

Categorical Decision Theory

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What is Decision Theory?

(2/62)

Individuals and societies must often make choices under uncertainty.
How should an agent decide when faced with such uncertainty?

This is the subject of a branch of economics called *Decision Theory*.

The foundations of decision theory were laid by Leonard J. Savage in 1954.

Savage modelled the decision problem as follows.

There is an (infinite) set \mathcal{S} of possible “states of the world”.

The true state is unknown.

\mathcal{S} represents all information which is unknown to the agent.

There is a set \mathcal{X} of possible “outcomes” (e.g. consumption bundles).

These are the things the agent ultimately cares about.

Each alternative defines a function $\alpha : \mathcal{S} \rightarrow \mathcal{X}$, called an *act*.

If the agent chooses the act α , and the true state of the world turns out to be s , then she will obtain the outcome $\alpha(s)$.

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Let \succeq be a *preference order* (a complete, transitive relation) on \mathcal{X}^S .

For any acts $\alpha, \beta \in \mathcal{X}^S$, the statement " $\alpha \succeq \beta$ " means, "If the agent had a choice, then she would choose α rather than β , *ex ante*."

Savage's Theorem. Suppose \succeq satisfies six axioms (encoding various criteria of "consistency" or "rationality"). Then there exists:

- ▶ a "cardinal utility" function $U : \mathcal{X} \rightarrow \mathbb{R}$, and
- ▶ a (finitely additive) probability measure P on \mathcal{S} ,

which provide a **subjective expected utility (SEU) representation** for \succeq . In other words, given any acts $\alpha, \beta \in \mathcal{X}^S$, we have

$$(\alpha \succeq \beta) \iff \left(\int_{\mathcal{S}} U[\alpha(s)] \, dP(s) \geq \int_{\mathcal{S}} U[\beta(s)] \, dP(s) \right).$$

Heuristically, U describes the agent's *ex post tastes* over outcomes in \mathcal{X} . Meanwhile, P describes her *ex ante beliefs* about states in \mathcal{S} .

Thus, Savage says any "rational" agent can be described as maximizing expected utility according to *some* system of preferences and beliefs.

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There are three ways we could improve on Savage's framework.

1. **Scope.** Savage assumed that \mathcal{S} and \mathcal{X} are arbitrary sets, and acts are arbitrary functions from \mathcal{S} to \mathcal{X} . (This can be extended to measurable spaces and measurable functions.)

But what if \mathcal{S} and \mathcal{X} are topological spaces, and acts must be continuous?

What if \mathcal{S} and \mathcal{X} are differentiable manifolds, and acts must be differentiable functions?

Want: a single theory which works in *all* of these environments (rather than multiple independent theories).

2. **Holism.** At different times, the same agent may face different sources of uncertainty (e.g. horse races, financial markets, weather, traffic) and different possible sets of outcomes (e.g. financial gains or losses, social status, physical (dis)comfort, physical danger), in different combinations.

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2. **Holism.**

3. **Endogenous/implicit states and outcomes.** Savage assumed that the agent could explicitly specify all possible “states of nature” and all possible “outcomes”, and could conceptualize each “act” as a function mapping states to outcomes.

This may be unrealistically demanding.

Also, even if people *do* represent decision problems this way, different people may adopt different representations of the same decision problem...

Want: A framework which does *not* require an explicit specification of the states and outcomes in advance.

Ideally, the statespace and outcome space should emerge “endogenously” from a description of the agent's preferences over acts.

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Plan:

- Part I. Savage structures; informal statement of main result.
- Part II. Partitions and probability.
- Part III. Concretization.
- Part IV. Products, spans and quasipreferences.
- Part V. Simple morphisms and SEU representations.
- Part VI. Formal statement of axioms and main result.

Part I.

Savage structures

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- ▶ *Associativity.* For all $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in [\mathcal{C}]$ and all $\alpha \in \vec{\mathcal{C}}(\mathcal{A}, \mathcal{B})$, $\beta \in \vec{\mathcal{C}}(\mathcal{B}, \mathcal{C})$, and $\gamma \in \vec{\mathcal{C}}(\mathcal{C}, \mathcal{D})$, $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.
- ▶ *Identity.* For every $\mathcal{A} \in [\mathcal{C}]$, there is an *identity* morphism $I_{\mathcal{A}} \in \vec{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ such that, for any $\mathcal{B} \in [\mathcal{C}]$, we have $I_{\mathcal{A}} \circ \phi = \phi$ for all $\phi \in \vec{\mathcal{C}}(\mathcal{B}, \mathcal{A})$, while $\phi \circ I_{\mathcal{A}} = \phi$ for all $\phi \in \vec{\mathcal{C}}(\mathcal{A}, \mathcal{B})$.

\mathcal{C} is a *concrete category* if the objects in $[\mathcal{C}]$ are *sets* (usually with some “structure”), the morphisms in $\vec{\mathcal{C}}(\mathcal{A}, \mathcal{B})$ are *functions* from \mathcal{A} to the set \mathcal{B} (which “preserve” this structure), and \circ is function composition.

Examples:

Set Objects are ordinary sets; morphisms are ordinary functions.

Meas Objects are measurable spaces; morphisms are measurable functions.

Top Objects are topological spaces; morphisms are continuous functions.

Diff Objects are differentiable manifolds; morphisms are diff’ble functions.

However, not all categories are concrete.

We will use the term *abstract category* to refer to a category which may or may not be concrete.

A *category* is a mathematical structure \mathcal{C} with three parts:

- ▶ A collection $[\mathcal{C}]$ of entities, called the *objects* of \mathcal{C} .
- ▶ For any $\mathcal{A}, \mathcal{B} \in [\mathcal{C}]$, a collection $\vec{\mathcal{C}}(\mathcal{A}, \mathcal{B})$ of entities, called *morphisms* from \mathcal{A} to \mathcal{B} .
- ▶ For any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in [\mathcal{C}]$, a *composition* operation \circ , such that, for any morphisms $\phi \in \vec{\mathcal{C}}(\mathcal{A}, \mathcal{B})$ and $\psi \in \vec{\mathcal{C}}(\mathcal{B}, \mathcal{C})$, we have $\psi \circ \phi \in \vec{\mathcal{C}}(\mathcal{A}, \mathcal{C})$.

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Let \mathcal{C} be a category. A *decision context* on \mathcal{C} is an ordered pair $(\mathcal{S}, \mathcal{X})$, where \mathcal{S} and \mathcal{X} are subcategories of \mathcal{C} .

We interpret the objects of the subcategory \mathcal{S} as “abstract state spaces”. (But they might not literally be spaces.) We will call them *state places*. For any $\mathcal{S}_1, \mathcal{S}_2 \in [\mathcal{S}]$, each $\phi \in \overrightarrow{\mathcal{S}}(\mathcal{S}_1, \mathcal{S}_2)$ is a \mathcal{C} -morphism from \mathcal{S}_1 to \mathcal{S}_2 that is somehow “compatible” with the agent’s beliefs about \mathcal{S}_1 and \mathcal{S}_2 (e.g. a measure-preserving transformation between two probability spaces).

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For every $\mathcal{S} \in [\mathcal{S}]$ and $\mathcal{X} \in [\mathcal{X}]$, let $\succeq_{\mathcal{X}}^{\mathcal{S}}$ be a preference order on $\vec{\mathcal{C}}(\mathcal{S}, \mathcal{X})$, representing the agent's *ex ante* preferences over acts.

The collection $\mathfrak{S} := \{\succeq_{\mathcal{X}}^{\mathcal{S}}; \mathcal{S} \in [\mathcal{S}] \text{ and } \mathcal{X} \in [\mathcal{X}]\}$ is a *Savage structure* if:

(BP) For any $\mathcal{S}_1, \mathcal{S}_2 \in [\mathcal{S}]$, any $\phi \in \vec{\mathcal{S}}(\mathcal{S}_1, \mathcal{S}_2)$, any $\mathcal{X} \in [\mathcal{X}]$, and any $\alpha, \beta \in \vec{\mathcal{C}}(\mathcal{S}_2, \mathcal{X})$, we have

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Goal. Find conditions under which a Savage structure admits a *subjective expected utility* (SEU) representation....

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Example. Let $\mathcal{C} := \text{Meas}$. Let \mathcal{S} be a collection of measurable spaces, each equipped with a probability measure.

For any $\mathcal{S}_1, \mathcal{S}_2 \in [\mathcal{S}]$, let $\vec{\mathcal{S}}(\mathcal{S}_1, \mathcal{S}_2)$ be the set of all measure-preserving functions from \mathcal{S}_1 into \mathcal{S}_2 . Then \mathcal{S} is a subcategory of \mathcal{C} .

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Very informally, we require $(\mathcal{S}, \mathcal{X})$ to satisfy three **structural conditions**:

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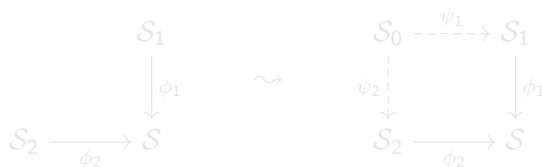
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Idea. \mathcal{S}_0 represents a *coupling* of the random variables represented by \mathcal{S}_1 and \mathcal{S}_2 , which are correlated through a “common observable” in \mathcal{S} .

Very informally, we require $(\mathcal{S}, \mathcal{X})$ to satisfy three structural conditions:

- (S1) Any state places \mathcal{S}_1 and \mathcal{S}_2 in \mathcal{S} have a product $\mathcal{S}_1 \times \mathcal{S}_2$ in \mathcal{S} .
Idea. $\mathcal{S}_1 \times \mathcal{S}_2$ encodes a *coupling* of the random variables represented by \mathcal{S}_1 and \mathcal{S}_2 .
- (S2) Any outcome places \mathcal{X}_1 and \mathcal{X}_2 in \mathcal{X} have a coproduct (roughly: a disjoint union) $\mathcal{X}_1 \amalg \mathcal{X}_2$ in \mathcal{X} .
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$$\begin{array}{ccc}
 \mathcal{S}_1 & & \mathcal{S}_0 \overset{\psi_1}{\dashrightarrow} \mathcal{S}_1 \\
 \downarrow \phi_1 & \rightsquigarrow & \downarrow \phi_1 \\
 \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S} & & \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S} \\
 & & \uparrow \psi_2
 \end{array}$$

Idea. \mathcal{S}_0 represents a *coupling* of the random variables represented by \mathcal{S}_1 and \mathcal{S}_2 , which are correlated through a “common observable” in \mathcal{S} .

We require the preferences defined by \mathfrak{S} to be *solvable*. Roughly speaking, this means that we can always find a compromise between two outcomes which is perfectly indifferent to some third alternative.

We will also require \mathfrak{S} to satisfy five axioms (stated very informally):

- (A1) On every outcome place in \mathcal{X} , there is a nontrivial *ex post* preference order, which governs the agent's preferences over “constant” acts.
- (A2) If one act α “statewise dominates” another act β (in terms of the *ex post* preferences), then the agent prefers α to β .
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Theorem. (Informal statement) *Let \mathcal{C} be any biconnected category. Let $(\mathcal{S}, \mathcal{X})$ be a decision context satisfying structural conditions (S1)-(S3).*

Let \mathfrak{S} be a solvable Savage structure on $(\mathcal{S}, \mathcal{X})$. Then:

\mathfrak{S} has a “subjective expected utility representation” if and only if it satisfies axioms (A1)-(A5).

In this representation, the “probabilistic beliefs” on each state place are unique. The “utility function” on each outcome place represents the agent’s “ex post preferences”, and is unique up to positive affine transform.

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Before we can formally state the theorem or the axioms, we must develop a theoretical framework in which these terms can be precisely defined....

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Part II

Partitions and Probability

Let \mathcal{X} and \mathcal{Y} be objects in a category \mathcal{C} , and let $\phi \in \vec{\mathcal{C}}(\mathcal{X}, \mathcal{Y})$.

ϕ is an *isomorphism* in \mathcal{C} if there is a morphism $\psi \in \vec{\mathcal{C}}(\mathcal{Y}, \mathcal{X})$ such that:

- ▶ $\psi \circ \phi = I_{\mathcal{X}}$ (the identity morphism on \mathcal{X}).
- ▶ $\phi \circ \psi = I_{\mathcal{Y}}$ (the identity morphism on \mathcal{Y}).

Examples. • If $\mathcal{C} = \text{Set}$, then isomorphisms are *bijections*.

• If $\mathcal{C} = \text{Meas}$, then isomorphisms are *bi-measurable bijections*.

• If $\mathcal{C} = \text{Top}$, then isomorphisms are *homeomorphisms*.

• If $\mathcal{C} = \text{Diff}$, then isomorphisms are *diffeomorphisms*.

We say that ϕ is a *monomorphism* (or is *monic*) if, for any other object $\mathcal{W} \in \mathcal{C}$, and any morphisms $\psi_1, \psi_2 \in \vec{\mathcal{C}}(\mathcal{W}, \mathcal{X})$, we have:

$$\left(\phi \circ \psi_1 = \phi \circ \psi_2 \right) \iff \left(\psi_1 = \psi_2 \right).$$

In most concrete categories, monomorphisms are *injective* morphisms.

Example. Let \mathcal{X} be a subobject of \mathcal{Y} (e.g. subspace, submanifold, etc.). Then the *inclusion morphism* $\mathcal{X} \hookrightarrow \mathcal{Y}$ is usually a monomorphism.

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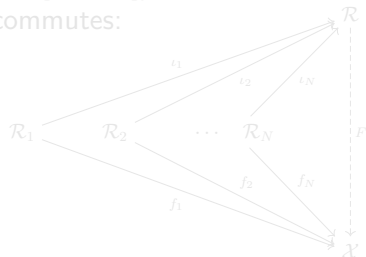
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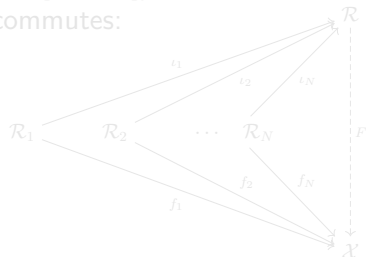


We then write $\mathcal{R} = \coprod_{n=1}^N \mathcal{R}_n$
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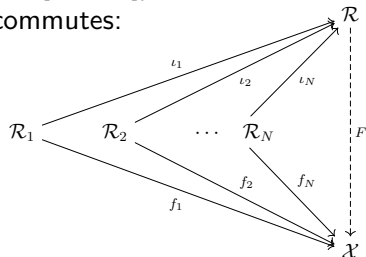


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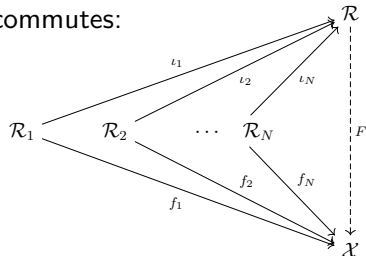


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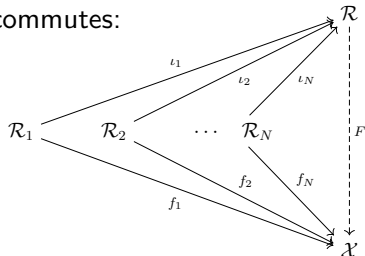


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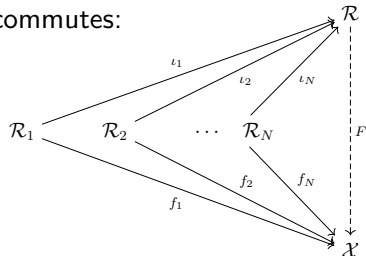


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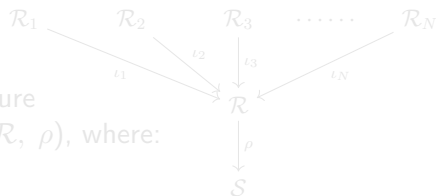
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An N -cell partition \mathcal{S} is a structure

$\mathcal{R} := (\mathcal{R}_1, \iota_1; \mathcal{R}_2, \iota_2; \dots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho)$, where:

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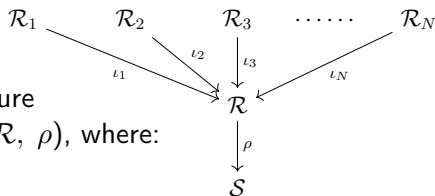
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(20/62)

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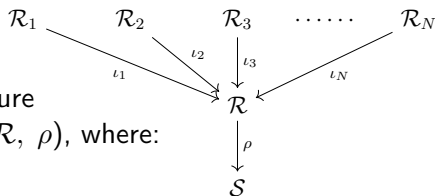
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Figure 1: Structure of $p(\mathcal{R}, \rho)$, where:

```

graph TD
    R1[R1] -- l1 --> R[R]
    R2[R2] -- l2 --> R
    R3[R3] -- l3 --> R
    dots[...] --> R
    RN[RN] -- lN --> R
    R -- rho --> S[S]

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(20/62)

Figure 1: Structure of the input space \mathcal{R} , where:

$$\mathcal{R} := (\mathcal{R}_1, \iota_1; \mathcal{R}_2, \iota_2; \dots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho), \text{ where:}$$

- ▶ \mathcal{R} is another stateplace in \mathcal{S} .
- ▶ $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_N$ are other objects in \mathcal{C} (the *cells* of \mathcal{R});
- ▶ $(\mathcal{R}; \iota_1, \iota_2, \dots, \iota_N)$ is a coproduct of $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_N$;

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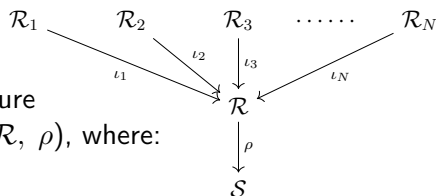
Figure 1: A directed graph representing a network. At the top, there are nodes labeled $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots, \mathcal{R}_N$. Arrows labeled $\iota_1, \iota_2, \iota_3, \dots, \iota_N$ point from these nodes to a central node labeled \mathcal{R} . From node \mathcal{R} , an arrow labeled ρ points down to a node labeled \mathcal{S} .

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Let S be an stateplace in \mathcal{S} .



An N -cell *partition* of S is a structure

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- ▶ $\rho \in \vec{\mathcal{S}}(\mathcal{R}, S)$ is a \mathcal{C} -monomorphism, called the *gluing morphism*.

Example 1. Suppose $\mathcal{C} = \text{Set}, \text{Meas}, \text{Top}, \text{or Diff}$.

Let S be an object in \mathcal{C} . Let $\mathcal{R}_1, \dots, \mathcal{R}_N$ be disjoint subsets of S which are *subobjects* of S in \mathcal{C} (measurable subsets, subspaces, submanifolds, etc.).

Let $\mathcal{R} := \mathcal{R}_1 \sqcup \dots \sqcup \mathcal{R}_N$ (with e.g. disjoint union topology, *not* subspace topology). Let $\iota_n : \mathcal{R}_n \hookrightarrow \mathcal{R}$ and $\rho : \mathcal{R} \hookrightarrow S$ be the inclusion maps.

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(20/62)

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(20/62)

Diagram illustrating the structure of the set \mathcal{R} . It shows a central node \mathcal{R} with arrows pointing to it from nodes $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots, \mathcal{R}_N$. The arrows are labeled $\iota_1, \iota_2, \iota_3, \dots, \iota_N$ respectively. Below \mathcal{R} , there is an arrow labeled ρ pointing to a node \mathcal{S} .

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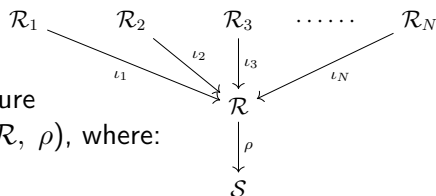
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An *adhesive* from \mathcal{R}' to \mathcal{R} is an ordered pair (η, ν) , where:



- $\eta \in \vec{\mathcal{S}}(\mathcal{R}', \mathcal{R})$ is an \mathcal{S} -morphism such that this diagram commutes:
- $$\begin{array}{ccc} \mathcal{R}' & & \mathcal{S} \\ \eta \downarrow & \searrow \rho' & \\ \mathcal{R} & \xrightarrow{\rho} & \mathcal{S} \end{array}$$
- $\nu : [1 \dots N'] \rightarrow [1 \dots N]$ is a surjection.
 - For any $m \in [1 \dots N']$, if $n = \nu(m)$, then there is a morphism $\eta_m \in \vec{\mathcal{C}}(\mathcal{R}'_m, \mathcal{R}_n)$ such that this diagram commutes:

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Heuristically, (η, ν) describes the way in which the cells of \mathcal{R}' are “glued together” to make the cells of \mathcal{R} . Note that (η, ν) is unique. We say that \mathcal{R}' is a *refinement* of \mathcal{R} , and write $\mathcal{R}' \trianglelefteq \mathcal{R}$.

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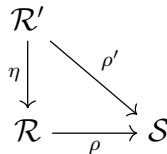
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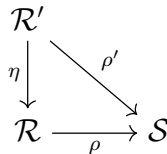


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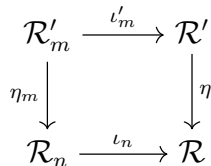
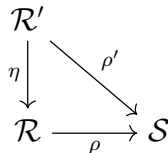


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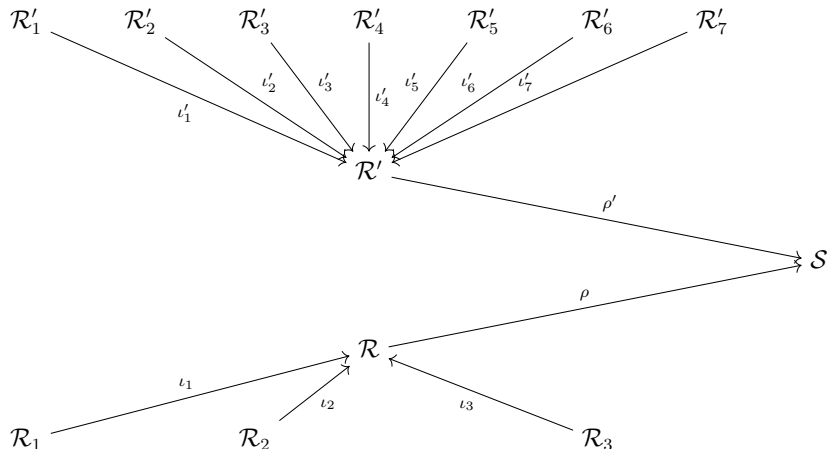
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Partition refinements (an illustrative example)

(22/62)

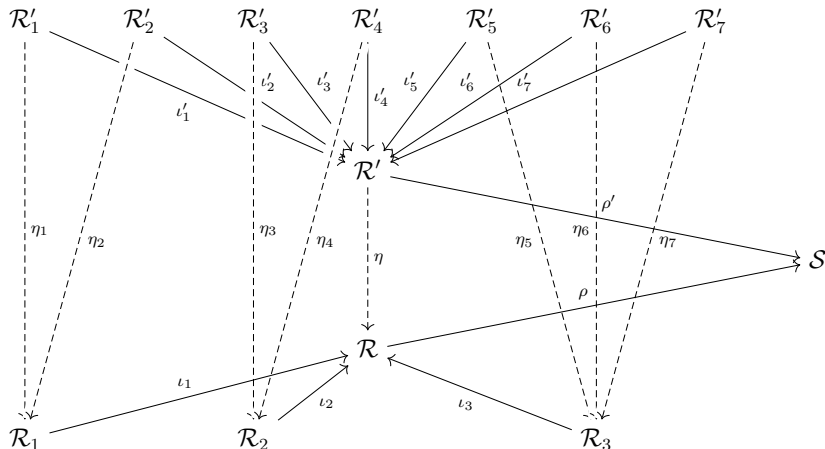
Here are two partitions $\mathcal{R} = (\mathcal{R}_1, \iota_1; \mathcal{R}_2, \iota_2; \mathcal{R}_3, \iota_3; \mathcal{R}, \rho)$ and $\mathcal{R}' = (\mathcal{R}'_1, \iota'_1; \mathcal{R}'_2, \iota'_2; \mathcal{R}'_3, \iota'_3; \mathcal{R}'_4, \iota'_4; \mathcal{R}'_5, \iota'_5; \mathcal{R}'_6, \iota'_6; \mathcal{R}'_7, \iota'_7; \mathcal{R}', \rho')$.



Partition refinements (an illustrative example)

(22/62)

Here is an adhesive (η, ν) making \mathcal{R}' a refinement of \mathcal{R} . In this case, $\nu(1) = \nu(2) = 1$, $\nu(3) = \nu(4) = 2$, and $\nu(5) = \nu(6) = \nu(7) = 3$.



Let \mathcal{R} , \mathcal{R}' and \mathcal{R}'' be three partitions an object \mathcal{S} in the category \mathcal{C} .

If \mathcal{R}' refines \mathcal{R} via the adhesive (η_1, ν_1) , and \mathcal{R}'' refines \mathcal{R}' via the adhesive (η_2, ν_2) , then \mathcal{R}'' refines \mathcal{R} via the adhesive $(\eta_1 \circ \eta_2, \nu_1 \circ \nu_2)$.

Thus, the set of all partitions of \mathcal{S} forms a *category*, $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, where the objects are the partitions and the morphisms are the adhesives.

We will need $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$ to satisfy the *Common Refinement Property*: For any $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, there exists $\mathcal{R} \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$ with $\mathcal{R} \sqsubseteq \mathcal{R}_1$ and $\mathcal{R} \sqsubseteq \mathcal{R}_2$.

Example 2. Suppose $\mathcal{S} = \mathcal{C} = \text{Set, Meas, Top, or Diff}$.

Then for any $\mathcal{S} \in [\mathcal{S}]$, the category $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$ contains the partitions described in Example 1 (with adhesives defined via inclusion maps).

In all cases, $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$ satisfies the Common Refinement Property.

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Then for any $\mathcal{S} \in [\mathcal{S}]$, the category $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$ contains the partitions described in Example 1 (with adhesives defined via inclusion maps).

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Example. Suppose \mathcal{R} is a subobject of \mathcal{S} , and ρ is the inclusion morphism. Let $\mathcal{R}' := \phi^{-1}(\mathcal{R})$ be the ϕ -preimage of \mathcal{R} in \mathcal{S}' . Let ρ' be the inclusion morphism. Then the right-hand diagram commutes.

For this reason, we call $(\mathcal{R}', \rho', \psi)$ a *partial preimage* of the left diagram.

Problem. The left-hand diagram might admit *many* such “partial” preimages. Not all of them count as “true” preimages....

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Problem. The left-hand diagram might admit *many* such “partial” preimages. Not all of them count as “true” preimages....

Idea. A *pullback* is a “maximal” partial preimage....

Let \mathcal{R} , \mathcal{S} and \mathcal{S}' be objects in \mathcal{C} . Consider the diagram on the left below.

$$\begin{array}{ccc} & \mathcal{S}' & \\ & \downarrow \phi & \\ \mathcal{R} & \xrightarrow{\rho} & \mathcal{S} \end{array} \qquad \begin{array}{ccccc} & \mathcal{R}' & \overset{\rho'}{\dashrightarrow} & \mathcal{S}' & \\ & \vdots \psi & & \downarrow \phi & \\ \mathcal{R} & \xrightarrow{\rho} & & \mathcal{S} & \end{array}$$

Question. Does there exist an object \mathcal{R}' , and morphisms $\rho' \in \vec{\mathcal{C}}(\mathcal{R}', \mathcal{S}')$ and $\psi \in \vec{\mathcal{C}}(\mathcal{R}', \mathcal{R})$, such that the right-hand diagram commutes?

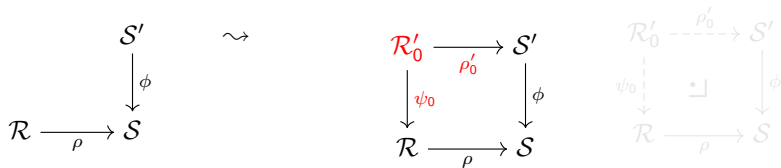
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A **pullback** of the left diagram is partial preimage $(\mathcal{R}'_0, \rho'_0, \psi_0)$ which is a *maximal* in the following sense. Given any other partial preimage $(\mathcal{R}', \rho', \psi)$, there is a unique morphism $\xi \in \vec{\mathcal{C}}(\mathcal{R}', \mathcal{R}'_0)$ making the centre diagram commute:

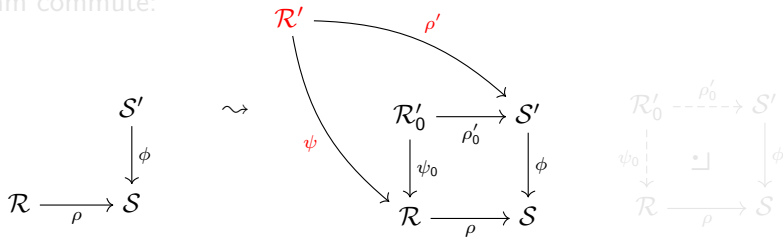


Such a maximal preimage might not exist, but if it does, then it is unique up to isomorphism. Thus, we say $(\mathcal{R}'_0, \rho'_0, \psi_0)$ is “the” pullback of the left diagram. This is indicated by the symbol “ \lrcorner ” in the right diagram.

Examples. Suppose $\mathcal{C} = \text{Set}, \text{Meas}, \text{Top}, \text{or Diff}$,

- If $\mathcal{R} \xrightarrow{\rho} \mathcal{S}$ is a subobject, then $\phi^{-1}(\mathcal{R}) \hookrightarrow \mathcal{S}'$ yields a pullback.
- If \mathcal{S} is a one-point space; then the pullback is just the Cartesian product $\mathcal{R} \times \mathcal{S}'$, with the appropriate product structure.

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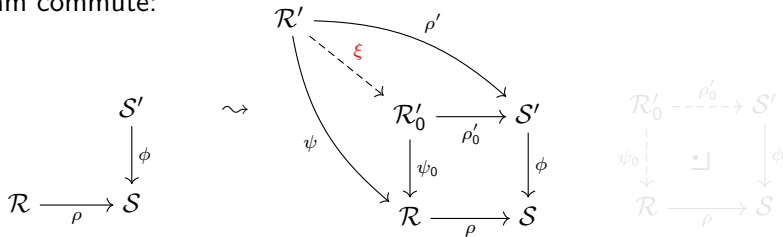
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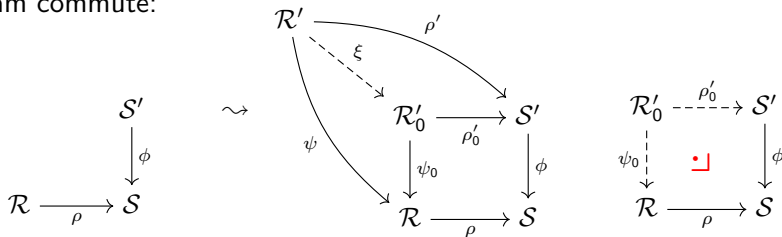
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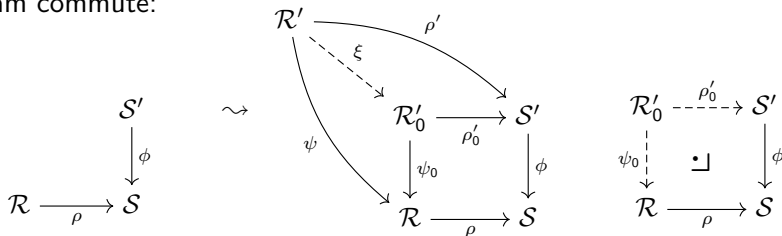
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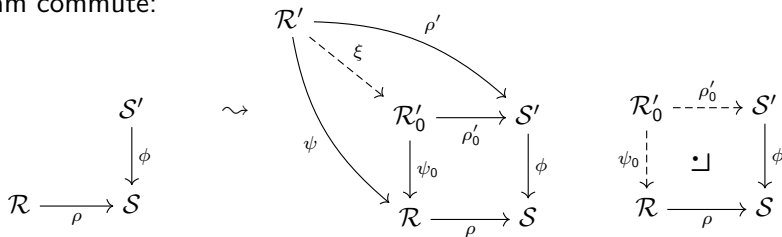


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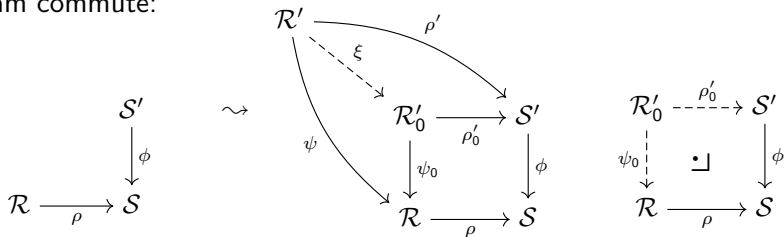
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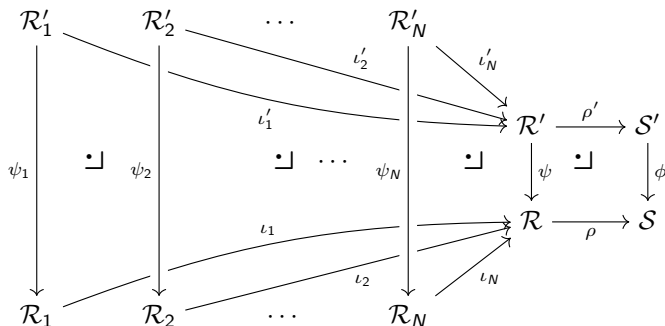
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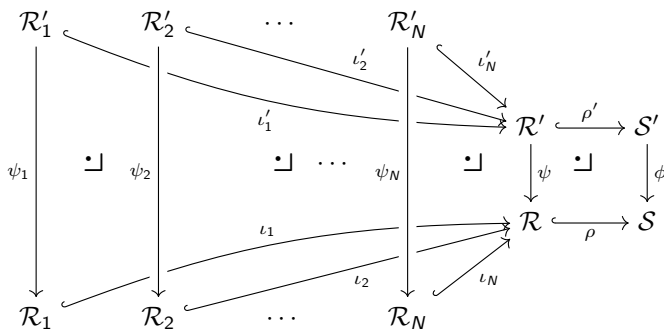
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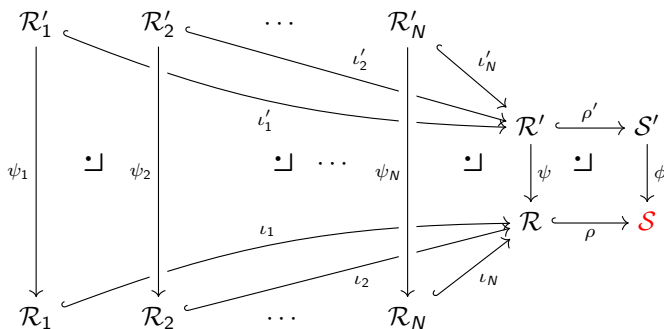




Example 4. Suppose $\mathcal{C} = \text{Set}, \text{Meas}, \text{or Top}$. Let $\mathcal{S} \in \mathcal{C}$.

Let $(\mathcal{R}_1, \iota_1; \dots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, as in Example 1. (i.e. $\mathcal{R}_n \xrightarrow{\iota_n} \mathcal{R}$ and $\mathcal{R} \xrightarrow{\rho} \mathcal{S}$ are inclusion morphisms). Let $\phi : \mathcal{S}' \rightarrow \mathcal{S}$ be a \mathcal{C} -morphism. Define $\mathcal{R}'_n := \phi^{-1}(\mathcal{R}_n) \subseteq \mathcal{S}'$ (for all $n \in [1 \dots N]$) and $\mathcal{R}' := \phi^{-1}(\mathcal{R}) = \mathcal{R}'_1 \sqcup \dots \sqcup \mathcal{R}'_N \subseteq \mathcal{S}'$. Let $\mathcal{R}'_n \xrightarrow{\iota'_n} \mathcal{R}'$ and $\mathcal{R}' \xrightarrow{\rho'} \mathcal{S}'$ be inclusion morphisms. Then $\mathcal{R}' := (\mathcal{R}'_1, \iota'_1; \dots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$ is a ϕ -preimage of \mathcal{R} .

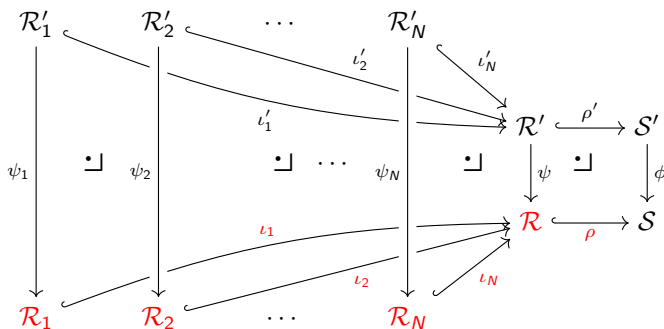
Proof. Let $\psi_n := \phi|_{\mathcal{R}'_n} : \mathcal{R}'_n \rightarrow \mathcal{R}_n$ ($\forall n \in [1 \dots N]$) and $\psi := \phi|_{\mathcal{R}'} : \mathcal{R}' \rightarrow \mathcal{R}$.



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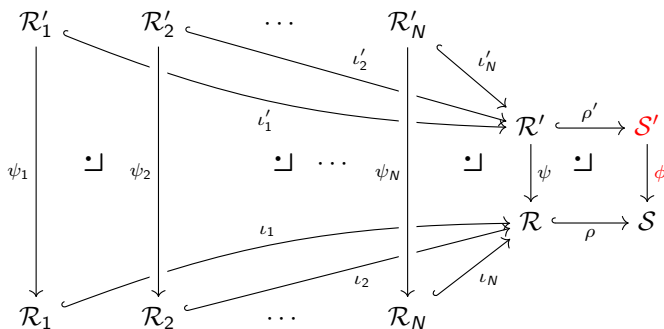
Let $(\mathcal{R}_1, \iota_1; \dots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, as in Example 1. (i.e. $\mathcal{R}_n \xrightarrow{\iota_n} \mathcal{R}$ and $\mathcal{R} \xrightarrow{\rho} \mathcal{S}$ are inclusion morphisms). Let $\phi : \mathcal{S}' \rightarrow \mathcal{S}$ be a \mathcal{C} -morphism.

Define $\mathcal{R}'_n := \phi^{-1}(\mathcal{R}_n) \subseteq \mathcal{S}'$ (for all $n \in [1 \dots N]$) and $\mathcal{R}' := \phi^{-1}(\mathcal{R}) =$

$\mathcal{R}'_1 \sqcup \dots \sqcup \mathcal{R}'_N \subseteq \mathcal{S}'$. Let $\mathcal{R}'_n \xrightarrow{\iota'_n} \mathcal{R}'$ and $\mathcal{R}' \xrightarrow{\rho'} \mathcal{S}'$ be inclusion morphisms.

Then $\mathcal{R}' := (\mathcal{R}'_1, \iota'_1; \dots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$ is a ϕ -preimage of \mathcal{R} .

Proof. Let $\psi_n := \phi|_{\mathcal{R}'_n} : \mathcal{R}'_n \rightarrow \mathcal{R}_n$ ($\forall n \in [1 \dots N]$) and $\psi := \phi|_{\mathcal{R}'} : \mathcal{R}' \rightarrow \mathcal{R}$.



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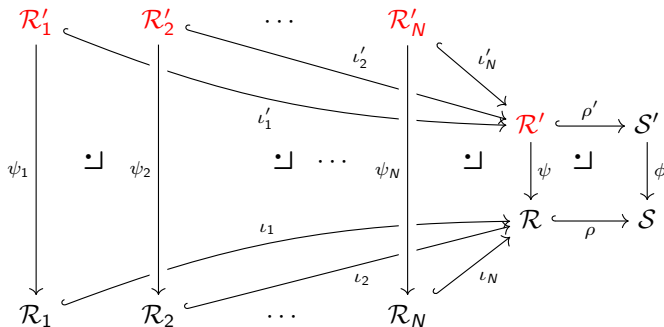
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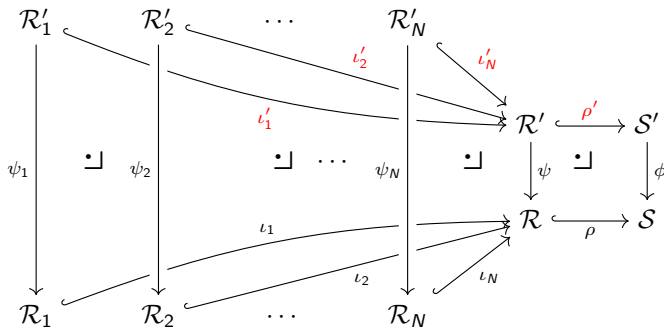


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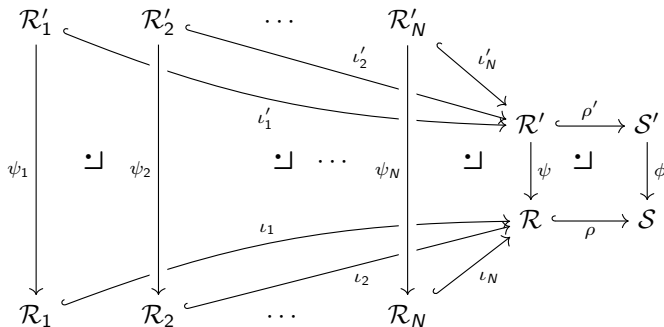
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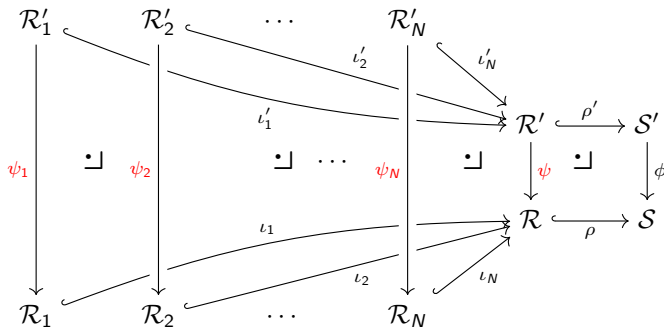
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Let \mathcal{S} be a subcategory of \mathcal{C} . Let $\mathcal{S}_1, \mathcal{S}_2 \in [\mathcal{S}]$, and let $\phi \in \vec{\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_2)$.

We will say that ϕ is \mathcal{S} -measurable if every \mathcal{S} -partition $\mathcal{R} \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S}_2)$ has a ϕ -preimage $\phi^{-1}(\mathcal{R})$ in $\mathfrak{R}_{\mathcal{S}}(\mathcal{S}_1)$.

Let \mathbf{P}_1 and \mathbf{P}_2 be probability structures on $\mathfrak{R}_{\mathcal{S}}(\mathcal{S}_1)$ and $\mathfrak{R}_{\mathcal{S}}(\mathcal{S}_2)$. We say ϕ is *probability-preserving* with respect to \mathbf{P}_1 and \mathbf{P}_2 if ϕ is measurable and, for every $\mathcal{R}_2 \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S}_2)$, if $\mathcal{R}_1 := \phi^{-1}(\mathcal{R}_2)$, then $\mathbf{p}_1^{\mathcal{R}_1} = \mathbf{p}_2^{\mathcal{R}_2}$.

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For any \mathcal{S} -partition $\mathcal{R} \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S}_2)$, define the preimage $\phi^{-1}(\mathcal{R})$ as in Example 4; then $\phi^{-1}(\mathcal{R}) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S}_1)$. Thus, ϕ is \mathcal{S} -measurable.

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Then ϕ is a probability-preserving morphism with respect to \mathbf{P}_1 and \mathbf{P}_2 .

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Part III

Concretization

Let \mathcal{C} be a category. An object B in \mathcal{C} is *null* if $\vec{\mathcal{C}}(\mathcal{A}, B) = \emptyset$ for all $\mathcal{A} \in [\mathcal{C}]$.

Example. The empty set \emptyset is the unique null object in the category \mathbf{Set} .

Let $B, C \in [\mathcal{C}]$ be non-null, and let $\kappa \in \vec{\mathcal{C}}(B, C)$.

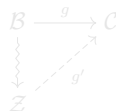
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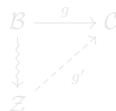
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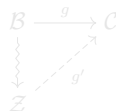
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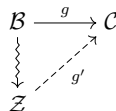
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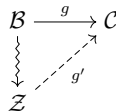
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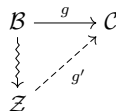
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The concretization functor (informal treatment)

(33/62)

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[skip details]

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(* This is not the case in some other concrete categories.)

But the concretization functor is well-defined even in an abstract category. We will refer to the elements of $\tilde{\mathcal{B}}$ as the *quasi-elements* of \mathcal{B} .

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Part IV

Products, spans,
and
quasipreferences

- ▶ Let \mathcal{X} be an outcome place in $[\mathcal{X}]$.
- ▶ A *span* on \mathcal{X} is a categorical construction which plays the role of a binary relation on \mathcal{X} .
- ▶ Let $[\triangleright]$ be a span on \mathcal{X} . Then we can define a binary relation $\widetilde{\triangleright}$ on $\widetilde{\mathcal{X}}$. For us, $\widetilde{\triangleright}$ will play the role of the *ex post preferences order*.
- ▶ For any state place $\mathcal{S} \in [\mathcal{S}]$, the span $[\triangleright]$ induces a binary relation $\underline{\triangleright}$ on $\vec{\mathcal{C}}(\mathcal{S}, \mathcal{X})$.
 $\underline{\triangleright}$ will play the role of the *statewise dominance order* induced by $\widetilde{\triangleright}$.
- ▶ If $[\triangleright]$ satisfies reasonable conditions, then $\underline{\triangleright}$ and $\widetilde{\triangleright}$ are reflexive and transitive, and $\widetilde{\triangleright}$ is also complete (i.e. it is a preference order on $\widetilde{\mathcal{X}}$).
- ▶ In this case, we say that $[\triangleright]$ is a *quasipreference* on \mathcal{X} .
- ▶ However, to save time, we will skip the details....

- ▶ Let \mathcal{X} be an outcome place in $[\mathcal{X}]$.
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- ▶ Let $[\triangleright]$ be a span on \mathcal{X} . Then we can define a binary relation $\widetilde{\triangleright}$ on $\widetilde{\mathcal{X}}$. For us, $\widetilde{\triangleright}$ will play the role of the *ex post preferences order*.
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Example. In most concrete categories, \mathcal{S} is the Cartesian product $\mathcal{S}_1 \times \mathcal{S}_2$ (equipped with the suitable “product” structure), while π_1 and π_2 are the coordinate projection maps (i.e. $\pi_1(s_1, s_2) = s_1$ and $\pi_2(s_1, s_2) = s_2$).

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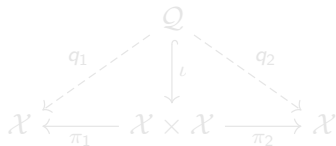
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Prototypical example. Suppose the product object $\mathcal{X} \times \mathcal{X}$ existed in \mathcal{C} . Let $\mathcal{Q} \xrightarrow{\iota} \mathcal{X} \times \mathcal{X}$ be a subobject of $\mathcal{X} \times \mathcal{X}$ (e.g. a *binary relation*). Construct the following commuting diagram:



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As this example shows, spans generalize binary relations.

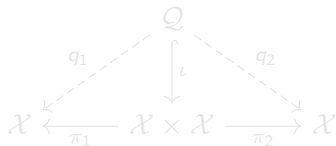
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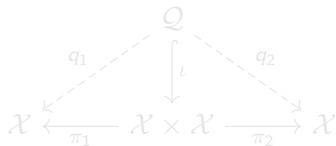
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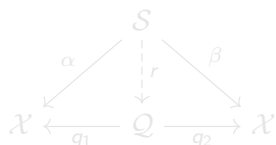
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Let $\alpha, \beta \in \vec{\mathcal{C}}(\mathcal{S}, \mathcal{X})$. Define $\alpha \triangleright \beta$ if there is a morphism $r \in \vec{\mathcal{C}}(\mathcal{S}, \mathcal{Q})$

which makes this diagram commute:



Example. Suppose $\mathcal{C} = \text{Set}$, and $\langle \triangleright \rangle$ represents a binary relation \triangleright on \mathcal{X} .

If $\alpha, \beta : \mathcal{S} \rightarrow \mathcal{X}$ are functions, then $(\alpha \triangleright \beta) \Leftrightarrow (\alpha(s) \triangleright \beta(s) \text{ for all } s \in \mathcal{S})$.

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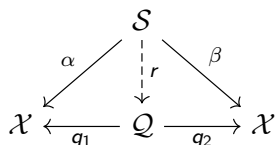
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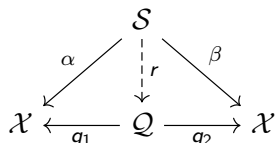
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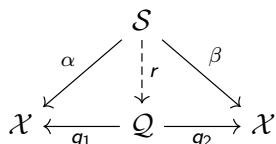
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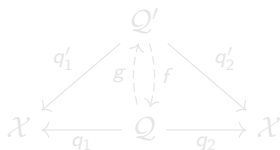
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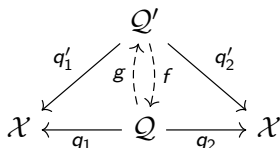
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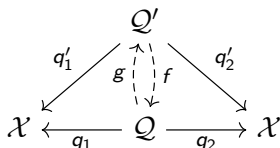
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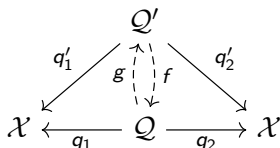
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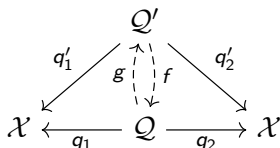
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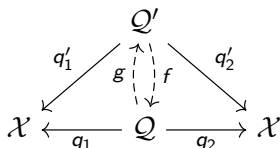
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$$(x \succeq y) \iff (u(x) \geq u(y)), \quad \text{for all } x, y \in \tilde{\mathcal{X}}.$$

Example. If $\mathcal{C} = \text{Set}$, then every real-valued function on $\tilde{\mathcal{X}}$ is a compatible utility function. But in other categories, this is not necessarily the case.

For example, let $\mathcal{C} = \text{Cpct}$, the category of compact spaces and continuous maps, and let $\mathcal{X} \in [\text{Cpct}]$.

Then a function $u : \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ is a Cpct -compatible utility function if and only if it is an increasing transform of a continuous, \mathbb{R} -valued function on \mathcal{X} .

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Example. If $\mathcal{C} = \text{Set}$, then every real-valued function on $\tilde{\mathcal{X}}$ is a compatible utility function. But in other categories, this is not necessarily the case.

For example, let $\mathcal{C} = \text{Cpct}$, the category of compact spaces and continuous maps, and let $\mathcal{X} \in [\text{Cpct}]$.

Then a function $u : \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ is a Cpct-compatible utility function if and only if it is an increasing transform of a continuous, \mathbb{R} -valued function on \mathcal{X} .

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Part V

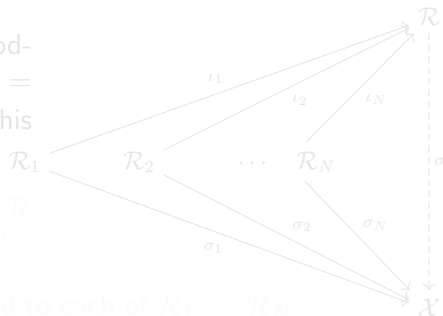
From
simple morphisms
to
SEU representations

Let $\mathcal{R} = (\mathcal{R}; \iota_1, \dots, \iota_N)$ be a coproduct of some objects $\mathcal{R}_1, \dots, \mathcal{R}_N \in [\mathcal{C}]$.

Let $\mathcal{X} \in [\mathcal{C}]$ be another object.

For all $n \in [1 \dots N]$, let $\sigma_n \in \mathcal{K}(\mathcal{R}_n, \mathcal{X})$ be a quasiconstant morphism.
Let $x_n \in \tilde{\mathcal{X}}$ be its \sim -equivalence class (the “value” of σ_n).

By the defining property of coproducts, there is a unique morphism $\sigma = [\sigma_1 | \dots | \sigma_N] \in \vec{\mathcal{C}}(\mathcal{R}, \mathcal{X})$ such that this diagram commutes:



Say σ is a *simple morphism* from \mathcal{R} into \mathcal{X} , and write “ $\sigma = [x_1 | \dots | x_N]$ ”.

Idea: σ is “constant” when restricted to each of $\mathcal{R}_1, \dots, \mathcal{R}_N$.

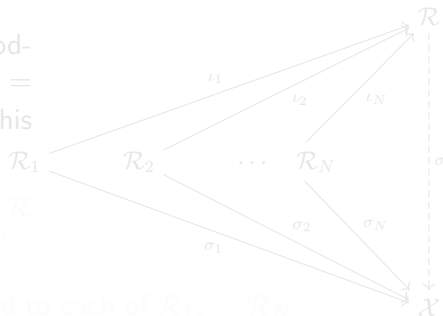
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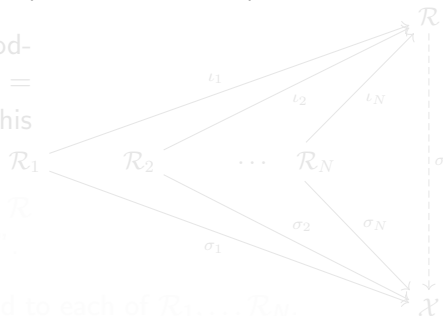
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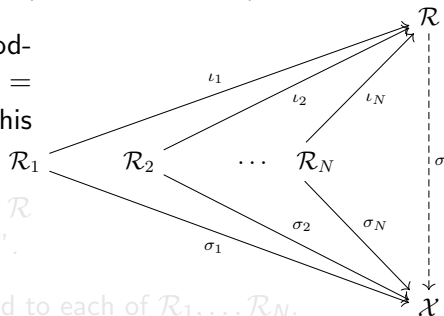
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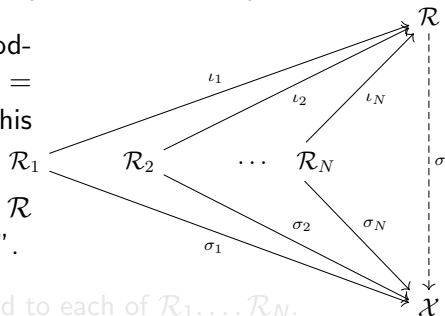
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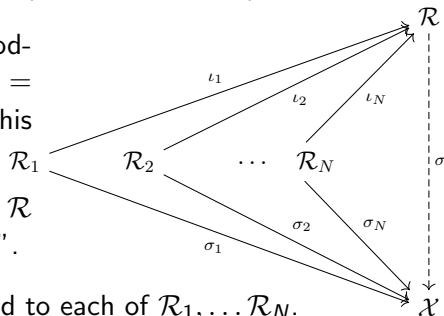
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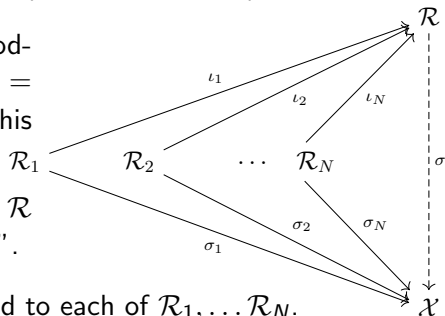
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Expected utility for any morphism (informal)

[skip summary] (45/62)

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We define $\underline{\mathbb{E}}_{\mathbf{P}}^u[\alpha]$ and $\overline{\mathbb{E}}_{\mathbf{P}}^u[\alpha]$, the *lower* and *upper expected utilities* of α with respect to u and \mathbf{P} , by approximating α “from below” and “from above” (in terms of \triangleright) by simple morphisms on partitions of \mathcal{S} .

If $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$ satisfies the Common Refinement Property, then $\overline{\mathbb{E}}_{\mathbf{P}}^u$ and $\underline{\mathbb{E}}_{\mathbf{P}}^u$ have most of the properties you would expect from a notion of “expected utility”.

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We define $\underline{\mathbb{E}}_{\mathbf{P}}^u[\alpha]$ and $\overline{\mathbb{E}}_{\mathbf{P}}^u[\alpha]$, the *lower* and *upper expected utilities* of α with respect to u and \mathbf{P} , by approximating α “from below” and “from above” (in terms of $[\triangleright]$) by simple morphisms on partitions of \mathcal{S} .

If $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$ satisfies the **Common Refinement Property**, then $\overline{\mathbb{E}}_{\mathbf{P}}^u$ and $\underline{\mathbb{E}}_{\mathbf{P}}^u$ have most of the properties you would expect from a notion of “expected utility”.

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Expected utility for any morphism (informal)

[skip summary] (45/62)

We define the *expected utility* of a simple morphism σ , with respect to u and \mathbf{P} :

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Now, let $u : \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ be a \mathcal{C} -compatible utility function, representing a quasipreference $[\succeq_u]$ on \mathcal{X} .

For any $\alpha \in \vec{\mathcal{C}}(\mathcal{S}, \mathcal{X})$, we define

$$\underline{\Sigma}_{\mathcal{S}}^u(\alpha) := \{(\sigma, \rho) \in \Sigma_{\mathcal{S}}(\mathcal{S}, \mathcal{X}) ; \sigma \preceq_u \alpha \circ \rho\}$$

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Interpretation. These are a lower estimate and an **upper estimate** of the expected utility of α with respect to u and \mathbf{P} .

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Let $(\mathcal{S}, \mathcal{X})$ be a **decision context** in a category \mathcal{C} .

For every $S \in [\mathcal{S}]$, let \mathbf{P}_S be a probability structure on $\mathfrak{R}_S(S)$.

For every $\mathcal{X} \in [\mathcal{X}]$, let $u_{\mathcal{X}} : \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ be a \mathcal{C} -compatible utility function.

The structure $[(\mathbf{P}_S)_{S \in [\mathcal{S}]}, (u_{\mathcal{X}})_{\mathcal{X} \in [\mathcal{X}]}]$ is an *SEU structure* on $(\mathcal{S}, \mathcal{X})$ if:

- (PP) For all $S_1, S_2 \in [\mathcal{S}]$, every measurable morphism in $\vec{\mathcal{S}}(S_1, S_2)$ is probability-preserving with respect to \mathbf{P}_{S_1} and \mathbf{P}_{S_2} .
- (UP) For all $\mathcal{X}_1, \mathcal{X}_2 \in [\mathcal{X}]$, and every $\phi \in \vec{\mathcal{X}}(\mathcal{X}_1, \mathcal{X}_2)$, the composition $u_{\mathcal{X}_2} \circ \tilde{\phi}$ is a *positive affine transformation* of $u_{\mathcal{X}_1}$ —that is, there exist $A > 0$ and $B \in \mathbb{R}$ such that $u_{\mathcal{X}_2}[\tilde{\phi}(x)] = A u_{\mathcal{X}_1}(x) + B$ for all $x \in \tilde{\mathcal{X}}_1$.

This SEU structure *represents* a Savage structure $\mathfrak{S} = (\succeq_{\mathcal{X}}^S)_{\mathcal{X} \in [\mathcal{X}]}$ if, for every $S \in [\mathcal{S}]$, and every $\mathcal{X} \in [\mathcal{X}]$, and all $\alpha, \beta \in \vec{\mathcal{C}}(S, \mathcal{X})$, we have

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For every $\mathcal{S} \in [\mathcal{S}]$, let $\mathbf{P}_{\mathcal{S}}$ be a **probability structure** on $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$.

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The structure $[(\mathbf{P}_{\mathcal{S}})_{\mathcal{S} \in [\mathcal{S}]}, (u_{\mathcal{X}})_{\mathcal{X} \in [\mathcal{X}]}]$ is an *SEU structure* on $(\mathcal{S}, \mathcal{X})$ if:

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Part VI

Formal statement of axioms and
main result

Structural conditions (S1)-(S3)

(50/62)

The decision context $(\mathcal{S}, \mathcal{X})$ must satisfy three structural conditions:

(S1) Every pair of state places in $[\mathcal{S}]$ have a product in the category \mathcal{S} .

(S2) Every pair of outcome places in $[\mathcal{X}]$ have a coproduct in \mathcal{X} .

(S3) Consider a pullback diagram in the category \mathcal{C} :

$$\begin{array}{ccc} \mathcal{S}_\top & \overset{\tau}{\dashrightarrow} & \mathcal{S}_\top \\ \lambda \downarrow & \lrcorner & \downarrow \rho \\ \mathcal{S}_\perp & \xrightarrow{\beta} & \mathcal{S}_\perp \end{array}$$

If \mathcal{S}_\top , \mathcal{S}_\perp , and \mathcal{S}_\perp are all in $[\mathcal{S}]$, and ρ and β are \mathcal{S} -morphisms, then \mathcal{S}_\top is also in $[\mathcal{S}]$, and τ and λ are also \mathcal{S} -morphisms.

Interpretation: Given any two “random variables” (e.g. any two state places \mathcal{S}_1 and \mathcal{S}_2), (S1) says we can *couple* them into a single “random variable” (namely $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$) such that \mathcal{S}_1 and \mathcal{S}_2 are “marginals” of \mathcal{S} . (\mathcal{S}_1 and \mathcal{S}_2 might not be *independent* random variables in this coupling.)

Given any two menus \mathcal{X}_1 and \mathcal{X}_2 of outcomes, (S2) says we can combine them into a single menu $(\mathcal{X}_1 \amalg \mathcal{X}_2)$. The agent’s preferences on this larger menu must agree with her preferences on the two submenus.

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Interpretation: Given any two “random variables” (e.g. any two state places \mathcal{S}_1 and \mathcal{S}_2), (S1) says we can *couple* them into a single “random variable” (namely $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$) such that \mathcal{S}_1 and \mathcal{S}_2 are “marginals” of \mathcal{S} . (\mathcal{S}_1 and \mathcal{S}_2 might not be *independent* random variables in this coupling.)

Given any two menus \mathcal{X}_1 and \mathcal{X}_2 of outcomes, (S2) says we can combine them into a single menu $(\mathcal{X}_1 \amalg \mathcal{X}_2)$. The agent’s preferences on this larger menu must agree with her preferences on the two submenus.

The decision context $(\mathcal{S}, \mathcal{X})$ must satisfy three structural conditions:

(S1) Every pair of state places in $[\mathcal{S}]$ have a product in the category \mathcal{S} .

(S2) Every pair of outcome places in $[\mathcal{X}]$ have a coproduct in \mathcal{X} .

(S3) Consider a pullback diagram in the category \mathcal{C} :

$$\begin{array}{ccc} \mathcal{S}_\top & \overset{\tau}{\dashrightarrow} & \mathcal{S}_\top \\ \lambda \downarrow & \lrcorner & \downarrow \rho \\ \mathcal{S}_\perp & \xrightarrow{\beta} & \mathcal{S}_\perp \end{array}$$

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Suppose \mathcal{C} is pullback-complete. Then (S3) is equivalent to:

(S3') For any $\mathcal{S}_\top, \mathcal{S}_\perp, \mathcal{S}_\sqcup \in [\mathcal{S}]$, and any $\beta \in \vec{\mathcal{S}}(\mathcal{S}_\perp, \mathcal{S}_\sqcup)$ and $\rho \in \vec{\mathcal{S}}(\mathcal{S}_\top, \mathcal{S}_\sqcup)$, there exists a fourth state place \mathcal{S}_\top , along with \mathcal{S} -morphisms τ and λ yielding the following pullback diagram in the category \mathcal{C} :

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This generalizes (S1). Suppose there are two sources of uncertainty, \mathcal{S}_\perp and \mathcal{S}_\top . The morphisms ρ and β are “measurements” of \mathcal{S}_\perp and \mathcal{S}_\top , taking values in \mathcal{S}_\sqcup . Suppose that \mathcal{S}_\perp and \mathcal{S}_\top are “correlated” in such a way that ρ and β always produce the same measurement value. Is there a way to explain this correlation? (S3') says “yes”: there a single, common, underlying source of uncertainty \mathcal{S}_\top , such that \mathcal{S}_\perp and \mathcal{S}_\top appear as “factors” of \mathcal{S}_\top (via the morphisms τ and λ).

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Let $\mathcal{R} = (\mathcal{R}; \iota_1, \dots, \iota_N)$ be a coproduct of objects $\mathcal{R}_1, \dots, \mathcal{R}_N \in [\mathcal{C}]$. Let

$$\Sigma(\mathcal{R}, \mathcal{X}) \quad := \quad \{\text{all simple morphisms from } \mathcal{R} \text{ to } \mathcal{X}\}.$$

For any simple morphism $\sigma \in \Sigma(\mathcal{R}, \mathcal{X})$, there exist quasielements $x_1, \dots, x_N \in \tilde{\mathcal{X}}$ such that $\sigma = [x_1 | \dots | x_N]$.

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A Savage structure $\mathfrak{S} = (\succeq_{\mathcal{X}}^{\mathcal{S}})^{\mathcal{S} \in [\mathcal{S}]}$ is *solvable* if, for any $\mathcal{R}_1, \dots, \mathcal{R}_N$ in $[\mathcal{C}]$ with a coproduct $\mathcal{R} = (\mathcal{R}; \iota_1, \dots, \iota_N)$ such that $\mathcal{R} \in [\mathcal{S}]$, any $\mathcal{X} \in [\mathcal{X}]$, any simple acts $\sigma, \tau \in \Sigma(\mathcal{R}, \mathcal{X})$, any $n \in [1 \dots N]$, and any $x, z \in \tilde{\mathcal{X}}$, if $(x_n | \sigma) \succ_{\mathcal{X}}^{\mathcal{R}} \tau \succ_{\mathcal{X}}^{\mathcal{R}} (z_n | \sigma)$, then there is $y \in \tilde{\mathcal{X}}$ with $(y_n | \sigma) \approx_{\mathcal{X}}^{\mathcal{R}} \tau$.

This is a standard condition in decision theory. It says that we can always find a “suitable compromise” y between any two elements x and z in $\tilde{\mathcal{X}}$.

In other words, $\tilde{\mathcal{X}}$ has “no gaps”.

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The Savage structure $\mathfrak{S} = (\succeq_{\mathcal{X}}^{\mathcal{S}})_{\mathcal{S} \in [\mathcal{S}]}$ must satisfy five axioms.....

Recall: $\mathcal{K}(\mathcal{S}, \mathcal{X})$ is the set of quasiconstant morphisms from \mathcal{S} to \mathcal{X} .
Heuristically, these represent “perfectly predictable” (i.e. “riskless”) acts.

Notation: For any $\kappa \in \mathcal{K}(\mathcal{S}, \mathcal{X})$, let $\bar{\kappa} \in \tilde{\mathcal{X}}$ denote its \sim -equivalence class.

For any $\mathcal{X} \in [\mathcal{X}]$, we require a quasipreference $[\succeq_{\mathcal{X}}^{\text{dom}}]$ on \mathcal{X} satisfying:

- (A1) (*Ex post preferences*) Let $\succeq_{\mathcal{X}}^{\text{xp}}$ be the preference order that $[\succeq_{\mathcal{X}}^{\text{dom}}]$ induces on $\tilde{\mathcal{X}}$. Then $\succeq_{\mathcal{X}}^{\text{xp}}$ is nontrivial, and for any $\mathcal{S} \in [\mathcal{S}]$ and any $\kappa_1, \kappa_2 \in \mathcal{K}(\mathcal{S}, \mathcal{X})$, we have $\kappa_1 \succeq_{\mathcal{X}}^{\mathcal{S}} \kappa_2$ if and only if $\bar{\kappa}_1 \succeq_{\mathcal{X}}^{\text{xp}} \bar{\kappa}_2$.
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Interpretation: $\succeq_{\mathcal{X}}^{\text{xp}}$ is the agent’s “ex post preference relation” on $\tilde{\mathcal{X}}$.

(A1) says that $\succeq_{\mathcal{X}}^{\text{xp}}$ governs the agent’s preferences over “riskless” acts.

If $\alpha \succeq_{\mathcal{X}}^{\text{dom}} \beta$, then α delivers a better *ex post* outcome than β in all circumstances. Then (A2) says the agent should prefer α over β *ex ante*.

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Recall. If $\mathcal{R} = (\mathcal{R}_1, \iota_1; \dots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho)$ is a partition of \mathcal{S} , then

$$\Sigma(\mathcal{R}, \mathcal{X}) := \{\text{all simple morphisms from } \mathcal{R} \text{ to } \mathcal{X}\}.$$

(A3) (*Simple density*) For any $\mathcal{S} \in \mathcal{S}$ and $\mathcal{X} \in \mathcal{X}$, and any $\alpha, \beta \in \vec{\mathcal{C}}(\mathcal{S}, \mathcal{X})$, if $\alpha \succ_{\mathcal{X}}^{\mathcal{S}} \beta$, then there exists a partition $\mathcal{R} = (\mathcal{R}_1, \iota_1; \dots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, and two simple acts $\alpha', \beta' \in \Sigma(\mathcal{R}, \mathcal{X})$ such that $\alpha \circ \rho \succeq_{\mathcal{X}}^{\text{dom}} \alpha' \succ_{\mathcal{X}}^{\mathcal{R}} \beta' \succeq_{\mathcal{X}}^{\text{dom}} \beta \circ \rho$.

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Axiom (A4): Tradeoff consistency

(55/62)

Let $\mathcal{R} = (\mathcal{R}; \iota_1, \dots, \iota_N)$ be a coproduct of some objects $\mathcal{R}_1, \dots, \mathcal{R}_N \in [\mathcal{C}]$ and suppose $\mathcal{R} \in \mathcal{S}$. Let $\mathcal{X} \in \mathcal{X}$.

For any $w, x, y, z \in \tilde{\mathcal{X}}$, write $(w \rightsquigarrow x) \cong (y \rightsquigarrow z)$ if there exists $\sigma, \tau \in \Sigma(\mathcal{R}, \mathcal{X})$ and $n \in [1 \dots N]$ such that $(w_n | \sigma) \approx_{\mathcal{R}}^{\mathcal{R}_n} (x_n | \tau)$ and $(y_n | \sigma) \approx_{\mathcal{R}}^{\mathcal{R}_n} (z_n | \tau)$.

Idea: The gain in changing w to x on \mathcal{R}_n is exactly equal to the gain in changing y to z on \mathcal{R}_n (because both are exactly cancelled by the loss of changing σ to τ on the complement of \mathcal{R}_n).

Thus, the “value difference” between w and x should be the same as the “value difference” between y and z .

Example: If \mathcal{R} and \mathcal{X} were sets, and $\succeq_{\mathcal{X}}^{\mathcal{R}}$ had an SEU representation with utility function $u : \mathcal{X} \rightarrow \mathbb{R}$, then

$(w \rightsquigarrow x) \cong (y \rightsquigarrow z)$ if and only if $u(w) - u(x) = u(y) - u(z)$.

The next axiom is due to Köbberling and Wakker (2003).

(A4) (Tradeoff Consistency) Let $(\mathcal{R}; \iota_1, \dots, \iota_N)$ be a coproduct of $\mathcal{R}_1, \dots, \mathcal{R}_N$, with $\mathcal{R} \in \mathcal{S}$. For any $\mathcal{X} \in \mathcal{X}$, and any $w, w', x, y, z \in \tilde{\mathcal{X}}$, if $(w \rightsquigarrow x) \cong (y \rightsquigarrow z)$ and $(w' \rightsquigarrow x) \cong (y \rightsquigarrow z)$, then $w \approx_{\mathcal{X}}^{\text{xp}} w'$.

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The sequence $(x_i)_{i=1}^{\infty}$ is an *infinite standard sequence* if $(x_i \rightsquigarrow x_{i+1}) \cong (x_j \rightsquigarrow x_{j+1})$ for all $i, j \in \mathbb{N}$.

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In this case, the utility-difference between x_i and x_{i+1} is effectively “infinitesimal” relative to the utility-difference between x_* and x^* .

Our last axiom is a standard condition in decision theory, which rules out such “infinitesimal” utility differences....

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SEU characterization theorem. (Formal statement)

Let \mathcal{C} be any biconnected category (e.g. Set, Meas, Top, Diff, etc.)

Let $(\mathcal{S}, \mathcal{X})$ be a decision structure satisfying structural conditions (S1)-(S3).

Let \mathfrak{S} be a solvable Savage structure on $(\mathcal{S}, \mathcal{X})$. Then:

- ▶ \mathfrak{S} has an SEU representation if and only if it satisfies (A1)-(A5).

Let $(\mathbf{P}_S)_{S \in \mathcal{S}}$ and $(u_X)_{X \in \mathcal{X}}$ be this SEU representation. Then:

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Prologue

What is Decision Theory?

Savage's Theorem

Desiderata I

Desiderata II

Outline

Part I. Savage structures

Definition: Category

Concrete categories

Decision Contexts

Savage structures

Definition

Exampel

Informal statement of axioms I

Informal statement of axioms II

Informal statement of main result I

Informal statement of main result II

Part II: Partitions and probability

Isomorphisms and monomorphisms

Coproducts

Partitions

Partition refinements

An illustrative example

Partition categories and common refinement

Probability structures

Partition preimages and measurability in a nutshell

Preimages and pullbacks

Partition preimages

Definition

Example

Measurable and probability-preserving morphisms

Part III. Concretization

Quasiconstant morphisms

The concretization functor....

Informal treatment

Formal treatment

Part IV. Products, spans, and quasipreferences

Executive summary

Products

- Spans
- Quasirelations and quasipreferences
- Compatible utility functions

Part V. From simple morphisms to SEU representations

- Simple morphisms
- Expected utility
 -for simple morphisms
 -for not-so-simple morphisms (informal)
- Virtual simple morphisms
- Expected utility for arbitrary morphisms (formal)
- Subjective expected utility representations

Part VI. Formal statement of axioms and main result

- Structural conditions (S1)-(S3)
- Solvability
- Axioms (A1) and (A2)
- Axiom (A3): Simple density
- Axiom (A4): Tradeoff consistency
- Axiom (A5): Archimedeanism
- SEU characterization theorem (formal statement)

Thank you