $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ as an implicative meet-semilattice.

Let $\mathcal{S}$ be a finitary and congruential logic satisfying (PDD) relative to a binary term $x \to y$ and let $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$. We are going to show that $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is an implicative meet semi-lattice, that is, its meet operation is residuated. This means that for every $\overline{X}, \overline{Y} \in \mathcal{P}_{\omega}(A)^*$ there exists a unique $\overline{Z} \in \mathcal{P}_{\omega}(A)^*$, denoted by $\overline{X} \to \overline{Y}$, such that for every $\overline{W} \in \mathcal{P}_{\omega}(A)^*$

$$\overline{W} \wedge \overline{X} \leq_{\mathcal{S}} \overline{Y} \quad \text{iff} \quad \overline{W} \leq_{\mathcal{S}} \overline{X} \to \overline{Y}.$$

We will refer to $\overline{X} \to \overline{Y}$ as the *residuum* of $\overline{X}$ relative to $\overline{Y}$ (w.r.t. the meet).

In order to show that $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is an implicative meet semi-lattice, we will use the following Lemma, proved in [18] in its order-dual version. We report its proof here for the reader's convenience.

Lemma 5.1 (Köhler-Pigozzi). *Let $\langle L, \wedge\rangle$ be a meet-semilattice and X be a set of generators X of L. If for every $a, b \in X$ the residuum $a \to b$ exists and belongs to X, then the residuum $a \to b$ exists for every $a, b \in L$.*

Proof. Let us argue by cases and show that if $a \in L$ and $b \in X$ then $a \to b$ exists. Since X is a set of generators, then $a = a_0 \wedge \ldots \wedge a_n$ for some $a_0, \ldots, a_n \in X$. Then, for every $c \in L$, $c \wedge a \leq b$ iff $(c \wedge a_0 \wedge \ldots \wedge a_{n-1}) \wedge a_n = c \wedge (a_0 \wedge \ldots \wedge a_n) \leq b$ iff $c \wedge a_0 \wedge \ldots \wedge a_{n-1} \leq a_n \to b$. So by applying the assumption n times, we obtain that

$$c \wedge a \leq b \quad \text{iff} \quad c \leq a_0 \to (\ldots(a_n \to b)\ldots). \tag{5.1}$$

Thus $a \to b = a_0 \to (\ldots(a_n \to b)\ldots)$.

Suppose now that $a \in L$ and $b \in L$. Assume that $b = b_0 \wedge \ldots \wedge b_m$ for some $b_0, \ldots, b_m \in X$. By the previous case, $a \to b_i$ exists for every $i \leq m$. Now, $c \wedge a \leq b$ iff $c \wedge a \leq b_0 \wedge \ldots \wedge b_m$ iff $c \wedge a \leq b_i$ for every $i \leq m$, iff $c \leq a \to b_i$ for every $i \leq m$. Thus,

$$c \wedge a \leq b \quad \text{iff} \quad c \leq (a \to b_0) \wedge \ldots \wedge (a \to b_m). \tag{5.2}$$

Hence, $a \to b = (a \to b_0) \wedge \ldots \wedge (a \to b_m)$. □

Let $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$. In order to apply Köhler-Pigozzi's Lemma to the meet semi-lattice $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$, recall that by Proposition 4.3 the set $\{\overline{a} : a \in A\}$ meet-generates $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$. Then, if $\to^{\boldsymbol{A}}$ is the interpretation of $\to$ in $\boldsymbol{A}$:

Lemma 5.2. *For every $a, b \in A$, $\overline{a \to^{\boldsymbol{A}} b}$ is the residuum in $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ of $\overline{a}$ relative to $\overline{b}$.*

Proof. Since by assumption $C^{\boldsymbol{A}}_{\mathcal{S}}$ satisfies (PDD), for every $\overline{X} \in \mathcal{P}_{\omega}(A)^*$, $\overline{X} \wedge \overline{a} \leq_{\mathcal{S}} \overline{b}$ iff $C^{\boldsymbol{A}}_{\mathcal{S}}(b) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(X, a)$ iff $a \to^{\boldsymbol{A}} b \in C^{\boldsymbol{A}}_{\mathcal{S}}(X)$ iff $\overline{X} \leq_{\mathcal{S}} \overline{a \to^{\boldsymbol{A}} b}$. □

As an immediate consequence of Lemmas 5.2 and 5.1 we then obtain:

Proposition 5.3. *$L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is a residuated meet semi-lattice.*

Let us denote by $\to^*$ the residuum of the meet in $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$:

Proposition 5.4. *The order embedding $j : \boldsymbol{A} \to L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is a $\to$-homomorphism.*

Proof. By Lemma 5.2, $j(a \to^{\boldsymbol{A}} b) = \overline{a \to^{\boldsymbol{A}} b} = \overline{a} \to^* \overline{b} = j(a) \to^* j(b)$. □

Let us now consider the π-extension of $\to^*$ to $\boldsymbol{A}^{\mathcal{S}}$, which is defined first on every $f \in K^{\mathcal{F}}(\boldsymbol{A}^{\mathcal{S}})$ and $i \in O^{\mathcal{I}}(\boldsymbol{A}^{\mathcal{S}})$, $\mathcal{F}$ being the set of poset-filters and $\mathcal{I}$ the set of poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$:

$$f \to^{\pi} i = \bigvee\{x \to y : x, y \in L^{\wedge}_{\mathcal{S}}(\boldsymbol{A}), f \leq x, y \leq i\}$$

and then, for every $u, v \in \boldsymbol{A}^{\mathcal{S}}$,

$$u \to^{\pi} v = \bigwedge\{f \to^{\pi} i : u \geq f \in K^{\mathcal{F}}(\boldsymbol{A}^{\mathcal{S}}) \text{ and } v \leq i \in O^{\mathcal{I}}(\boldsymbol{A}^{\mathcal{S}})\}.$$

Proposition 5.5. *The composition* $k := (m \circ j) : \boldsymbol{A} \to (L^{\wedge}_{\mathcal{S}}(\boldsymbol{A}))^{\sigma}$ *of the embedding* $j : \boldsymbol{A} \to L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ *and the canonical embedding* $m : L^{\wedge}_{\mathcal{S}}(\boldsymbol{A}) \to (L^{\wedge}_{\mathcal{S}}(\boldsymbol{A}))^{\sigma}$ *is a* $\to$*-homomorphism, that is, for every* $a, b \in A$,

$$k(a \to^{\boldsymbol{A}} b) = k(a) \to^{\pi} k(b).$$

Proof. By construction, $\to^{\pi}$ is an extension of $\to^{*}$; hence m is a $\to$-homomorphism. The statement follows from this and Proposition 5.4. □

By Proposition 3.6 in [3], $\to^{\pi}$ is the residuum of the σ-extension of the meet of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$. On the other hand, the σ-extension of the meet of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is the meet of $\boldsymbol{A}^{\mathcal{S}}$ (cf. [9]); therefore we get:

Proposition 5.6. $\langle \boldsymbol{A}^{\mathcal{S}}, \to^{\pi} \rangle$ *is a complete Heyting algebra. So its* $\to$*-reduct is a Hilbert algebra.*

Every complete Heyting algebra satisfies the $(\wedge, \bigvee)$-distributive law. Then by Proposition 4.21, $\boldsymbol{A}^{\mathcal{S}}$ is completely distributive, which implies that $\boldsymbol{A}^{\mathcal{S}}$ satisfies also the $(\vee, \bigwedge)$-distributive law, and so by Proposition 4.20, $\boldsymbol{A}^{\mathcal{S}}$ is the $(\mathcal{F}, \mathcal{I})$-completion of $\boldsymbol{A}$, for the collection $\mathcal{F}$ of $\mathcal{S}$-filters and the collection $\mathcal{I}$ of the non-empty up-directed $\mathcal{S}$-ideals of $\boldsymbol{A}$. Therefore,

Theorem 5.7. *The* $(\mathcal{F}, \mathcal{I})$*-completion* $\langle \boldsymbol{A}^{\mathcal{S}}, \to^{\pi} \rangle$ *of a Hilbert algebra* $\langle A, \to \rangle$*, for the collection* $\mathcal{F}$ *of* $\mathcal{S}$*-filters and the collection* $\mathcal{I}$ *of the non-empty up-directed* $\mathcal{S}$*-ideals of* $\boldsymbol{A}$*, is a Hilbert algebra.*

This justifies the definition of the canonical extension of each Hilbert algebra $\boldsymbol{A} = \langle A, \to \rangle$ as $\langle \boldsymbol{A}^{\mathcal{S}}, \to^{\pi} \rangle$.

5.2. **An internal description of the π-extension.** The canonicity of Hilbert algebras was shown in a nonstandard way, the standard way being the much stronger proof that axioms H1–H4, possibly independently of one another, are canonical. Instead we derived it in one step, as a byproduct of the fact that the meet-semilattice $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is residuated. The standard proof of canonicity would be based on an internal description of the residuum operation $\to^{*}$ and of its extension $\to^{\pi}$ restricted to $\mathcal{F}$-closed and $\mathcal{I}$-open elements of $\boldsymbol{A}^{\mathcal{S}}$. In this section we are going to provide this internal description, that will be crucial for proving the canonicity of axiomatic extensions of Hilbert algebras (an instance of which will be the content of the next section: indeed, we will show that Tarski algebras are canonical). In order to provide this internal description, we will not use the abstract characterization of $\boldsymbol{A}^{\mathcal{S}}$, but rather the specific way in which $\boldsymbol{A}^{\mathcal{S}}$ is obtained by the construction described in Section 3. To this end we will first introduce some notation: for every sequence $a_0, \ldots, a_n$ of elements of A and every $b \in A$ let us inductively define the element $(a_n, \ldots, a_0; b) \in A$ as follows:

$$(a_0; b) := a_0 \to^{\boldsymbol{A}} b \quad \text{and} \quad (a_{i+1}, \ldots, a_0; b) := a_{i+1} \to^{\boldsymbol{A}} (a_i, \ldots, a_0; b).$$

So for instance, $(a_2, a_1, a_0; b) = a_2 \to^{\boldsymbol{A}} (a_1 \to^{\boldsymbol{A}} (a_0 \to^{\boldsymbol{A}} b))$.
Since $\mathcal{S}$ satisfies (PDD), for every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ the operation $\to^{\boldsymbol{A}}$ is order reversing in the first coordinate and order preserving in the second coordinate w.r.t. the partial order $\leq^{\boldsymbol{A}}_{\mathcal{S}}$. Moreover:

(1) $(a_n, \ldots, a_0; b) = (a_{p(n)}, \ldots, a_{p(0)}; b)$ for every permutation p of the indexes $\{0, \ldots, n\}$,
(2) $(a_n, \ldots, a_0; b) \leq^{\boldsymbol{A}}_{\mathcal{S}} (a_{n+1}, a_n, \ldots, a_0; b)$

(3) $(a, a; b) = (a; b)$

Because of property (1) above, we can introduce the following notation for every nonempty and finite $X = \{a_0, \ldots, a_n\} \subseteq A$ and every $b \in A$:

$$X \to b := (a_n, \ldots, a_0; b) \quad \text{and} \quad \emptyset \to b := b$$

Note that:

(a) By Property (2), for every $X, Y \subseteq_\omega A$ and every $b \in A$, if $X \subseteq Y$ then $X \to b \leq^{\boldsymbol{A}}_{\mathcal{S}} Y \to b$,

(b) for every $X \subseteq_\omega A$ and every $a, b \in A$, $X \cup \{b\} \to a = b \to (X \to a)$.

For every $X, Y \subseteq_\omega A$ let us define

$$X \to Y := \{X \to b : b \in Y\},$$

and then let us define the binary operation $\to$ in $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ as follows:

$$\overline{X} \to \overline{Y} := \overline{X \to Y} = \overline{\{X \to b : b \in Y\}}.$$

Note that if X or Y is empty, then $\overline{X} \to \overline{Y} = \overline{\emptyset}$. This definition does not depend on the choice of the representatives X and Y, as it is shown in the next lemma:

Lemma 5.8. *For every $X, Y, Z \subseteq_\omega A$,*

$$C^{\boldsymbol{A}}_{\mathcal{S}}(Y) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Z \cup X) \quad \textit{iff} \quad C^{\boldsymbol{A}}_{\mathcal{S}}(\{X \to b : b \in Y\}) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Z).$$

Hence, if $\overline{X} = \overline{X'}$ and $\overline{Y} = \overline{Y'}$ then $\overline{\{X \to b : b \in Y\}} = \overline{\{X' \to b' : b' \in Y'\}}$.

Proof. Assume that $C^{\boldsymbol{A}}_{\mathcal{S}}(Y) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Z \cup X)$: if $Y = \emptyset$, then $\{X \to b : b \in Y\} = \emptyset$, so $C^{\boldsymbol{A}}_{\mathcal{S}}(\{X \to b : b \in Y\}) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Z)$. If $Y \neq \emptyset$, let $b \in Y$ and let us show that $X \to b \in Z$. By assumptions $b \in Y \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Y) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Z \cup X)$; so, by (PDD), $X \to b \in C^{\boldsymbol{A}}_{\mathcal{S}}(Z)$. Conversely, if $C^{\boldsymbol{A}}_{\mathcal{S}}(\{X \to b : b \in Y\}) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Z)$ and $a \in Y$, then $X \to a \in C^{\boldsymbol{A}}_{\mathcal{S}}(\{X \to b : b \in Y\}) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Z)$. Hence by (PDD), $a \in C^{\boldsymbol{A}}_{\mathcal{S}}(Z \cup X)$. As for the second part of the statement, we will only show the left-to-right inclusion: by the first part of the statement, it is enough to prove that $C^{\boldsymbol{A}}_{\mathcal{S}}(Y) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(\{X' \to b' : b' \in Y'\} \cup X)$. Since $C^{\boldsymbol{A}}_{\mathcal{S}}(Y) = C^{\boldsymbol{A}}_{\mathcal{S}}(Y')$, it is enough to show that $Y' \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(\{X' \to b' : b' \in Y'\} \cup X)$, so if $a \in Y'$ then $X' \to a' \in C^{\boldsymbol{A}}_{\mathcal{S}}(\{X' \to b' : b' \in Y'\} \cup X)$, so by (PDD), $a' \in C^{\boldsymbol{A}}_{\mathcal{S}}(\{X' \to b' : b' \in Y'\} \cup X \cup X') = C^{\boldsymbol{A}}_{\mathcal{S}}(\{X' \to b' : b' \in Y'\} \cup X)$, as desired. □

Recall that the residuum of the meet-semilattice $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ was denoted by $\to^*$. The next proposition says that $\to$ is the internal description of $\to^*$:

Proposition 5.9. *For every $\overline{X}, \overline{Y}, \overline{Z} \in \mathcal{P}_\omega(A)^*$,*

$$\overline{Z} \wedge \overline{X} \leq_{\mathcal{S}} \overline{Y} \quad \textit{iff} \quad \overline{Z} \leq_{\mathcal{S}} \overline{X} \to \overline{Y}.$$

Hence, $\to^*$ coincides with $\to$.

Proof. $\overline{Z} \wedge \overline{X} \leq_{\mathcal{S}} \overline{Y}$ iff $\overline{Z \cup X} \leq_{\mathcal{S}} \overline{Y}$ iff $C^{\boldsymbol{A}}_{\mathcal{S}}(Y) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Z \cup X)$ iff $C^{\boldsymbol{A}}_{\mathcal{S}}(\{X \to b : b \in Y\}) \subseteq C^{\boldsymbol{A}}_{\mathcal{S}}(Z)$ iff $\overline{Z} \leq_{\mathcal{S}} \overline{\{X \to b : b \in Y\}}$ iff $\overline{Z} \leq_{\mathcal{S}} \overline{X} \to \overline{Y}$. □

Next, let us give an internal description of the extension $\to^\pi$ restricted to the $\mathcal{F}$-closed and $\mathcal{I}$-open elements of $\boldsymbol{A}^{\mathcal{S}}$, $\mathcal{F}$ and $\mathcal{I}$ being the collections of poset-filters and poset-ideals of $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ respectively. To simplify the notation, let us abbreviate $m[F]$ as $[F]$ and $m[I]$ as $[I]$ for every $F \in \mathcal{F}$ and $I \in \mathcal{I}$. Then, by definition,

$$[F] \to^\pi [I] = \bigvee \{[\overline{X} \to \overline{Y}] : \overline{X} \in F, \overline{Y} \in I\}$$

and
$$[I] \to^{\pi} [F] = \bigwedge\{[G] \to^{\pi} [J] : [G] \leq [I] \text{ and } [F] \leq [J]\}.$$

Proposition 5.10. *For every $F \in \mathcal{F}$ and $I \in \mathcal{I}$,*
$$\begin{aligned}[F] \to^{\pi} [I] &= [\{\overline{Z} : (\exists \overline{X} \in F)\ \overline{Z} \wedge \overline{X} \in I\}] \\ &= [\{\overline{Z} : (\exists \overline{X} \in F)(\exists \overline{Y} \in I)\ \overline{Z} \leq_{\mathcal{S}} \overline{X} \to \overline{Y}\}].\end{aligned}$$

Proof. Let us preliminarily show that $\mathbb{X} = \{\overline{Z} : (\exists \overline{X} \in F)\ \overline{X \cup Z} \in I\} \in \mathcal{I}$. If $\overline{Z'} \leq_{\mathcal{S}} \overline{Z} \in \mathcal{I}$, then $C_{\mathcal{S}}^{\boldsymbol{A}}(Z) \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(Z')$ and $\overline{X \cup Z} \in I$ for some $\overline{X} \in F$. Therefore $C_{\mathcal{S}}^{\boldsymbol{A}}(X \cup Z) \subseteq C_{\mathcal{S}}^{\boldsymbol{A}}(X \cup Z')$, so $\overline{X \cup Z'} \leq_{\mathcal{S}} \overline{X \cup Z}$. Hence, $\overline{X \cup Z'} \in I$, which shows that $\overline{Z'} \in \mathbb{X}$. If $\overline{Z}, \overline{Z'} \in \mathbb{X}$, then $\overline{X \cup Z}, \overline{X' \cup Z'} \in I$ for some $\overline{X}, \overline{X'} \in F$. Since I is up-directed, then $\overline{X \cup Z}, \overline{X' \cup Z'} \leq_{\mathcal{S}} \overline{Y}$ for some $\overline{Y} \in I$. Then, $\overline{Z} \leq_{\mathcal{S}} \overline{X} \to \overline{Y} \leq_{\mathcal{S}} (\overline{X} \wedge \overline{X'}) \to \overline{Y}$ and $\overline{Z'} \leq_{\mathcal{S}} \overline{X'} \to \overline{Y} \leq_{\mathcal{S}} (\overline{X} \wedge \overline{X'}) \to \overline{Y}$. Since $(\overline{X} \wedge \overline{X'}) \wedge ((\overline{X} \wedge \overline{X'}) \to \overline{Y}) \leq_{\mathcal{S}} \overline{Y} \in I$, then $(\overline{X} \wedge \overline{X'}) \wedge ((\overline{X} \wedge \overline{X'}) \to \overline{Y}) \in I$. Since $\overline{X} \wedge \overline{X'} \in F$, we get $(\overline{X} \wedge \overline{X'}) \to \overline{Y} \in \mathbb{X}$, which concludes the proof that $\mathbb{X} \in \mathcal{I}$.
To show that $[F] \to^{\pi} [I] \leq [\mathbb{X}]$, it is enough to show that for every $\overline{X} \in F$ and $\overline{Y} \in I$, $[\overline{X} \to \overline{Y}] \leq [\mathbb{X}]$: indeed, note that $\overline{X} \to \overline{Y} \in \mathbb{X}$, because $\overline{X} \wedge (\overline{X} \to \overline{Y}) \leq_{\mathcal{S}} \overline{Y} \in I$.
Now to show the first equality, it is enough to show that if $u \in A^{\mathcal{S}}$ and $[\overline{X} \to \overline{Y}] \leq u$ for every $\overline{X} \in F$ and every $\overline{Y} \in I$, then $[\mathbb{X}] \leq u$. By denseness, $u = \bigwedge\{[H] : H \in \mathcal{I}, u \leq [H]\}$, so it is enough to show that if $H \in \mathcal{I}$ and $u \leq [H]$, then $[\mathbb{X}] \leq [H]$, i.e. that $\mathbb{X} \subseteq H$. If $\overline{Z} \in \mathbb{X}$, then $\overline{X \cup Z} \in I$ for some $\overline{X} \in F$. Then $[\overline{X} \to \overline{X \cup Z}] \leq u \leq [H]$, and so $\overline{X} \to \overline{X \cup Z} \in H$. Since $\overline{X} \wedge \overline{Z} \leq_{\mathcal{S}} \overline{X} \wedge \overline{Z}$, then $\overline{Z} \leq_{\mathcal{S}} \overline{X} \to \overline{X \cup Z}$, which implies that $\overline{Z} \in H$. As for the second equality, it is enough to show that for every $\overline{Z}$, $\overline{Z} \wedge \overline{X} \in I$ for some $\overline{X} \in F$ iff $\overline{Z} \leq_{\mathcal{S}} \overline{X} \to \overline{Y}$ for some $\overline{X} \in F$ and some $\overline{Y} \in I$. Both directions are easy consequences of Proposition 5.9. ☐

Proposition 5.11. *For every $F \in \mathcal{F}$ and $I \in \mathcal{I}$,*
$$[I] \to^{\pi} [F] = [\{\overline{Z} : (\exists \overline{X} \in F)(\exists \overline{Y} \in I)\ \overline{Y} \to \overline{X} \leq_{\mathcal{S}} \overline{Z}\}]$$

Proof. Let us show that $\mathbb{X} = \{\overline{Z} : (\exists \overline{X} \in F)(\exists \overline{Y} \in I)\ \overline{Y} \to \overline{X} \leq_{\mathcal{S}} \overline{Z}\} \in \mathcal{F}$. By construction, $\mathbb{X}$ is an up-set. If $\overline{Z}, \overline{Z'} \in \mathbb{X}$, then $\overline{Y} \to \overline{X} \leq_{\mathcal{S}} \overline{Z}$ and $\overline{Y'} \to \overline{X'} \leq_{\mathcal{S}} \overline{Z'}$ for some $\overline{X}, \overline{X'} \in F$ and $\overline{Y}, \overline{Y'} \in I$. Since I is up-directed, then $\overline{Y}, \overline{Y'} \leq_{\mathcal{S}} \overline{Y''}$ for some $\overline{Y''} \in I$. Then $\overline{Y''} \to \overline{X \cup X'} \leq_{\mathcal{S}} \overline{Y} \to \overline{X \cup X'}$ and $\overline{Y''} \to \overline{X \cup X'} \leq_{\mathcal{S}} \overline{Y'} \to \overline{X \cup X'}$. Moreover $\overline{Y} \to \overline{X \cup X'} \leq_{\mathcal{S}} \overline{Y} \to \overline{X}$ and $\overline{Y'} \to \overline{X \cup X'} \leq_{\mathcal{S}} \overline{Y'} \to \overline{X}$. Hence, $\overline{Y''} \to (\overline{X} \wedge \overline{X'}) = \overline{Y''} \to \overline{X \cup X'} \leq_{\mathcal{S}} \overline{Z} \wedge \overline{Z'}$. Therefore, $\overline{Z} \wedge \overline{Z'} \in \mathbb{X}$, which finishes the proof that $\mathbb{X} \in \mathcal{F}$.
Since $\mathbb{X} \in \mathcal{F}$, then (cf. [14]) $[\mathbb{X}] = \bigwedge\{[\overline{Z}] : \overline{Z} \in \mathbb{X}\}$. Therefore, in order to show that
$$\bigwedge\{[G] \to^{\pi} [J] : [G] \leq [I], G \in \mathcal{F} \text{ and } [F] \leq [J], J \in \mathcal{I}\} = [\mathbb{X}],$$
it is enough to show that (a) for every $\overline{Z} \in \mathbb{X}$, $[G] \to^{\pi} [J] \leq [\overline{Z}]$ for some $G \in \mathcal{F}, J \in \mathcal{I}$ such that $[G] \leq [I]$ and $[F] \leq [J]$, and (b) if $G \in \mathcal{F}$, $J \in \mathcal{I}$ such that $[G] \leq [I]$ and $[F] \leq [J]$, then $[G] \to^{\pi} [J] \leq [\overline{Z}]$ for some $\overline{Z} \in \mathbb{X}$.
(a): if $\overline{Z} \in \mathbb{X}$, then $\overline{Y} \to \overline{X} \leq_{\mathcal{S}} \overline{Z}$ for some $\overline{X} \in F$ and $\overline{Y} \in I$. Then take $G = \overline{X}$ and $J = \overline{Y}$: indeed, $[\overline{Y}] \to [\overline{X}] = [\overline{Y} \to \overline{X}] \leq [\overline{Z}]$ and moreover $[F] \leq [\overline{X}]$ and $[\overline{Y}] \leq [I]$.
(b): if $G \in \mathcal{F}$, $J \in \mathcal{I}$ such that $[G] \leq [I]$ and $[F] \leq [J]$, then $G \cap I \neq \emptyset \neq F \cap J$, so $\overline{Y} \in G$ and $\overline{X} \in J$ for some $\overline{X} \in F, \overline{Y} \in I$. Then $[G] \to^{\pi} [J] \leq [\overline{Y}] \to [\overline{X}]$ and $\overline{Y} \to \overline{X} \in \mathbb{X}$. ☐

6. Tarski algebras are canonical

A *Tarski algebra* is a Hilbert algebra $\langle A, \to\rangle$ that satisfies the equation

T: $(x \to y) \to x \approx x$.

In this section we will prove the canonicity of Tarski algebras by showing that for every Hilbert algebra $\boldsymbol{A}$,

$$\boldsymbol{A} \models T \text{ implies that } \boldsymbol{A}^{\mathcal{S}} \models T.$$

Lemma 6.1. *For every Tarski algebra $\boldsymbol{A}$, every $X \subseteq_\omega A$ and every $a \in A$,*

$$\overline{(a \to X) \to a} = \overline{a}.$$

Proof. By induction on the cardinality of X. If $X = \emptyset$,

$$\overline{(a \to X) \to a} = \overline{(a \to \emptyset) \to a} = \overline{\emptyset \to a} = \overline{a}.$$

If $X = \{b\}$, then $\overline{(a \to X) \to a} = \overline{(a \to b) \to a} = \overline{a}$, because the equation $(a \to b) \to a = a$ holds in every Tarski algebra. Suppose now that the statement is true for every X of cardinality $n > 0$, and let us show it holds for every X of cardinality $n+1$. If $X = \{b_0, \ldots, b_n\}$, then by inductive hypothesis, $\overline{(a \to X) \to a} = \overline{(a \to b_n) \to (\{a \to b_{n-1}, \ldots, a \to b_0\} \to a)} = \overline{(a \to b_n) \to a} = \overline{a}$. □

Lemma 6.2. *For every Tarski algebra $\boldsymbol{A}$ and every $X, Y \subseteq_\omega A$,*

$$(\overline{Y} \to \overline{X}) \to \overline{Y} \leq \overline{Y}.$$

Proof. Since $\overline{Y} = \bigwedge_{a \in Y} \overline{a}$, it is enough to show that for every $a \in Y$,

$$(\overline{Y} \to \overline{X}) \to \overline{Y} \leq \overline{a}.$$

If $a \in Y$, then $\overline{Y} \leq \overline{a}$ and so $\overline{a} \to \overline{X} \leq \overline{Y} \to \overline{X}$. Hence, $(\overline{Y} \to \overline{X}) \to \overline{a} \leq (\overline{a} \to \overline{X}) \to \overline{a} = \overline{a}$, the last equality holding by the lemma above. Moreover, $(\overline{Y} \to \overline{X}) \to \overline{Y} \leq (\overline{Y} \to \overline{X}) \to \overline{a}$. Thus, $(\overline{Y} \to \overline{X}) \to \overline{Y} \leq \overline{a}$. □

Corollary 6.3. *If $\boldsymbol{A}$ is a Tarski algebra, then $L^{\wedge}_{\mathcal{S}}(\boldsymbol{A})$ is a Tarski algebra.*

Proof. $\langle L^{\wedge}_{\mathcal{S}}(\boldsymbol{A}), \to\rangle$ as a subalgebra of $\langle \boldsymbol{A}^{\mathcal{S}}, \to^\pi\rangle$, is a Hilbert algebra. For every $X, Y \subseteq_\omega A$, $\overline{Y} \wedge (\overline{Y} \to \overline{X}) \leq \overline{Y}$, which implies that $\overline{Y} \leq (\overline{Y} \to \overline{X}) \to \overline{Y}$. This, together with the lemma above, concludes the proof. □

Theorem 6.4. *For every Tarski algebra $\boldsymbol{A}$ and every u, v in $A^{\mathcal{S}}$,*

$$(u \to^\pi v) \to^\pi u = u.$$

Proof. Since $u \wedge (u \to^\pi v) \leq u$, then $u \leq (u \to^\pi v) \to^\pi u$. Conversely, let us preliminarily show that for every $u, v \in A^{\mathcal{S}}$, $(u \to^\pi v) \to^\pi u \leq u$ iff for every $I \in \mathcal{I}$ and every $G \in \mathcal{F}$,

$$([I] \to^\pi [G]) \to^\pi [I] \leq [I].$$

Indeed, by density, $u = \bigwedge\{[I] : I \in \mathcal{I}, u \leq [I]\}$, so it is enough to show that, if $u \leq [I]$ and $[G] \leq v$, then $(u \to^\pi v) \to^\pi u \leq ([I] \to^\pi [G]) \to [I]$. By assumptions, $[I] \to^\pi [G] \leq [I] \to^\pi v \leq u \to^\pi v$, hence $(u \to^\pi v) \to^\pi u \leq ([I] \to^\pi [G]) \to^\pi u \leq ([I] \to^\pi [G]) \to^\pi [I]$.

Let us show that for every $I \in \mathcal{I}$ and every $G \in \mathcal{F}$, $([I] \to^\pi [G]) \to^\pi [I] \leq [I]$: By Proposition 5.11, $([I] \to^\pi [G]) \to^\pi [I] = [\mathbb{X}] \to^\pi [G]$, where

$$\mathbb{X} = \{\overline{Z} : (\exists \overline{X} \in G)(\exists \overline{Y} \in I)\ \overline{Y} \to \overline{X} \leq_{\mathcal{S}} \overline{Z}\} \in \mathcal{F}.$$

So, by Propositions 5.10 and 5.11:

$$\begin{aligned}([I] \to^\pi [G]) \to^\pi [I] &= [\mathbb{X}] \to^\pi [I] \\ &= [\{\overline{W} : (\exists \overline{Z} \in \mathbb{X})(\exists \overline{Y} \in I)\ \overline{W} \leq_{\mathcal{S}} \overline{Z} \to \overline{Y}\}] \\ &= [\{\overline{W} : (\exists \overline{X} \in G)(\exists \overline{Y}, \exists Y' \in I)\ \overline{W} \leq_{\mathcal{S}} (\overline{Y'} \to \overline{X}) \to \overline{Y}\}].\end{aligned}$$

and moreover,

$$\mathbb{Y} = \{\overline{W} : (\exists \overline{X} \in G)(\exists \overline{Y}, \exists Y' \in I)\ \overline{W} \leq_{\mathcal{S}} (\overline{Y'} \to \overline{X}) \to \overline{Y}\} \in \mathcal{I}.$$

Hence, to show that $[\mathbb{Y}] \leq [I]$, we need to show that $\mathbb{Y} \subseteq I$. If $\overline{W} \in \mathbb{Y}$, then $\overline{W} \leq_{\mathcal{S}} (\overline{Y'} \to \overline{X}) \to \overline{Y}$ for some $\overline{X} \in G$, $\overline{Y}, \overline{Y'} \in I$. Since I is up-directed, then $\overline{Y}, \overline{Y'} \leq_{\mathcal{S}} \overline{Y''}$ for some $\overline{Y''} \in I$. Thus, $\overline{Y''} \to \overline{X} \leq_{\mathcal{S}} \overline{Y'} \to \overline{X}$, and so, using the lemma above,

$$\overline{W} \leq_{\mathcal{S}} (\overline{Y'} \to \overline{X}) \to \overline{Y} \leq_{\mathcal{S}} (\overline{Y''} \to \overline{X}) \to \overline{Y} \leq_{\mathcal{S}} (\overline{Y''} \to \overline{X}) \to \overline{Y''} \leq_{\mathcal{S}} \overline{Y''},$$

which implies that $\overline{W} \in I$. □

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