Categorical Decision Theory

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What is Decision Theory?

Individuals and societies must often make choices under uncertainty.
How should an agent decide when faced with such uncertainty?

This is the subject of a branch of economics called *Decision Theory*.

The foundations of decision theory were laid by Leonard J. Savage in 1954.

Savage modelled the decision problem as follows.

There is an (infinite) set $\mathcal{S}$ of possible "states of the world".

The true state is unknown.

$\mathcal{S}$ represents all information which is unknown to the agent.

There is a set $\mathcal{X}$ of possible "outcomes" (e.g. consumption bundles).
These are the things the agent ultimately cares about.

Each alternative defines a function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$, called an *act*.

If the agent chooses the act α, and the true state of the world turns out to be s, then she will obtain the outcome $\alpha(s)$.

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- *a "cardinal utility" function $U : \mathcal{X} \longrightarrow \mathbb{R}$, and*
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which provide a **subjective expected utility** (SEU) **representation** *for $\succeq$.*
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$$\left(\alpha \succeq \beta\right) \iff \left(\int_{\mathcal{S}} U[\alpha(s)]\, \mathrm{d}P(s) \geq \int_{\mathcal{S}} U[\beta(s)]\, \mathrm{d}P(s)\right).$$

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Desiderata

There are three ways we could improve on Savage's framework.

1. Scope. Savage assumed that $\mathcal{S}$ and $\mathcal{X}$ are arbitrary sets, and acts are arbitrary functions from $\mathcal{S}$ to $\mathcal{X}$. (This can be extended to measurable spaces and measurable functions.)

But what if $\mathcal{S}$ and $\mathcal{X}$ are topological spaces, and acts must be continuous?

What if $\mathcal{S}$ and $\mathcal{X}$ are differentiable manifolds, and acts must be differentiable functions?

Want: a single theory which works in *all* of these environments (rather than multiple independent theories).

2. Holism. At different times, the same agent may face different sources of uncertainty (e.g. horse races, financial markets, weather, traffic) and different possible sets of outcomes (e.g. financial gains or losses, social status, physical (dis)comfort, physical danger), in different combinations.

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1. **Scope.**
2. **Holism.**
3. **Endogenous/implicit states and outcomes.** Savage assumed that the agent could explicitly specify all possible "states of nature" and all possible "outcomes", and could conceptualize each "act" as a function mapping states to outcomes.

This may be unrealistically demanding.

Also, even if people *do* represent decision problems this way, different people may adopt different representations of the same decision problem...

Want: A framework which does *not* require an explicit specification of the states and outcomes in advance.

Ideally, the statespace and outcome space should emerge "endogenously" from a description of the agent's preferences over acts.

Idea. Reformulate classical decision theory using the tools of category theory, and obtain a theorem which satisfies these three desiderata.

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1. **Scope.**
2. **Holism.**
3. **Endogenous/implicit states and outcomes.** Savage assumed that the agent could explicitly specify all possible "states of nature" and all possible "outcomes", and could conceptualize each "act" as a function mapping states to outcomes.

This may be unrealistically demanding.

Also, even if people *do* represent decision problems this way, different people may adopt different representations of the same decision problem...

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Plan:

Part I. Savage structures; informal statement of main result.

Part II. Partitions and probability.

Part III. Concretization.

Part IV. Products, spans and quasipreferences.

Part V. Simple morphisms and SEU representations.

Part VI. Formal statement of axioms and main result.

Part I.

Savage structures

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However, not all categories are concrete. We will use the term *abstract category* to refer to a category which may or may not be concrete.

A *category* is a mathematical structure $\mathcal{C}$ with three parts:

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- For any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in [\mathcal{C}]$, a *composition* operation $\circ$, such that, for any morphisms $\phi \in \overrightarrow{\mathcal{C}}(\mathcal{A}, \mathcal{B})$ and $\psi \in \overrightarrow{\mathcal{C}}(\mathcal{B}, \mathcal{C})$, we have $\psi \circ \phi \in \overrightarrow{\mathcal{C}}(\mathcal{A}, \mathcal{C})$.

The composition operation has two key algebraic properties:

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$\mathcal{C}$ is a *concrete category* if the objects in $[\mathcal{C}]$ are *sets* (usually with some "structure"), the morphisms in $\overrightarrow{\mathcal{C}}(\mathcal{A}, \mathcal{B})$ are *functions* from $\mathcal{A}$ to the set $\mathcal{B}$ (which "preserve" this structure), and $\circ$ is function composition.

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The collection $\mathfrak{S} := \{\succeq_{\mathcal{X}}^{\mathcal{S}};\ \mathcal{S} \in [\mathcal{S}] \text{ and } \mathcal{X} \in [\mathcal{X}]\}$ is a *Savage structure* if:

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Informal statement of axioms

Very informally, we require $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$ to satisfy three structural conditions:

(S1) Any state places $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\boldsymbol{\mathcal{S}}$ have a product $\mathcal{S}_1 \times \mathcal{S}_2$ in $\boldsymbol{\mathcal{S}}$.

Idea. $\mathcal{S}_1 \times \mathcal{S}_2$ encodes a *coupling* of the random variables represented by $\mathcal{S}_1$ and $\mathcal{S}_2$.

(S2) Any outcome places $\mathcal{X}_1$ and $\mathcal{X}_2$ in $\boldsymbol{\mathcal{X}}$ have a coproduct (roughly: a disjoint union) $\mathcal{X}_1 \amalg \mathcal{X}_2$ in $\boldsymbol{\mathcal{X}}$.

(S3) Given any three stateplaces $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ in $[\boldsymbol{\mathcal{S}}]$, and any $\boldsymbol{\mathcal{S}}$-morphisms ϕ_1 and ϕ_2 as shown in the left-hand diagram below, there exists a fourth state place $\mathcal{S}_0$ in $[\boldsymbol{\mathcal{S}}]$, and $\boldsymbol{\mathcal{S}}$-morphisms ψ_1 and ψ_2 such that the right-hand diagram below commutes. Furthermore, $\mathcal{S}_0$ is the "maximal" state place with this property (i.e. it is a pullback in $\boldsymbol{\mathcal{C}}$).

$$\begin{tikzcd} & \mathcal{S}_1 \arrow[d, "\phi_1"] \\ \mathcal{S}_2 \arrow[r, "\phi_2"'] & \mathcal{S} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{S}_0 \arrow[r, dashed, "\psi_1"] \arrow[d, dashed, "\psi_2"'] & \mathcal{S}_1 \arrow[d, "\phi_1"] \\ \mathcal{S}_2 \arrow[r, "\phi_2"'] & \mathcal{S} \end{tikzcd}$$

Idea. $\mathcal{S}_0$ represents a *coupling* of the random variables represented by $\mathcal{S}_1$ and $\mathcal{S}_2$, which are correlated through a "common observable" in $\mathcal{S}$.

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Idea. $\mathcal{S}_0$ represents a *coupling* of the random variables represented by $\mathcal{S}_1$ and $\mathcal{S}_2$, which are correlated through a "common observable" in $\mathcal{S}$.

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Very informally, we require $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$ to satisfy three structural conditions:

(S1) Any state places $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\boldsymbol{\mathcal{S}}$ have a product $\mathcal{S}_1 \times \mathcal{S}_2$ in $\boldsymbol{\mathcal{S}}$.

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(S2) Any outcome places $\mathcal{X}_1$ and $\mathcal{X}_2$ in $\boldsymbol{\mathcal{X}}$ have a coproduct (roughly: a disjoint union) $\mathcal{X}_1 \sqcup \mathcal{X}_2$ in $\boldsymbol{\mathcal{X}}$.

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(S1) Any two state places in $\boldsymbol{\mathcal{S}}$ have a product in $\boldsymbol{\mathcal{S}}$.
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Theorem. (Informal statement) *Let $\mathcal{C}$ be any biconnected category. Let $(\mathcal{S}, \mathcal{X})$ be a decision context satisfying structural conditions* (S1)-(S3). *Let $\mathfrak{S}$ be a solvable Savage structure on $(\mathcal{S}, \mathcal{X})$. Then:*
$\mathfrak{S}$ has a "subjective expected utility representation" if and only if it satisfies axioms (A1)-(A5).
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Many terms in the axioms and theorem appear in quotation marks, because they have not yet been formally defined.

The meaning of these terms is fairly obvious in a concrete category like Set or Meas.... but it is not clear what they even *mean* in an abstract category.

Before we can formally state the theorem or the axioms, we must develop a theoretical framework in which these terms can be precisely defined....

Informal statement of main result

Theorem. (Informal statement) *Let $\mathcal{C}$ be any biconnected category. Let $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$ be a decision context satisfying structural conditions* (S1)-(S3). *Let $\mathfrak{S}$ be a "solvable" Savage structure on $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$. Then:*
$\mathfrak{S}$ has a "subjective expected utility representation" if and only if it satisfies axioms (A1)-(A5).
In this representation, the "probabilistic beliefs" on each state place are unique. The "utility function" on each outcome place represents the agent's "ex post preferences", and is unique up to positive affine transformations.

Many terms in the axioms and theorem appear in quotation marks, because they have not yet been formally defined.

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Part II

Partitions and Probability

Let $\mathcal{X}$ and $\mathcal{Y}$ be objects in a category $\mathcal{C}$, and let $\phi \in \overrightarrow{\mathcal{C}}(\mathcal{X}, \mathcal{Y})$.

ϕ is an *isomorphism* in $\mathcal{C}$ if there is a morphism $\psi \in \overrightarrow{\mathcal{C}}(\mathcal{Y}, \mathcal{X})$ such that:

- $\psi \circ \phi = I_{\mathcal{X}}$ (the identity morphism on $\mathcal{X}$).
- $\phi \circ \psi = I_{\mathcal{Y}}$ (the identity morphism on $\mathcal{Y}$).

Examples. • If $\mathcal{C} = \mathrm{Set}$, then isomorphisms are *bijections*.
• If $\mathcal{C} = \mathrm{Meas}$, then isomorphisms are *bi-measurable bijections*.
• If $\mathcal{C} = \mathrm{Top}$, then isomorphisms are *homeomorphisms*.
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We say that ϕ is a *monomorphism* (or is *monic*) if, for any other object $\mathcal{W} \in \mathcal{C}$, and any morphisms $\psi_1, \psi_2 \in \overrightarrow{\mathcal{C}}(\mathcal{W}, \mathcal{X})$, we have:

$$\Big(\phi \circ \psi_1 = \phi \circ \psi_2\Big) \iff \Big(\psi_1 = \psi_2\Big).$$

In most concrete categories, monomorphisms are *injective* morphisms.

Example. Let $\mathcal{X}$ be a subobject of $\mathcal{Y}$ (e.g. subspace, submanifold, etc.). Then the *inclusion morphism* $\mathcal{X} \hookrightarrow \mathcal{Y}$ is usually a monomorphism.

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Coproducts

Let $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_N$ be objects in category $\boldsymbol{\mathcal{C}}$. A *coproduct* of $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_N$ is a structure $\boldsymbol{\mathcal{R}} = (\mathcal{R}; \iota_1, \ldots, \iota_N)$, where $\mathcal{R}$ is an object in $\boldsymbol{\mathcal{C}}$, and $\iota_n \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{R}_n, \mathcal{R})$ for all $n \in [1 \ldots N]$, with the following property: For *any* other $\mathcal{X} \in [\boldsymbol{\mathcal{C}}]$, and *any* morphisms $f_n \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{R}_n, \mathcal{X})$ (for all $n \in [1 \ldots N]$), there is a unique $F \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{R}, \mathcal{X})$ such that the next diagram commutes:

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We then write $\boldsymbol{\mathcal{R}} = \displaystyle\coprod_{n=1}^{N} \mathcal{R}_n$

and $F = [f_1 | f_2 | \cdots | f_N]$.

Note. $\mathcal{R}_1, \ldots, \mathcal{R}_N$ might not *have* a coproduct in a category $\boldsymbol{\mathcal{C}}$. But if they do, then it is essentially unique up to canonical isomorphism.

Example. In the categories Set, Meas, Top and Diff, the coproduct is just the *disjoint union* of the objects $\mathcal{R}_1, \ldots, \mathcal{R}_N$ (with the appropriate measurable/topological/differentiable structure).

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Example. In the categories Set, Meas, Top and Diff, the coproduct is just the *disjoint union* of the objects $\mathcal{R}_1, \ldots, \mathcal{R}_N$ (with the appropriate measurable/topological/differentiable structure).

Partitions

Let $\mathcal{S}$ be an stateplace in $\boldsymbol{\mathcal{S}}$.

$$
\begin{tikzcd}
\mathcal{R}_1 \arrow[drr, "\iota_1"'] & \mathcal{R}_2 \arrow[dr, "\iota_2"] & \mathcal{R}_3 \arrow[d, "\iota_3"] & \cdots\cdots & \mathcal{R}_N \arrow[dll, "\iota_N"'] \\
 & & \mathcal{R} \arrow[d, "\rho"] & & \\
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\end{tikzcd}
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An *N-cell partition* of $\mathcal{S}$ is a structure $\boldsymbol{\mathcal{R}} := (\mathcal{R}_1, \iota_1; \mathcal{R}_2, \iota_2; \ldots; \mathcal{R}_N, \iota_N;\ \mathcal{R},\ \rho)$, where:

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Partition refinements

Let $\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho)$ and $\boldsymbol{\mathcal{R}}' = (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_{N'}, \iota'_{N'}; \mathcal{R}', \rho')$ be two partitions of $\mathcal{S}$, with $N' \geq N$.

An *adhesive* from $\boldsymbol{\mathcal{R}}'$ to $\boldsymbol{\mathcal{R}}$ is an ordered pair (η, ν), where:

- $\eta \in \overrightarrow{\mathcal{S}}(\mathcal{R}', \mathcal{R})$ is an $\mathcal{S}$-morphism such that this diagram commutes:

$$\begin{tikzcd} \mathcal{R}' \arrow[d, "\eta"'] \arrow[dr, "\rho'"] & \\ \mathcal{R} \arrow[r, "\rho"'] & \mathcal{S} \end{tikzcd}$$

- $\nu : [1 \ldots N'] \longrightarrow [1 \ldots N]$ is a surjection.

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Partition refinements

Let $\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho)$ and $\boldsymbol{\mathcal{R}}' = (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_{N'}, \iota'_{N'}; \mathcal{R}', \rho')$ be two partitions of $\mathcal{S}$, with $N' \geq N$.

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Partition refinements (an illustrative example)

Here are two partitions $\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \iota_1;\ \mathcal{R}_2, \iota_2;\ \mathcal{R}_3, \iota_3;\ \mathcal{R}, \rho)$ and $\boldsymbol{\mathcal{R}'} = (\mathcal{R}'_1, \iota'_1;\ \mathcal{R}'_2, \iota'_2;\ \mathcal{R}'_3, \iota'_3;\ \mathcal{R}'_4, \iota'_4;\ \mathcal{R}'_5, \iota'_5;\ \mathcal{R}'_6, \iota'_6;\ \mathcal{R}'_7, \iota'_7;\ \mathcal{R}', \rho')$.

$$
\begin{tikzcd}
\mathcal{R}'_1 \arrow[drrr, "\iota'_1"] & \mathcal{R}'_2 \arrow[drr, "\iota'_2"] & \mathcal{R}'_3 \arrow[dr, "\iota'_3"] & \mathcal{R}'_4 \arrow[d, "\iota'_4"] & \mathcal{R}'_5 \arrow[dl, "\iota'_5"'] & \mathcal{R}'_6 \arrow[dll, "\iota'_6"'] & \mathcal{R}'_7 \arrow[dlll, "\iota'_7"'] & \\
 & & & \mathcal{R}' \arrow[drrrr, "\rho'"] & & & & \\
 & & & & & & & \mathcal{S} \\
 & & & \mathcal{R} \arrow[urrrr, "\rho"] & & & & \\
\mathcal{R}_1 \arrow[urrr, "\iota_1"] & & \mathcal{R}_2 \arrow[ur, "\iota_2"'] & & & \mathcal{R}_3 \arrow[ull, "\iota_3"] & &
\end{tikzcd}
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Partition refinements (an illustrative example)

Here is an adhesive (η, ν) making $\boldsymbol{\mathcal{R}}'$ a refinement of $\boldsymbol{\mathcal{R}}$. In this case, $\nu(1) = \nu(2) = 1$, $\nu(3) = \nu(4) = 2$, and $\nu(5) = \nu(6) = \nu(7) = 3$.

$$
\begin{tikzcd}
\mathcal{R}'_1 \arrow[drrr, "\iota'_1"] \arrow[dddd, dashed, "\eta_1"] & \mathcal{R}'_2 \arrow[drr, "\iota'_2"] \arrow[ddddl, dashed, "\eta_2"] & \mathcal{R}'_3 \arrow[dr, "\iota'_3"] \arrow[dddd, dashed, "\eta_3"] & \mathcal{R}'_4 \arrow[d, "\iota'_4"] \arrow[ddddl, dashed, "\eta_4"] & \mathcal{R}'_5 \arrow[dl, "\iota'_5"'] \arrow[ddddr, dashed, "\eta_5"'] & \mathcal{R}'_6 \arrow[dll, "\iota'_6"'] \arrow[dddd, dashed, "\eta_6"'] & \mathcal{R}'_7 \arrow[dlll, "\iota'_7"'] \arrow[ddddl, dashed, "\eta_7"] & \\
 & & & \mathcal{R}' \arrow[dd, dashed, "\eta"] \arrow[drrrr, "\rho'"] & & & & \\
 & & & & & & & \mathcal{S} \\
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Partition categories and common refinement

Let $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$ be three partitions an object $\mathcal{S}$ in the category $\mathcal{C}$.

If $\mathcal{R}'$ refines $\mathcal{R}$ via the adhesive (η_1, ν_1), and $\mathcal{R}''$ refines $\mathcal{R}'$ via the adhesive (η_2, ν_2), then $\mathcal{R}''$ refines $\mathcal{R}$ via the adhesive $(\eta_1 \circ \eta_2, \nu_1 \circ \nu_2)$.

Thus, the set of all partitions of $\mathcal{S}$ forms a *category*, $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, where the objects are the partitions and the morphisms are the adhesives.

We will need $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$ to satisfy the *Common Refinement Property*: For any $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, there exists $\mathcal{R} \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$ with $\mathcal{R} \trianglelefteq \mathcal{R}_1$ and $\mathcal{R} \trianglelefteq \mathcal{R}_2$.

Example 2. Suppose $\mathcal{S} = \mathcal{C} = \mathrm{Set}, \mathrm{Meas}, \mathrm{Top}$, or Diff.

Then for any $\mathcal{S} \in [\mathcal{S}]$, the category $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$ contains the partitions described in Example 1 (with adhesives defined via inclusion maps).

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For all $N \in \mathbb{N}$, let $\Delta^N := \{\mathbf{p} \in \mathbb{R}_+^N;\ \ p_1 + \cdots + p_N = 1\}$ be the N-dimensional probability simplex. Let $\mathcal{S} \in \boldsymbol{\mathcal{S}}$.

A *probability structure* on $\mathcal{S}$ is a system $\mathbf{P} := \{\mathbf{p}^{\mathcal{R}};\ \ \mathcal{R} \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})\}$, where

- For each N-cell partition $\mathcal{R}$ in $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})$, we have $\mathbf{p}^{\mathcal{R}} \in \Delta^N$; and
- For any partitions $\mathcal{R}' \trianglelefteq \mathcal{R}$ in $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}$, if (η, ν) is the (unique) adhesive from $\mathcal{R}'$ to $\mathcal{R}$, and $\mathbf{p}^{\mathcal{R}} = (p_1, \ldots, p_N)$ and $\mathbf{p}^{\mathcal{R}'} = (p'_1, \ldots, p'_{N'})$, then

$$p_n \quad = \quad \sum_{m \in \nu^{-1}\{n\}} p'_m, \qquad \text{for all } n \in [1 \ldots N]. \qquad \text{(Additivity)}$$

Idea: $\mathbf{P}$ assigns an additive "probability" to subobjects of $\mathcal{S}$, but *only if they appear as a cell of some partition in* $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})$.

Example 3. Let $\mathcal{S} \in [\mathrm{Meas}]$, and define $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})$ as in Example 2.

Then any probability measure μ on $\mathcal{S}$ induces a probability structure on $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})$, in the obvious way.
(For every $\mathcal{R} = (\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho)$ in $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})$, let $\mathbf{p}^{\mathcal{R}} := (\mu[\mathcal{R}_1], \ldots, \mu[\mathcal{R}_N])$.)

Probability structures

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For all $N \in \mathbb{N}$, let $\Delta^N := \{\mathbf{p} \in \mathbb{R}_+^N;\ p_1 + \cdots + p_N = 1\}$ be the N-dimensional probability simplex. Let $\mathcal{S} \in \boldsymbol{\mathcal{S}}$.

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Idea: $\mathbf{P}$ assigns an additive "probability" to subobjects of $\mathcal{S}$, but *only if they appear as a cell of some partition in $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})$.*

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- A *pullback* is a categorical construction which plays the role of an *inverse image*.
- Using pullbacks, we can define the *preimage* of any partition in $\mathfrak{R}_{\mathcal{S}}(\mathcal{S}_2)$ under any morphism $\phi \in \overrightarrow{\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_2)$.
- We then define ϕ to be *measurable* if every partition in $\mathfrak{R}_{\mathcal{S}}(\mathcal{S}_2)$ has a ϕ-preimage in $\mathfrak{R}_{\mathcal{S}}(\mathcal{S}_1)$.
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Measurability in a nutshell

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Partial preimages

Let $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{S}'$ be objects in $\mathcal{C}$. Consider the diagram on the left below.

$$\begin{tikzcd} & \mathcal{S}' \arrow[d, "\phi"] \\ \mathcal{R} \arrow[r, "\rho"'] & \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[r, dashed, "\rho'"] & \mathcal{S}' \arrow[d, "\phi"] \\ \mathcal{R} \arrow[r, "\rho"'] & \mathcal{S} \end{tikzcd}$$

Question. Does there exist an object $\mathcal{R}'$, and morphisms $\rho' \in \overrightarrow{\mathcal{C}}(\mathcal{R}', \mathcal{S}')$ and $\psi \in \overrightarrow{\mathcal{C}}(\mathcal{R}', \mathcal{R})$, such that the right-hand diagram commutes?

Example. Suppose $\mathcal{R}$ is a subobject of $\mathcal{S}$, and ρ is the inclusion morphism. Let $\mathcal{R}' := \phi^{-1}(\mathcal{R})$ be the ϕ-preimage of $\mathcal{R}$ in $\mathcal{S}'$. Let ρ' be the inclusion morphism. Then the right-hand diagram commutes.

For this reason, we call $(\mathcal{R}', \rho', \psi)$ a *partial preimage* of the left diagram.

Problem. The left-hand diagram might admit *many* such "partial" preimages. Not all of them count as "true" preimages....

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Pullbacks

[skip]

A *pullback* of the left diagram is partial preimage $(\mathcal{R}'_0, \rho'_0, \psi_0)$ which is a *maximal* in the following sense. Given any other partial preimage $(\mathcal{R}', \rho', \psi)$, there is a unique morphism $\xi \in \overrightarrow{\mathcal{C}}(\mathcal{R}', \mathcal{R}'_0)$ making the centre diagram commute:

$$\begin{tikzcd} & \mathcal{S}' \arrow[d, "\phi"] \\ \mathcal{R} \arrow[r, "\rho"'] & \mathcal{S} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{R}'_0 \arrow[r, "\rho'_0"'] \arrow[d, "\psi_0"] & \mathcal{S}' \arrow[d, "\phi"] \\ \mathcal{R} \arrow[r, "\rho"'] & \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{R}'_0 \arrow[r, dashed, "\rho'_0"] \arrow[d, dashed, "\psi_0"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}' \arrow[d, "\phi"] \\ \mathcal{R} \arrow[r, "\rho"'] & \mathcal{S} \end{tikzcd}$$

Such a maximal preimage might not exist, but if it does, then it is unique up to isomorphism. Thus, we say $(\mathcal{R}'_0, \rho'_0, \psi_0)$ is "the" pullback of the left diagram. This is indicated by the symbol "$\lrcorner$" in the right diagram.

Examples. Suppose $\mathcal{C} = \mathrm{Set}, \mathrm{Meas}, \mathrm{Top}$, or Diff,

(a) If $\mathcal{R} \overset{\rho}{\hookrightarrow} \mathcal{S}$ is a subobject, then $\phi^{-1}(\mathcal{R}) \hookrightarrow \mathcal{S}'$ yields a pullback.

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\mathcal{R} \arrow[r, "\rho"'] & \mathcal{S}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathcal{R}' \arrow[dr, dashed, "\xi"] \arrow[drr, bend left, "\rho'"] \arrow[ddr, bend right, "\psi"'] & & \\
 & \mathcal{R}'_0 \arrow[r, "\rho'_0"'] \arrow[d, "\psi_0"] & \mathcal{S}' \arrow[d, "\phi"] \\
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\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R}'_0 \arrow[r, dashed, "\rho'_0"] \arrow[d, dashed, "\psi_0"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}' \arrow[d, "\phi"] \\
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\end{tikzcd}
$$

Such a maximal preimage might not exist, but if it does, then it is unique up to isomorphism. Thus, we say $(\mathcal{R}'_0, \rho'_0, \psi_0)$ is "the" pullback of the left diagram. This is indicated by the symbol "$\lrcorner$" in the right diagram.

Examples. Suppose $\mathcal{C} = \mathrm{Set}$, Meas, Top, or Diff,
(a) If $\mathcal{R} \overset{\rho}{\hookrightarrow} \mathcal{S}$ is a subobject, then $\phi^{-1}(\mathcal{R}) \hookrightarrow \mathcal{S}'$ yields a pullback.
(b) If $\mathcal{S}$ is a one-point space; then the pullback is just the Cartesian product $\mathcal{R} \times \mathcal{S}'$, with the appropriate product structure.

Pullbacks

[skip]

A *pullback* of the left diagram is partial preimage $(\mathcal{R}_0', \rho_0', \psi_0)$ which is a *maximal* in the following sense. Given any other partial preimage $(\mathcal{R}', \rho', \psi)$, there is a unique morphism $\xi \in \overrightarrow{\mathcal{C}}(\mathcal{R}', \mathcal{R}_0')$ making the centre diagram commute:

$$\begin{tikzcd} & \mathcal{S}' \arrow[d, "\phi"] \\ \mathcal{R} \arrow[r, "\rho"'] & \mathcal{S} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{R}' \arrow[dr, dashed, "\xi"] \arrow[drr, bend left, "\rho'"] \arrow[ddr, bend right, "\psi"'] & & \\ & \mathcal{R}_0' \arrow[r, "\rho_0'"'] \arrow[d, "\psi_0"] & \mathcal{S}' \arrow[d, "\phi"] \\ & \mathcal{R} \arrow[r, "\rho"'] & \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{R}_0' \arrow[r, dashed, "\rho_0'"] \arrow[d, dashed, "\psi_0"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}' \arrow[d, "\phi"] \\ \mathcal{R} \arrow[r, "\rho"'] & \mathcal{S} \end{tikzcd}$$

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Partition preimages :Definition [skip to example]

Let $\mathcal{S}$ and $\mathcal{S}'$ be two state places in $\boldsymbol{\mathcal{S}}$, and let $\phi \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}', \mathcal{S})$.

Let $\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho)$ be a partition of $\mathcal{S}$.

A *ϕ-preimage* of the partition $\boldsymbol{\mathcal{R}}$ (if it exists) is constructed as follows.

1. Let $\mathcal{R}' \in [\boldsymbol{\mathcal{S}}]$ and $\rho' \in \overrightarrow{\boldsymbol{\mathcal{S}}}(\mathcal{R}', \mathcal{S}')$ satisfy the pullback diagram below.

2. For all $n \in [1 \ldots N]$, let $\mathcal{R}'_n$ be an object and let $\iota'_n \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{R}'_n, \mathcal{R}')$ be a morphism satisfying the left-hand pullback diagram below.

3. Suppose that $(\mathcal{R}'; \iota'_1, \ldots, \iota'_N)$ is a coproduct of $\mathcal{R}'_1, \ldots, \mathcal{R}'_N$.

Let $\boldsymbol{\mathcal{R}}' := (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$. Then $\boldsymbol{\mathcal{R}}' \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}')$, and we get a commuting diagram:

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\mathcal{R}_n \arrow[r, "\iota_n"'] & \mathcal{R}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
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Partition preimages :Definition

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3. Suppose that $(\mathcal{R}'; \iota'_1, \ldots, \iota'_N)$ is a coproduct of $\mathcal{R}'_1, \ldots, \mathcal{R}'_N$.

Let $\color{red}{\boldsymbol{\mathcal{R}}'}$ $:= (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$. Then $\boldsymbol{\mathcal{R}}' \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}')$, and we get a commuting diagram:

$$
\begin{tikzcd}
\mathcal{R}'_1 \arrow[d, "\psi_1"'] \arrow[rrrr, bend left=10, "\iota'_1"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{R}'_2 \arrow[d, "\psi_2"'] \arrow[rrr, "\iota'_2"] \arrow[dr, phantom, "\lrcorner"] & \cdots \arrow[d, phantom, "\cdots"] & \mathcal{R}'_N \arrow[d, "\psi_N"'] \arrow[r, "\iota'_N"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{R}' \arrow[r, "\rho'"] \arrow[d, "\psi"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}' \arrow[d, "\phi"] \\
\mathcal{R}_1 \arrow[rrrr, bend right=10, "\iota_1"] & \mathcal{R}_2 \arrow[rrr, "\iota_2"'] & \cdots & \mathcal{R}_N \arrow[r, "\iota_N"'] & \mathcal{R} \arrow[r, "\rho"'] & \mathcal{S}
\end{tikzcd}
$$

Partition preimages: example

$$
\begin{tikzcd}
\mathcal{R}'_1 \arrow[ddd, "\psi_1"'] \arrow[drrrr, hook, bend right=10, "\iota'_1"'] \arrow[dddr, phantom, "\lrcorner"] & \mathcal{R}'_2 \arrow[ddd, "\psi_2"'] \arrow[drrr, hook, "\iota'_2"] \arrow[dddr, phantom, "\lrcorner"] & \cdots \arrow[dddr, phantom, "\cdots"] & \mathcal{R}'_N \arrow[ddd, "\psi_N"'] \arrow[dr, hook, "\iota'_N"] \arrow[dddr, phantom, "\lrcorner"] & & \\
& & & & \mathcal{R}' \arrow[r, hook, "\rho'"] \arrow[d, "\psi"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}' \arrow[d, "\phi"] \\
& & & & \mathcal{R} \arrow[r, hook, "\rho"'] & \mathcal{S} \\
\mathcal{R}_1 \arrow[urrrr, hook, bend left=10, "\iota_1"] & \mathcal{R}_2 \arrow[urrr, hook, "\iota_2"'] & \cdots & \mathcal{R}_N \arrow[ur, hook, "\iota_N"'] & &
\end{tikzcd}
$$

Example 4. Suppose $\mathcal{C} = \mathrm{Set}$, Meas, or Top. Let $\mathcal{S} \in \mathcal{C}$. Let $(\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho) \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})$, as in Example 1. (i.e. $\mathcal{R}_n \overset{\iota_n}{\hookrightarrow} \mathcal{R}$ and $\mathcal{R} \overset{\rho}{\hookrightarrow} \mathcal{S}$ are inclusion morphisms). Let $\phi : \mathcal{S}' \longrightarrow \mathcal{S}$ be a $\mathcal{C}$-morphism. Define $\mathcal{R}'_n := \phi^{-1}(\mathcal{R}_n) \subseteq \mathcal{S}'$ (for all $n \in [1 \ldots N]$) and $\mathcal{R}' := \phi^{-1}(\mathcal{R}) = \mathcal{R}'_1 \sqcup \ldots \sqcup \mathcal{R}'_N \subseteq \mathcal{S}'$. Let $\mathcal{R}'_n \overset{\iota'_n}{\hookrightarrow} \mathcal{R}'$ and $\mathcal{R}' \overset{\rho'}{\hookrightarrow} \mathcal{S}'$ be inclusion morphisms. Then $\boldsymbol{\mathcal{R}}' := (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$ is a ϕ-preimage of $\boldsymbol{\mathcal{R}}$.

Proof. Let $\psi_n := \phi_{\upharpoonright \mathcal{R}'_n} : \mathcal{R}'_n \longrightarrow \mathcal{R}_n$ ($\forall\, n \in [1 \ldots N]$) and $\psi := \phi_{\upharpoonright \mathcal{R}'} : \mathcal{R}' \to \mathcal{R}$.

$$
\begin{tikzcd}
\mathcal{R}'_1 \arrow[ddd, "\psi_1"'] \arrow[drrrr, hook, "\iota'_1"'] & \mathcal{R}'_2 \arrow[ddd, "\psi_2"'] \arrow[drrr, hook, "\iota'_2"] & \cdots & \mathcal{R}'_N \arrow[ddd, "\psi_N"'] \arrow[dr, hook, "\iota'_N"] & & \\
& & & & \mathcal{R}' \arrow[r, hook, "\rho'"] \arrow[d, "\psi"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}' \arrow[d, "\phi"] \\
& & & & \mathcal{R} \arrow[r, hook, "\rho"'] & {\color{red}\mathcal{S}} \\
\mathcal{R}_1 \arrow[urrrr, hook, "\iota_1"] & \mathcal{R}_2 \arrow[urrr, hook, "\iota_2"'] & \cdots & \mathcal{R}_N \arrow[ur, hook, "\iota_N"'] & &
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Example 4. Suppose $\mathcal{C} = \mathrm{Set}$, Meas, or Top. Let ${\color{red}\mathcal{S}} \in \mathcal{C}$. Let $(\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, as in Example 1. (i.e. $\mathcal{R}_n \overset{\iota_n}{\hookrightarrow} \mathcal{R}$ and $\mathcal{R} \overset{\rho}{\hookrightarrow} \mathcal{S}$ are inclusion morphisms). Let $\phi : \mathcal{S}' \longrightarrow \mathcal{S}$ be a $\mathcal{C}$-morphism. Define $\mathcal{R}'_n := \phi^{-1}(\mathcal{R}_n) \subseteq \mathcal{S}'$ (for all $n \in [1...N]$) and $\mathcal{R}' := \phi^{-1}(\mathcal{R}) = \mathcal{R}'_1 \sqcup ... \sqcup \mathcal{R}'_N \subseteq \mathcal{S}'$. Let $\mathcal{R}'_n \overset{\iota'_n}{\hookrightarrow} \mathcal{R}'$ and $\mathcal{R}' \overset{\rho'}{\hookrightarrow} \mathcal{S}'$ be inclusion morphisms.

Then $\boldsymbol{\mathcal{R}}' := (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$ is a ϕ-preimage of $\boldsymbol{\mathcal{R}}$.

Proof. Let $\psi_n := \phi_{\restriction \mathcal{R}'_n} : \mathcal{R}'_n \longrightarrow \mathcal{R}_n$ ($\forall\, n \in [1...N]$) and $\psi := \phi_{\restriction \mathcal{R}'} : \mathcal{R}' \to \mathcal{R}$.

Partition preimages: example

$$
\begin{tikzcd}
\mathcal{R}'_1 \arrow[rrrrd, hook, "\iota'_1"] \arrow[ddd, "\psi_1"'] \arrow[rddd, phantom, "\lrcorner"] & \mathcal{R}'_2 \arrow[rrrd, hook, "\iota'_2"] \arrow[ddd, "\psi_2"'] \arrow[rddd, phantom, "\lrcorner"] & \cdots & \mathcal{R}'_N \arrow[rd, hook, "\iota'_N"] \arrow[ddd, "\psi_N"'] \arrow[rdd, phantom, "\lrcorner"] & & \\
& & & & \mathcal{R}' \arrow[r, hook, "\rho'"] \arrow[d, "\psi"] \arrow[rd, phantom, "\lrcorner"] & \mathcal{S}' \arrow[d, "\phi"] \\
& & & & \mathcal{R} \arrow[r, hook, "\rho"'] & \mathcal{S} \\
\mathcal{R}_1 \arrow[rrrru, hook, "\iota_1"] & \mathcal{R}_2 \arrow[rrru, hook, "\iota_2"] & \cdots & \mathcal{R}_N \arrow[ru, hook, "\iota_N"'] & &
\end{tikzcd}
$$

Example 4. Suppose $\mathcal{C} = \mathrm{Set}$, Meas, or Top. Let $\mathcal{S} \in \mathcal{C}$. Let $(\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, as in Example 1. (i.e. $\mathcal{R}_n \overset{\iota_n}{\hookrightarrow} \mathcal{R}$ and $\mathcal{R} \overset{\rho}{\hookrightarrow} \mathcal{S}$ are inclusion morphisms). Let $\phi : \mathcal{S}' \longrightarrow \mathcal{S}$ be a $\mathcal{C}$-morphism. Define $\mathcal{R}'_n := \phi^{-1}(\mathcal{R}_n) \subseteq \mathcal{S}'$ (for all $n \in [1...N]$) and $\mathcal{R}' := \phi^{-1}(\mathcal{R}) = \mathcal{R}'_1 \sqcup \ldots \sqcup \mathcal{R}'_N \subseteq \mathcal{S}'$. Let $\mathcal{R}'_n \overset{\iota'_n}{\hookrightarrow} \mathcal{R}'$ and $\mathcal{R}' \overset{\rho'}{\hookrightarrow} \mathcal{S}'$ be inclusion morphisms. Then $\boldsymbol{\mathcal{R}}' := (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$ is a ϕ-preimage of $\boldsymbol{\mathcal{R}}$.

Proof. Let $\psi_n := \phi_{\restriction \mathcal{R}'_n} : \mathcal{R}'_n \longrightarrow \mathcal{R}_n$ ($\forall\, n \in [1...N]$) and $\psi := \phi_{\restriction \mathcal{R}'} : \mathcal{R}' \to \mathcal{R}$.

$$
\begin{tikzcd}
\mathcal{R}'_1 \arrow[ddd, "\psi_1"'] \arrow[drrrr, hook, "\iota'_1"] \arrow[dddr, phantom, "\lrcorner"] & \mathcal{R}'_2 \arrow[ddd, "\psi_2"'] \arrow[drrr, hook, "\iota'_2"] \arrow[dddr, phantom, "\lrcorner"] & \cdots & \mathcal{R}'_N \arrow[ddd, "\psi_N"'] \arrow[dr, hook, "\iota'_N"] \arrow[dddr, phantom, "\lrcorner"] & & \\
 & & & & \mathcal{R}' \arrow[r, hook, "\rho'"] \arrow[d, "\psi"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}' \arrow[d, "\phi"] \\
 & & & & \mathcal{R} \arrow[r, hook, "\rho"'] & \mathcal{S} \\
\mathcal{R}_1 \arrow[urrrr, hook, "\iota_1"] & \mathcal{R}_2 \arrow[urrr, hook, "\iota_2"] & \cdots & \mathcal{R}_N \arrow[ur, hook, "\iota_N"'] & &
\end{tikzcd}
$$

Example 4. Suppose $\mathcal{C} = \mathrm{Set}$, Meas, or Top. Let $\mathcal{S} \in \mathcal{C}$.
Let $(\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, as in Example 1. (i.e. $\mathcal{R}_n \overset{\iota_n}{\hookrightarrow} \mathcal{R}$ and $\mathcal{R} \overset{\rho}{\hookrightarrow} \mathcal{S}$ are inclusion morphisms). Let $\phi : \mathcal{S}' \longrightarrow \mathcal{S}$ be a $\mathcal{C}$-morphism.
Define $\mathcal{R}'_n := \phi^{-1}(\mathcal{R}_n) \subseteq \mathcal{S}'$ (for all $n \in [1...N]$) and $\mathcal{R}' := \phi^{-1}(\mathcal{R}) = \mathcal{R}'_1 \sqcup ... \sqcup \mathcal{R}'_N \subseteq \mathcal{S}'$. Let $\mathcal{R}'_n \overset{\iota'_n}{\hookrightarrow} \mathcal{R}'$ and $\mathcal{R}' \overset{\rho'}{\hookrightarrow} \mathcal{S}'$ be inclusion morphisms.
Then $\boldsymbol{\mathcal{R}}' := (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$ is a ϕ-preimage of $\boldsymbol{\mathcal{R}}$.
Proof. Let $\psi_n := \phi_{\upharpoonright \mathcal{R}'_n} : \mathcal{R}'_n \longrightarrow \mathcal{R}_n$ ($\forall\, n \in [1...N]$) and $\psi := \phi_{\upharpoonright \mathcal{R}'} : \mathcal{R}' \to \mathcal{R}$.

Partition preimages: example

$$
\begin{tikzcd}
{\color{red}\mathcal{R}'_1} \arrow[rrrrd, hook, "\iota'_1"] \arrow[ddd, "\psi_1"'] & {\color{red}\mathcal{R}'_2} \arrow[rrrd, hook, "\iota'_2"] \arrow[ddd, "\psi_2"'] & \cdots & {\color{red}\mathcal{R}'_N} \arrow[rd, hook, "\iota'_N"] \arrow[ddd, "\psi_N"'] & & \\
& & & & {\color{red}\mathcal{R}'} \arrow[r, hook, "\rho'"] \arrow[d, "\psi"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{S}' \arrow[d, "\phi"] \\
& & & & \mathcal{R} \arrow[r, hook, "\rho"'] & \mathcal{S} \\
\mathcal{R}_1 \arrow[rrrru, hook, "\iota_1"] & \mathcal{R}_2 \arrow[rrru, hook, "\iota_2"'] & \cdots & \mathcal{R}_N \arrow[ru, hook, "\iota_N"'] & &
\end{tikzcd}
$$

Example 4. Suppose $\mathcal{C} = \mathrm{Set}$, Meas, or Top. Let $\mathcal{S} \in \mathcal{C}$. Let $(\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, as in Example 1. (i.e. $\mathcal{R}_n \overset{\iota_n}{\hookrightarrow} \mathcal{R}$ and $\mathcal{R} \overset{\rho}{\hookrightarrow} \mathcal{S}$ are inclusion morphisms). Let $\phi : \mathcal{S}' \longrightarrow \mathcal{S}$ be a $\mathcal{C}$-morphism. Define ${\color{red}\mathcal{R}'_n} := \phi^{-1}(\mathcal{R}_n) \subseteq \mathcal{S}'$ (for all $n \in [1...N]$) and ${\color{red}\mathcal{R}'} := \phi^{-1}(\mathcal{R}) = \mathcal{R}'_1 \sqcup ... \sqcup \mathcal{R}'_N \subseteq \mathcal{S}'$. Let $\mathcal{R}'_n \overset{\iota'_n}{\hookrightarrow} \mathcal{R}'$ and $\mathcal{R}' \overset{\rho'}{\hookrightarrow} \mathcal{S}'$ be inclusion morphisms. Then $\boldsymbol{\mathcal{R}}' := (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$ is a ϕ-preimage of $\boldsymbol{\mathcal{R}}$.

Proof. Let $\psi_n := \phi_{\upharpoonright \mathcal{R}'_n} : \mathcal{R}'_n \longrightarrow \mathcal{R}_n$ ($\forall\ n \in [1...N]$) and $\psi := \phi_{\upharpoonright \mathcal{R}'} : \mathcal{R}' \to \mathcal{R}$.

$$\begin{tikzcd}
\mathcal{R}'_1 \arrow[dddd, "\psi_1"'] \arrow[drrrr, hook, "\iota'_1"] & \mathcal{R}'_2 \arrow[dddd, "\psi_2"'] \arrow[drrr, hook, "\iota'_2"] & \cdots & \mathcal{R}'_N \arrow[dddd, "\psi_N"'] \arrow[dr, hook, "\iota'_N"] & & \\
& & & & \mathcal{R}' \arrow[r, hook, "\rho'"] \arrow[dd, "\psi"] \arrow[ddrr, phantom, "\lrcorner", very near start] & \mathcal{S}' \arrow[dd, "\phi"] \\
& & & & & \\
& & & & \mathcal{R} \arrow[r, hook, "\rho"'] & \mathcal{S} \\
\mathcal{R}_1 \arrow[urrrr, hook, "\iota_1"] & \mathcal{R}_2 \arrow[urrr, hook, "\iota_2"'] & \cdots & \mathcal{R}_N \arrow[ur, hook, "\iota_N"'] & &
\end{tikzcd}$$

Example 4. Suppose $\mathcal{C} = \mathrm{Set}$, Meas, or Top. Let $\mathcal{S} \in \mathcal{C}$. Let $(\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, as in Example 1. (i.e. $\mathcal{R}_n \overset{\iota_n}{\hookrightarrow} \mathcal{R}$ and $\mathcal{R} \overset{\rho}{\hookrightarrow} \mathcal{S}$ are inclusion morphisms). Let $\phi : \mathcal{S}' \longrightarrow \mathcal{S}$ be a $\mathcal{C}$-morphism. Define $\mathcal{R}'_n := \phi^{-1}(\mathcal{R}_n) \subseteq \mathcal{S}'$ (for all $n \in [1...N]$) and $\mathcal{R}' := \phi^{-1}(\mathcal{R}) = \mathcal{R}'_1 \sqcup \ldots \sqcup \mathcal{R}'_N \subseteq \mathcal{S}'$. Let $\mathcal{R}'_n \overset{\iota'_n}{\hookrightarrow} \mathcal{R}'$ and $\mathcal{R}' \overset{\rho'}{\hookrightarrow} \mathcal{S}'$ be inclusion morphisms.

Then $\mathcal{R}' := (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$ is a ϕ-preimage of $\mathcal{R}$.

Proof. Let $\psi_n := \phi_{\restriction \mathcal{R}'_n} : \mathcal{R}'_n \longrightarrow \mathcal{R}_n$ ($\forall\, n \in [1...N]$) and $\psi := \phi_{\restriction \mathcal{R}'} : \mathcal{R}' \to \mathcal{R}$.

Partition preimages: example

$$
\begin{tikzcd}
\mathcal{R}'_1 \arrow[ddd, "\psi_1"'] \arrow[drrrr, hook, "\iota'_1"] & \mathcal{R}'_2 \arrow[ddd, "\psi_2"'] \arrow[drrr, hook, "\iota'_2"] & \cdots & \mathcal{R}'_N \arrow[ddd, "\psi_N"'] \arrow[dr, hook, "\iota'_N"] & & \\
& & & & \mathcal{R}' \arrow[d, "\psi"] \arrow[r, hook, "\rho'"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}' \arrow[d, "\phi"] \\
& & & & \mathcal{R} \arrow[r, hook, "\rho"'] & \mathcal{S} \\
\mathcal{R}_1 \arrow[urrrr, hook, "\iota_1"] \arrow[uuur, phantom, "\lrcorner"] & \mathcal{R}_2 \arrow[urrr, hook, "\iota_2"'] & \cdots & \mathcal{R}_N \arrow[ur, hook, "\iota_N"'] & &
\end{tikzcd}
$$

Example 4. Suppose $\mathcal{C} = \mathrm{Set}$, Meas, or Top. Let $\mathcal{S} \in \mathcal{C}$. Let $(\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, as in Example 1. (i.e. $\mathcal{R}_n \overset{\iota_n}{\hookrightarrow} \mathcal{R}$ and $\mathcal{R} \overset{\rho}{\hookrightarrow} \mathcal{S}$ are inclusion morphisms). Let $\phi : \mathcal{S}' \longrightarrow \mathcal{S}$ be a $\mathcal{C}$-morphism. Define $\mathcal{R}'_n := \phi^{-1}(\mathcal{R}_n) \subseteq \mathcal{S}'$ (for all $n \in [1...N]$) and $\mathcal{R}' := \phi^{-1}(\mathcal{R}) = \mathcal{R}'_1 \sqcup ... \sqcup \mathcal{R}'_N \subseteq \mathcal{S}'$. Let $\mathcal{R}'_n \overset{\iota'_n}{\hookrightarrow} \mathcal{R}'$ and $\mathcal{R}' \overset{\rho'}{\hookrightarrow} \mathcal{S}'$ be inclusion morphisms. Then $\mathcal{R}' := (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$ is a ϕ-preimage of $\boldsymbol{\mathcal{R}}$.

Proof. Let $\psi_n := \phi_{\restriction \mathcal{R}'_n} : \mathcal{R}'_n \longrightarrow \mathcal{R}_n$ ($\forall\ n \in [1...N]$) and $\psi := \phi_{\restriction \mathcal{R}'} : \mathcal{R}' \to \mathcal{R}$.

$$
\begin{tikzcd}
\mathcal{R}'_1 \arrow[ddd, "\color{red}\psi_1"'] \arrow[drrrr, hook, bend right, "\iota'_1"] \arrow[dddr, phantom, "\lrcorner"] & \mathcal{R}'_2 \arrow[ddd, "\color{red}\psi_2"'] \arrow[drrr, hook, "\iota'_2"] \arrow[dddr, phantom, "\lrcorner"] & \cdots & \mathcal{R}'_N \arrow[ddd, "\color{red}\psi_N"'] \arrow[dr, hook, "\iota'_N"] \arrow[dddr, phantom, "\lrcorner"] & & \\
& & & & \mathcal{R}' \arrow[r, hook, "\rho'"] \arrow[d, "\color{red}\psi"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}' \arrow[d, "\phi"] \\
& & & & \mathcal{R} \arrow[r, hook, "\rho"'] & \mathcal{S} \\
\mathcal{R}_1 \arrow[urrrr, hook, bend left, "\iota_1"] & \mathcal{R}_2 \arrow[urrr, hook, "\iota_2"'] & \cdots & \mathcal{R}_N \arrow[ur, hook, "\iota_N"'] & &
\end{tikzcd}
$$

Example 4. Suppose $\mathcal{C} = \mathrm{Set}$, Meas, or Top. Let $\mathcal{S} \in \mathcal{C}$.
Let $(\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})$, as in Example 1. (i.e. $\mathcal{R}_n \overset{\iota_n}{\hookrightarrow} \mathcal{R}$ and $\mathcal{R} \overset{\rho}{\hookrightarrow} \mathcal{S}$ are inclusion morphisms). Let $\phi : \mathcal{S}' \longrightarrow \mathcal{S}$ be a $\mathcal{C}$-morphism. Define $\mathcal{R}'_n := \phi^{-1}(\mathcal{R}_n) \subseteq \mathcal{S}'$ (for all $n \in [1...N]$) and $\mathcal{R}' := \phi^{-1}(\mathcal{R}) = \mathcal{R}'_1 \sqcup ... \sqcup \mathcal{R}'_N \subseteq \mathcal{S}'$. Let $\mathcal{R}'_n \overset{\iota'_n}{\hookrightarrow} \mathcal{R}'$ and $\mathcal{R}' \overset{\rho'}{\hookrightarrow} \mathcal{S}'$ be inclusion morphisms. Then $\boldsymbol{\mathcal{R}}' := (\mathcal{R}'_1, \iota'_1; \ldots; \mathcal{R}'_N, \iota'_N; \mathcal{R}', \rho')$ is a ϕ-preimage of $\boldsymbol{\mathcal{R}}$.

Proof. Let $\psi_n := \phi_{\upharpoonright \mathcal{R}'_n} : \mathcal{R}'_n \longrightarrow \mathcal{R}_n$ ($\forall\ n \in [1...N]$) and $\psi := \phi_{\upharpoonright \mathcal{R}'} : \mathcal{R}' \to \mathcal{R}$.

Measurable and probability-preserving morphisms.

Let $\boldsymbol{\mathcal{S}}$ be a subcategory of $\boldsymbol{\mathcal{C}}$. Let $\mathcal{S}_1$, $\mathcal{S}_2 \in [\boldsymbol{\mathcal{S}}]$, and let $\phi \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}_1, \mathcal{S}_2)$.

We will say that ϕ is $\boldsymbol{\mathcal{S}}$*-measurable* if every $\boldsymbol{\mathcal{S}}$-partition $\boldsymbol{\mathcal{R}} \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$ has a ϕ-preimage $\phi^{-1}(\boldsymbol{\mathcal{R}})$ in $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$.

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be probability structures on $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$ and $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$. We say ϕ is *probability-preserving* with respect to $\mathbf{P}_1$ and $\mathbf{P}_2$ if ϕ is measurable and, for every $\boldsymbol{\mathcal{R}}_2 \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$, if $\boldsymbol{\mathcal{R}}_1 := \phi^{-1}(\boldsymbol{\mathcal{R}}_2)$, then $\mathbf{p}_1^{\boldsymbol{\mathcal{R}}_1} = \mathbf{p}_2^{\boldsymbol{\mathcal{R}}_2}$.

Example 5. Let $\boldsymbol{\mathcal{S}}$ be a subcategory of Meas. Let $\mathcal{S}_1, \mathcal{S}_2 \in [\boldsymbol{\mathcal{S}}]$, and let $\phi : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ be a measurable function.
For any $\boldsymbol{\mathcal{S}}$-partition $\boldsymbol{\mathcal{R}} \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$, define the preimage $\phi^{-1}(\boldsymbol{\mathcal{R}})$ as in Example 4; then $\phi^{-1}(\boldsymbol{\mathcal{R}}) \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$. Thus, ϕ is $\boldsymbol{\mathcal{S}}$-measurable.

Let μ_1 and μ_2 be probability measures on $\mathcal{S}_1$ and $\mathcal{S}_2$; and use these to define probability structures $\mathbf{P}_1$ and $\mathbf{P}_2$ on $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$ and $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$ as in Example 3. Suppose ϕ is *measure-preserving* with respect to μ_1 and μ_2 (i.e. $\mu_1[\phi^{-1}(\mathcal{R})] = \mu_2[\mathcal{R}]$ for every measurable subset $\mathcal{R} \subseteq \mathcal{S}_2$).
Then ϕ is a probability-preserving morphism with respect to $\mathbf{P}_1$ and $\mathbf{P}_2$.

Let $\boldsymbol{\mathcal{S}}$ be a subcategory of $\boldsymbol{\mathcal{C}}$. Let $\mathcal{S}_1$, $\mathcal{S}_2 \in [\boldsymbol{\mathcal{S}}]$, and let $\phi \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}_1, \mathcal{S}_2)$.

We will say that ϕ is *$\boldsymbol{\mathcal{S}}$-measurable* if every $\boldsymbol{\mathcal{S}}$-partition $\boldsymbol{\mathcal{R}} \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$ has a ϕ-preimage $\phi^{-1}(\boldsymbol{\mathcal{R}})$ in $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$.

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be probability structures on $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$ and $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$. We say ϕ is *probability-preserving* with respect to $\mathbf{P}_1$ and $\mathbf{P}_2$ if ϕ is measurable and, for every $\mathcal{R}_2 \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$, if $\mathcal{R}_1 := \phi^{-1}(\mathcal{R}_2)$, then $\mathbf{p}_1^{\mathcal{R}_1} = \mathbf{p}_2^{\mathcal{R}_2}$.

Example 5. Let $\mathcal{S}$ be a subcategory of Meas. Let $\mathcal{S}_1, \mathcal{S}_2 \in [\mathcal{S}]$, and let $\phi : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ be a measurable function.
For any $\mathcal{S}$-partition $\mathcal{R} \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S}_2)$, define the preimage $\phi^{-1}(\mathcal{R})$ as in Example 4; then $\phi^{-1}(\mathcal{R}) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S}_1)$. Thus, ϕ is $\mathcal{S}$-measurable.

Let μ_1 and μ_2 be probability measures on $\mathcal{S}_1$ and $\mathcal{S}_2$; and use these to define probability structures $\mathbf{P}_1$ and $\mathbf{P}_2$ on $\mathfrak{R}_{\mathcal{S}}(\mathcal{S}_1)$ and $\mathfrak{R}_{\mathcal{S}}(\mathcal{S}_2)$ as in Example 3. Suppose ϕ is *measure-preserving* with respect to μ_1 and μ_2 (i.e. $\mu_1[\phi^{-1}(\mathcal{R})] = \mu_2[\mathcal{R}]$ for every measurable subset $\mathcal{R} \subseteq \mathcal{S}_2$).
Then ϕ is a probability-preserving morphism with respect to $\mathbf{P}_1$ and $\mathbf{P}_2$.

Let $\boldsymbol{\mathcal{S}}$ be a subcategory of $\boldsymbol{\mathcal{C}}$. Let $\mathcal{S}_1$, $\mathcal{S}_2 \in [\boldsymbol{\mathcal{S}}]$, and let $\phi \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}_1, \mathcal{S}_2)$.

We will say that ϕ is $\boldsymbol{\mathcal{S}}$*-measurable* if every $\boldsymbol{\mathcal{S}}$-partition $\boldsymbol{\mathcal{R}} \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$ has a ϕ-preimage $\phi^{-1}(\boldsymbol{\mathcal{R}})$ in $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$.

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be probability structures on $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$ and $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$. We say ϕ is *probability-preserving* with respect to $\mathbf{P}_1$ and $\mathbf{P}_2$ if ϕ is measurable and, for every $\boldsymbol{\mathcal{R}}_2 \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$, if $\boldsymbol{\mathcal{R}}_1 := \phi^{-1}(\boldsymbol{\mathcal{R}}_2)$, then $\mathbf{p}_1^{\boldsymbol{\mathcal{R}}_1} = \mathbf{p}_2^{\boldsymbol{\mathcal{R}}_2}$.

Example 5. Let $\mathcal{S}$ be a subcategory of Meas. Let $\mathcal{S}_1, \mathcal{S}_2 \in [\mathcal{S}]$, and let $\phi : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ be a measurable function.
For any $\mathcal{S}$-partition $\mathcal{R} \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S}_2)$, define the preimage $\phi^{-1}(\mathcal{R})$ as in Example 4; then $\phi^{-1}(\mathcal{R}) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S}_1)$. Thus, ϕ is $\mathcal{S}$-measurable.

Let μ_1 and μ_2 be probability measures on $\mathcal{S}_1$ and $\mathcal{S}_2$; and use these to define probability structures $\mathbf{P}_1$ and $\mathbf{P}_2$ on $\mathfrak{R}_{\mathcal{S}}(\mathcal{S}_1)$ and $\mathfrak{R}_{\mathcal{S}}(\mathcal{S}_2)$ as in Example 3.
Suppose ϕ is *measure-preserving* with respect to μ_1 and μ_2 (i.e. $\mu_1[\phi^{-1}(\mathcal{R})] = \mu_2[\mathcal{R}]$ for every measurable subset $\mathcal{R} \subseteq \mathcal{S}_2$).
Then ϕ is a probability-preserving morphism with respect to $\mathbf{P}_1$ and $\mathbf{P}_2$.

Let $\boldsymbol{\mathcal{S}}$ be a subcategory of $\boldsymbol{\mathcal{C}}$. Let $\mathcal{S}_1$, $\mathcal{S}_2 \in [\boldsymbol{\mathcal{S}}]$, and let $\phi \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}_1, \mathcal{S}_2)$.

We will say that ϕ is $\boldsymbol{\mathcal{S}}$*-measurable* if every $\boldsymbol{\mathcal{S}}$-partition $\boldsymbol{\mathcal{R}} \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$ has a ϕ-preimage $\phi^{-1}(\boldsymbol{\mathcal{R}})$ in $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$.

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be probability structures on $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$ and $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$. We say ϕ is *probability-preserving* with respect to $\mathbf{P}_1$ and $\mathbf{P}_2$ if ϕ is measurable and, for every $\boldsymbol{\mathcal{R}}_2 \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$, if $\boldsymbol{\mathcal{R}}_1 := \phi^{-1}(\boldsymbol{\mathcal{R}}_2)$, then $\mathbf{p}_1^{\boldsymbol{\mathcal{R}}_1} = \mathbf{p}_2^{\boldsymbol{\mathcal{R}}_2}$.

Example 5. Let $\boldsymbol{\mathcal{S}}$ be a subcategory of Meas. Let $\mathcal{S}_1, \mathcal{S}_2 \in [\boldsymbol{\mathcal{S}}]$, and let $\phi : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ be a measurable function.

For any $\boldsymbol{\mathcal{S}}$-partition $\boldsymbol{\mathcal{R}} \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$, define the preimage $\phi^{-1}(\boldsymbol{\mathcal{R}})$ as in Example 4; then $\phi^{-1}(\boldsymbol{\mathcal{R}}) \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$. Thus, ϕ is $\boldsymbol{\mathcal{S}}$-measurable.

Let μ_1 and μ_2 be probability measures on $\mathcal{S}_1$ and $\mathcal{S}_2$; and use these to define probability structures $\mathbf{P}_1$ and $\mathbf{P}_2$ on $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$ and $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$ as in Example 3. Suppose ϕ is *measure-preserving* with respect to μ_1 and μ_2 (i.e. $\mu_1[\phi^{-1}(\mathcal{R})] = \mu_2[\mathcal{R}]$ for every measurable subset $\mathcal{R} \subseteq \mathcal{S}_2$).
Then ϕ is a probability-preserving morphism with respect to $\mathbf{P}_1$ and $\mathbf{P}_2$.

Let $\boldsymbol{\mathcal{S}}$ be a subcategory of $\boldsymbol{\mathcal{C}}$. Let $\mathcal{S}_1$, $\mathcal{S}_2 \in [\boldsymbol{\mathcal{S}}]$, and let $\phi \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}_1, \mathcal{S}_2)$.

We will say that ϕ is $\boldsymbol{\mathcal{S}}$*-measurable* if every $\boldsymbol{\mathcal{S}}$-partition $\boldsymbol{\mathcal{R}} \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$ has a ϕ-preimage $\phi^{-1}(\boldsymbol{\mathcal{R}})$ in $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$.

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be probability structures on $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$ and $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$. We say ϕ is *probability-preserving* with respect to $\mathbf{P}_1$ and $\mathbf{P}_2$ if ϕ is measurable and, for every $\boldsymbol{\mathcal{R}}_2 \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2)$, if $\boldsymbol{\mathcal{R}}_1 := \phi^{-1}(\boldsymbol{\mathcal{R}}_2)$, then $\mathbf{p}_1^{\boldsymbol{\mathcal{R}}_1} = \mathbf{p}_2^{\boldsymbol{\mathcal{R}}_2}$.

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Part III

Concretization

Quasiconstant morphisms

Let $\mathcal{C}$ be a category. An object $\mathcal{B}$ in $\mathcal{C}$ is *null* if $\overrightarrow{\mathcal{C}}(\mathcal{A},\mathcal{B}) = \emptyset$ for all $\mathcal{A} \in [\mathcal{C}]$.
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Let $\mathcal{B}, \mathcal{C} \in [\mathcal{C}]$ be non-null, and let $\kappa \in \overrightarrow{\mathcal{C}}(\mathcal{B},\mathcal{C})$.

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(a) In a concrete category, any constant morphism is quasiconstant.
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We will use these quasiconstant morphisms to construct a "concrete" representation of $\mathcal{C}$....

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We say $\mathcal{C}$ is *biconnected* if $\overrightarrow{\mathcal{C}}(\mathcal{A},\mathcal{B})$ is nonempty for all non-null $\mathcal{A},\mathcal{B} \in \mathcal{C}$.

Example. Set, Meas, Top, Diff, etc. are biconnected.

Suppose $\mathcal{C}$ is a biconnected category. We can use quasiconstant morphisms to define a *concretization functor* from $\mathcal{C}$ into Set, as follows...

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Part IV

Products, spans, and quasipreferences

Spans: executive summary

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- Let $\mathcal{X}$ be an outcome place in $[\boldsymbol{\mathcal{X}}]$.
- A *span* on $\mathcal{X}$ is a categorical construction which plays the role of a binary relation on $\mathcal{X}$.
- Let $[\trianglerighteq]$ be a span on $\mathcal{X}$. Then we can define a binary relation $\widetilde{\trianglerighteq}$ on $\widetilde{\mathcal{X}}$. For us, $\widetilde{\trianglerighteq}$ will play the role of the *ex post preferences order*.
- For any state place $\mathcal{S} \in [\boldsymbol{\mathcal{S}}]$, the span $[\trianglerighteq]$ induces a binary relation $\trianglerighteq$ on $\overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{X})$. $\trianglerighteq$ will play the role of the *statewise dominance order* induced by $\widetilde{\trianglerighteq}$.
- If $[\trianglerighteq]$ satisfies reasonable conditions, then $\trianglerighteq$ and $\widetilde{\trianglerighteq}$ are reflexive and transitive, and $\widetilde{\trianglerighteq}$ is also complete (i.e. it is a preference order on $\widetilde{\mathcal{X}}$).
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Products

Let $\mathcal{C}$ be a category, and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be objects in $\mathcal{C}$.

A *product* of $\mathcal{S}_1$ and $\mathcal{S}_2$ is a triple $(\mathcal{S}; \pi_1, \pi_2)$, where $\mathcal{S}$ is another object in $\mathcal{C}$, and where $\pi_1 \in \overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{S}_1)$ and $\pi_2 \in \overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{S}_2)$ are morphisms (called *projections*) with the following property: for *any* other object $\mathcal{R}$ in $\mathcal{C}$, and *any* morphisms $f_1 \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{S}_1)$ and $f_2 \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{S}_2)$, there is a unique morphism $F \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{S})$ such that the following diagram commutes:

$$\begin{tikzcd}
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\mathcal{S}_1 & \mathcal{S} \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathcal{S}_2
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Example. In most concrete categories, $\mathcal{S}$ is the Cartesian product $\mathcal{S}_1 \times \mathcal{S}_2$ (equipped with the suitable "product" structure), while π_1 and π_2 are the coordinate projection maps (i.e. $\pi_1(s_1, s_2) = s_1$ and $\pi_2(s_1, s_2) = s_2$). For any $f_1 \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{S}_1)$ and $f_2 \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{S}_2)$, we get a function $F : \mathcal{R} \longrightarrow \mathcal{S}_1 \times \mathcal{S}_2$ defined by $F(r) := (f_1(r), f_2(r))$, for all $r \in \mathcal{R}$.

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A *product* of $\mathcal{S}_1$ and $\mathcal{S}_2$ is a triple $(\mathcal{S}; \pi_1, \pi_2)$, where $\mathcal{S}$ is another object in $\mathcal{C}$, and where $\pi_1 \in \overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{S}_1)$ and $\pi_2 \in \overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{S}_2)$ are morphisms (called *projections*) with the following property: for *any* other object $\mathcal{R}$ in $\mathcal{C}$, and *any* morphisms $f_1 \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{S}_1)$ and $f_2 \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{S}_2)$, there is a unique morphism $F \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{S})$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{R} \arrow[dl, "f_1"'] \arrow[d, dashed, "F"] \arrow[dr, "f_2"] & \\
\mathcal{S}_1 & \mathcal{S} \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathcal{S}_2
\end{tikzcd}$$

Example. In most concrete categories, $\mathcal{S}$ is the Cartesian product $\mathcal{S}_1 \times \mathcal{S}_2$ (equipped with the suitable "product" structure), while π_1 and π_2 are the coordinate projection maps (i.e. $\pi_1(s_1, s_2) = s_1$ and $\pi_2(s_1, s_2) = s_2$).
For any $f_1 \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{S}_1)$ and $f_2 \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{S}_2)$, we get a function $F : \mathcal{R} \longrightarrow \mathcal{S}_1 \times \mathcal{S}_2$ defined by $F(r) := (f_1(r), f_2(r))$, for all $r \in \mathcal{R}$.

Spans

Let $\mathcal{X} \in [\mathcal{C}]$. A *span* on $\mathcal{X}$ is a structure $\langle\triangleright\rangle = (\mathcal{Q}; q_1, q_2)$, where $\mathcal{Q}$ is another object in $\mathcal{C}$, and where $q_1, q_2 \in \overrightarrow{\mathcal{C}}(\mathcal{Q}, \mathcal{X})$.

Prototypical example. Suppose the product object $\mathcal{X} \times \mathcal{X}$ existed in $\mathcal{C}$. Let $\mathcal{Q} \overset{\iota}{\hookrightarrow} \mathcal{X} \times \mathcal{X}$ be a subobject of $\mathcal{X} \times \mathcal{X}$ (e.g. a *binary relation*). Construct the following commuting diagram:

$$\begin{tikzcd} & \mathcal{Q} \arrow[dl, dashed, "q_1"'] \arrow[d, hook, "\iota"] \arrow[dr, dashed, "q_2"] & \\ \mathcal{X} & \mathcal{X} \times \mathcal{X} \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathcal{X} \end{tikzcd}$$

Then $(\mathcal{Q}; q_1, q_2)$ is a span on $\mathcal{X}$.

As this example shows, spans generalize binary relations.

Indeed, if $\mathcal{C} = \mathrm{Set}$, then spans are equivalent to binary relations.

In other categories, the link from spans to relations on $\mathcal{X}$ is more subtle.

However, each span determines binary relations on morphisms and quasielements, as we now explain....

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Let $\alpha, \beta \in \overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{X})$. Define $\alpha \trianglerighteq \beta$ if there is a morphism $r \in \overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{Q})$

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Example. Suppose $\boldsymbol{\mathcal{C}} = \mathrm{Set}$, and $\langle\trianglerighteq\rangle$ represents a binary relation $\trianglerighteq$ on $\mathcal{X}$. If $\alpha, \beta : \mathcal{S} \longrightarrow \mathcal{X}$ are functions, then $(\alpha \trianglerighteq \beta) \Leftrightarrow \Big(\alpha(s) \trianglerighteq \beta(s)$ for all $s \in \mathcal{S}\Big)$. If $\trianglerighteq$ is a preference order on $\mathcal{X}$, this says that α *statewise dominates* β.

Given any quasielements $x_1, x_2 \in \widetilde{\mathcal{X}}$, and any $\mathcal{S} \in [\boldsymbol{\mathcal{C}}]_+$, we define

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This defines a relation $\widetilde{\trianglerighteq}$ on $\widetilde{\mathcal{X}}$ (independent of the choice of $\mathcal{S}$). If $\mathcal{C} = \mathrm{Set}$, then *every* binary relation on $\widetilde{\mathcal{X}}$ comes from a span in this way.

For us, $\widetilde{\trianglerighteq}$ will play the role of the *ex post* preference relation, and $\trianglerighteq$ will be the "statewise dominance" relation induced by $\widetilde{\trianglerighteq}$.

Spans define binary relations on morphisms

Let $\langle \trianglerighteq \rangle = (\mathcal{Q}; q_1, q_2)$ be a span on $\mathcal{X}$, and let $\mathcal{S}$ be another object in $\boldsymbol{\mathcal{C}}$.
Let $\alpha, \beta \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}, \mathcal{X})$. Define $\alpha \trianglerighteq \beta$ if there is a morphism $r \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}, \mathcal{Q})$

which makes this diagram commute:

$$\begin{tikzcd}
& \mathcal{S} \arrow[dl, "\alpha"'] \arrow[d, dashed, "r"] \arrow[dr, "\beta"] & \\
\mathcal{X} & \mathcal{Q} \arrow[l, "q_1"] \arrow[r, "q_2"'] & \mathcal{X}
\end{tikzcd}$$

Example. Suppose $\boldsymbol{\mathcal{C}} = \mathrm{Set}$, and $\langle \trianglerighteq \rangle$ represents a binary relation $\trianglerighteq$ on $\mathcal{X}$.
If $\alpha, \beta : \mathcal{S} \longrightarrow \mathcal{X}$ are functions, then $(\alpha \trianglerighteq \beta) \Leftrightarrow \Big(\alpha(s) \trianglerighteq \beta(s) \text{ for all } s \in \mathcal{S}\Big)$.
If $\trianglerighteq$ is a preference order on $\mathcal{X}$, this says that α *statewise dominates* β.

Given any quasielements $x_1, x_2 \in \widetilde{\mathcal{X}}$, and any $\mathcal{S} \in [\boldsymbol{\mathcal{C}}]_+$, we define

$$\Big(x_1 \,\widetilde{\trianglerighteq}\, x_2\Big) \iff \Big(\exists\ \kappa_1, \kappa_2 \in \mathcal{K}(\mathcal{S}, \mathcal{X}) \text{ with } x_1 = \overline{\kappa}_1,\ \ x_2 = \overline{\kappa}_2, \text{ and } \kappa_1 \trianglerighteq \kappa_2\Big).$$

This defines a relation $\widetilde{\trianglerighteq}$ on $\widetilde{\mathcal{X}}$ (independent of the choice of $\mathcal{S}$).
If $\boldsymbol{\mathcal{C}} = \mathrm{Set}$, then *every* binary relation on $\widetilde{\mathcal{X}}$ comes from a span in this way.
For us, $\widetilde{\trianglerighteq}$ will play the role of the *ex post* preference relation, and $\trianglerighteq$ will be the "statewise dominance" relation induced by $\widetilde{\trianglerighteq}$.

Let $\langle \trianglerighteq \rangle = (\mathcal{Q}; q_1, q_2)$ and $\langle \trianglerighteq' \rangle = (\mathcal{Q}'; q_1', q_2')$ be two spans on $\mathcal{X}$.

We say that $\langle \trianglerighteq \rangle$ and $\langle \trianglerighteq' \rangle$ are *equivalent* if there are morphisms $f \in \overrightarrow{\mathcal{C}}(\mathcal{Q}', \mathcal{Q})$ and $g \in \overrightarrow{\mathcal{C}}(\mathcal{Q}, \mathcal{Q}')$ such that this diagram commutes:

$$
\begin{tikzcd}
 & \mathcal{Q}' \arrow[dl, "q_1'"'] \arrow[dr, "q_2'"] \arrow[d, dashed, bend left, "f"] & \\
\mathcal{X} & \mathcal{Q} \arrow[l, "q_1"] \arrow[r, "q_2"'] \arrow[u, dashed, bend left, "g"] & \mathcal{X}
\end{tikzcd}
$$

Let $[\trianglerighteq]$ denote the equivalence class of $\langle \trianglerighteq \rangle$. We call $[\trianglerighteq]$ a *quasirelation*.

If two spans are equivalent, then they induce the same relation $\trianglerighteq$ on $\overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{X})$, and the same relation $\widetilde{\trianglerighteq}$ on $\widetilde{\mathcal{X}}$.

Thus, $\trianglerighteq$ and $\widetilde{\trianglerighteq}$ can be associated to the entire quasirelation $[\trianglerighteq]$.

If $[\trianglerighteq]$ satisfies reasonable conditions, then $\trianglerighteq$ and $\widetilde{\trianglerighteq}$ are reflexive and transitive, and $\widetilde{\trianglerighteq}$ is also complete (i.e. it is a preference order on $\widetilde{\mathcal{X}}$).

In this case, we say that $[\trianglerighteq]$ is a *quasipreference* on $\mathcal{X}$.

Let $\langle \trianglerighteq \rangle = (\mathcal{Q}; q_1, q_2)$ and $\langle \trianglerighteq' \rangle = (\mathcal{Q}'; q_1', q_2')$ be two spans on $\mathcal{X}$.
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Quasirelations and quasipreferences

Let $\langle\trianglerighteq\rangle = (\mathcal{Q}; q_1, q_2)$ and $\langle\trianglerighteq'\rangle = (\mathcal{Q}'; q_1', q_2')$ be two spans on $\mathcal{X}$.
We say that $\langle\trianglerighteq\rangle$ and $\langle\trianglerighteq'\rangle$ are *equivalent* if there are morphisms $f \in \overrightarrow{\mathcal{C}}(\mathcal{Q}', \mathcal{Q})$ and $g \in \overrightarrow{\mathcal{C}}(\mathcal{Q}, \mathcal{Q}')$ such that this diagram commutes:

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 & \mathcal{Q}' \arrow[dl, "q_1'"'] \arrow[dr, "q_2'"] \arrow[d, dashed, bend left, "f"] & \\
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If two spans are equivalent, then they induce the same relation $\trianglerighteq$ on $\overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{X})$, and the same relation $\widetilde{\trianglerighteq}$ on $\widetilde{\mathcal{X}}$.

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Quasirelations and quasipreferences

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In this case, we say that $[\trianglerighteq]$ is a *quasipreference* on $\mathcal{X}$.

Quasirelations and quasipreferences

Let $\langle \unrhd \rangle = (\mathcal{Q}; q_1, q_2)$ and $\langle \unrhd' \rangle = (\mathcal{Q}'; q_1', q_2')$ be two spans on $\mathcal{X}$.
We say that $\langle \unrhd \rangle$ and $\langle \unrhd' \rangle$ are *equivalent* if there are morphisms $f \in \overrightarrow{\mathcal{C}}(\mathcal{Q}', \mathcal{Q})$ and $g \in \overrightarrow{\mathcal{C}}(\mathcal{Q}, \mathcal{Q}')$ such that this diagram commutes:

$$
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 & \mathcal{Q}' \arrow[dl, "q_1'"'] \arrow[dr, "q_2'"] \arrow[d, dashed, bend left, "f"] & \\
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\end{tikzcd}
$$

Let $[\unrhd]$ denote the equivalence class of $\langle \unrhd \rangle$. We call $[\unrhd]$ a *quasirelation*.

If two spans are equivalent, then they induce the same relation $\unrhd$ on $\overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{X})$, and the same relation $\widetilde{\unrhd}$ on $\widetilde{\mathcal{X}}$.

Thus, $\unrhd$ and $\widetilde{\unrhd}$ can be associated to the entire quasirelation $[\unrhd]$.

If $[\unrhd]$ satisfies reasonable conditions, then $\unrhd$ and $\widetilde{\unrhd}$ are reflexive and transitive, and $\widetilde{\unrhd}$ is also complete (i.e. it is a preference order on $\widetilde{\mathcal{X}}$).

In this case, we say that $[\unrhd]$ is a *quasipreference* on $\mathcal{X}$.

Let $\mathcal{X} \in [\mathcal{C}]$. A function $u : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ is a *$\mathcal{C}$-compatible utility function* if there is a quasipreference $[\trianglerighteq]$ on $\mathcal{X}$ for which u is an *ordinal representation*:

$$\left(x \widetilde{\trianglerighteq} y\right) \iff \left(u(x) \geq u(y)\right), \qquad \text{for all } x, y \in \widetilde{\mathcal{X}}.$$

Example. If $\mathcal{C} = \mathrm{Set}$, then *every* real-valued function on $\widetilde{\mathcal{X}}$ is a compatible utility function. But in other categories, this is not necessarily the case.

For example, let $\mathcal{C} = \mathrm{Cpct}$, the category of compact spaces and continuous maps, and let $\mathcal{X} \in [\mathrm{Cpct}]$.

Then a function $u : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ is a Cpct-compatible utility function if and only if it is an increasing transform of a continuous, $\mathbb{R}$-valued function on $\mathcal{X}$.

(This means, in particular, that u must be Borel-measurable.)

Compatible utility functions

Let $\mathcal{X} \in [\mathcal{C}]$. A function $u : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ is a *$\mathcal{C}$-compatible utility function* if there is a quasipreference $[\trianglerighteq]$ on $\mathcal{X}$ for which u is an *ordinal representation*:

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Let $\mathcal{X} \in [\mathcal{C}]$. A function $u : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ is a $\mathcal{C}$*-compatible utility function* if there is a quasipreference $[\trianglerighteq]$ on $\mathcal{X}$ for which u is an *ordinal representation*:

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(This means, in particular, that u must be Borel-measurable.)

Part V

From simple morphisms to SEU representations

Simple morphisms

Let $\boldsymbol{\mathcal{R}} = (\mathcal{R};\ \iota_1, \ldots, \iota_N)$ be a coproduct of some objects $\mathcal{R}_1, \ldots, \mathcal{R}_N \in [\boldsymbol{\mathcal{C}}]$.

Let $\mathcal{X} \in [\mathcal{C}]$ be another object.

For all $n \in [1 \ldots N]$, let $\sigma_n \in \mathcal{K}(\mathcal{R}_n, \mathcal{X})$ be a quasiconstant morphism. Let $x_n \in \widetilde{\mathcal{X}}$ be its $\sim$-equivalence class (the "value" of σ_n).

By the defining property of coproducts, there is a unique morphism $\sigma = [\sigma_1|\cdots|\sigma_N] \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{X})$ such that this diagram commutes:

$$\begin{tikzcd} & & & & \mathcal{R} \arrow[dd, dashed, "\sigma"] \\ \mathcal{R}_1 \arrow[urrrr, "\iota_1"] \arrow[drrrr, "\sigma_1"'] & \mathcal{R}_2 \arrow[urrr, "\iota_2"] \arrow[drrr, "\sigma_2"'] & \cdots & \mathcal{R}_N \arrow[ur, "\iota_N"] \arrow[dr, "\sigma_N"'] & \\ & & & & \mathcal{X} \end{tikzcd}$$

Say σ is a *simple morphism* from $\mathcal{R}$ into $\mathcal{X}$, and write "$\sigma = [x_1|\cdots|x_N]$".

Idea: σ is "constant" when restricted to each of $\mathcal{R}_1, \ldots, \mathcal{R}_N$.

Let $\Sigma(\mathcal{R}, \mathcal{X})$ be the set of all simple morphisms from $\mathcal{R}$ to $\mathcal{X}$ which are compatible with the coproduct structure of $\mathcal{R}$.

Simple morphisms

Let $\boldsymbol{\mathcal{R}} = (\mathcal{R};\ \iota_1, \ldots, \iota_N)$ be a coproduct of some objects $\mathcal{R}_1, \ldots, \mathcal{R}_N \in [\boldsymbol{\mathcal{C}}]$.

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$$\begin{tikzcd}
 & & & & \mathcal{R} \arrow[dd, dashed, "\sigma"] \\
\mathcal{R}_1 \arrow[urrrr, "\iota_1"] \arrow[drrrr, "\sigma_1"'] & \mathcal{R}_2 \arrow[urrr, "\iota_2"] \arrow[drrr, "\sigma_2"] & \cdots & \mathcal{R}_N \arrow[ur, "\iota_N"] \arrow[dr, "\sigma_N"] & \\
 & & & & \mathcal{X}
\end{tikzcd}$$

Say σ is a *simple morphism* from $\mathcal{R}$ into $\mathcal{X}$, and write "$\sigma = [x_1 | \cdots | x_N]$".

Idea: σ is "constant" when restricted to each of $\mathcal{R}_1, \ldots \mathcal{R}_N$.

Let $\Sigma(\boldsymbol{\mathcal{R}}, \mathcal{X})$ be the set of all simple morphisms from $\mathcal{R}$ to $\mathcal{X}$ which are compatible with the coproduct structure of $\boldsymbol{\mathcal{R}}$.

Simple morphisms

Let $\boldsymbol{\mathcal{R}} = (\mathcal{R};\ \iota_1, \ldots, \iota_N)$ be a coproduct of some objects $\mathcal{R}_1, \ldots, \mathcal{R}_N \in [\boldsymbol{\mathcal{C}}]$.

Let $\mathcal{X} \in [\boldsymbol{\mathcal{C}}]$ be another object.

For all $n \in [1 \ldots N]$, let $\sigma_n \in \mathcal{K}(\mathcal{R}_n, \mathcal{X})$ be a quasiconstant morphism. Let $x_n \in \widetilde{\mathcal{X}}$ be its $\sim$-equivalence class (the "value" of σ_n).

By the defining property of coproducts, there is a unique morphism $\sigma = [\sigma_1|\cdots|\sigma_N] \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{R}, \mathcal{X})$ such that this diagram commutes:

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Let $\mathbf{P} = (\mathbf{p}^{\mathcal{R}})_{\boldsymbol{\mathcal{R}} \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})}$ be a probability structure on $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})$.

Let $\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N;\ \mathcal{R};\ \rho) \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})$ (a partition of $\mathcal{S}$).

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Suppose $\sigma = [x_1 | \cdots | x_N]$, for some quasielements $x_1, \ldots, x_N \in \widetilde{\mathcal{X}}$

Let $u : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ be a real-valued function (e.g. a "utility function").

We define the *expected utility* of σ, with respect to u and $\mathbf{P}$, as follows:

$$\mathbb{E}^u_{\mathbf{P}}[\sigma] \quad := \quad \sum_{n=1}^{N} p_n^{\boldsymbol{\mathcal{R}}}\, u(x_n),$$

where $\mathbf{p}^{\boldsymbol{\mathcal{R}}} = (p_1^{\boldsymbol{\mathcal{R}}}, \ldots, p_N^{\boldsymbol{\mathcal{R}}})$.

Expected utility for simple morphisms

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Now, let $u : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ be a $\mathcal{C}$-compatible utility function, representing a quasipreference $[\trianglerighteq]$. Meanwhile, let $\alpha \in \overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{X})$ be any morphism.

We define $\underline{\mathbb{E}}^u_{\mathbf{P}}[\alpha]$ and $\overline{\mathbb{E}}^u_{\mathbf{P}}[\alpha]$, the *lower* and *upper expected utilities* of α with respect to u and $\mathbf{P}$, by approximating α "from below" and "from above" (in terms of $[\trianglerighteq]$) by simple morphisms on partitions of $\mathcal{S}$.

If $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$ satisfies the Common Refinement Property, then $\overline{\mathbb{E}}^u_{\mathbf{P}}$ and $\underline{\mathbb{E}}^u_{\mathbf{P}}$ have most of the properties you would expect from a notion of "expected utility".

If $\underline{\mathbb{E}}^u_{\mathbf{P}}[\alpha] = \overline{\mathbb{E}}^u_{\mathbf{P}}[\alpha]$, then we denote their common value by $\mathbb{E}^u_{\mathbf{P}}[\alpha]$, and we say that α is $(u, \mathbf{P})$-*integrable*.

(But in fact, we don't need $(u, \mathbf{P})$-integrable morphisms.) [skip details]

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Expected utility for any morphism (informal)

We define the *expected utility* of a simple morphism σ, with respect to u and $\mathbf{P}$:

$$\mathbb{E}^u_{\mathbf{P}}[\sigma] \quad := \quad \sum_{n=1}^{N} p_n^{\mathcal{R}}\, u(x_n),$$

where $\mathbf{p}^{\mathcal{R}} = (p_1^{\mathcal{R}}, \ldots, p_N^{\mathcal{R}})$, and where $x_1, \ldots, x_N \in \widetilde{\mathcal{X}}$ are such that $\sigma = [x_1|\cdots|x_N]$.

Now, let $u : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ be a $\mathcal{C}$-compatible utility function, representing a quasipreference $[\trianglerighteq]$. Meanwhile, let $\alpha \in \overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{X})$ be any morphism.

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Now, let $u : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ be a $\mathcal{C}$-compatible utility function, representing a quasipreference $[\underline{\triangleright}]$. Meanwhile, let $\alpha \in \overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{X})$ be any morphism.

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Let $\mathcal{X} \in [\mathcal{C}]$ and let $\sigma \in \overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{X})$. Thus, $\sigma' := \sigma \circ \rho \in \overrightarrow{\mathcal{C}}(\mathcal{R}, \mathcal{X})$.

If σ' is a simple morphism on $\mathcal{R}$, then we will say that σ is a *simple morphism on $\mathcal{S}$ subordinate to the partition $\boldsymbol{\mathcal{R}}$.*

Problem. In many categories (e.g. Top, Diff), the only simple morphisms on $\mathcal{S}$ are the constant functions....

Solution. Treat the simple morphisms in $\Sigma_{\mathcal{S}}(\boldsymbol{\mathcal{R}}, \mathcal{X})$ as "virtual" simple morphisms on $\mathcal{S}$ itself.

Formally, a *virtual simple morphism* on $\mathcal{S}$ is a structure (σ, ρ), where $\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N;\ \mathcal{R};\ \rho)$ is a partition of $\mathcal{S}$, and $\sigma \in \Sigma(\boldsymbol{\mathcal{R}}, \mathcal{X})$.

Let $\Sigma_{\mathcal{S}}(\mathcal{S}, \mathcal{X})$ be the set of all virtual simple morphisms from $\mathcal{S}$ into $\mathcal{X}$ arising from partitions in $\mathfrak{R}_{\mathcal{S}}(\mathcal{S})$. Formally,

$$\Sigma_{\mathcal{S}}(\mathcal{S}, \mathcal{X}) := \{(\sigma, \rho)\ ;\ \boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N;\ \mathcal{R}, \rho) \in \mathfrak{R}_{\mathcal{S}}(\mathcal{S})\ \ \&\ \ \sigma \in \Sigma(\boldsymbol{\mathcal{R}}, \mathcal{X})\}.$$

Virtual simple morphisms

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Expected utility for arbitrary morphisms

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Subjective expected utility representations

Let $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$ be a decision context in a category $\boldsymbol{\mathcal{C}}$.

For every $\mathcal{S} \in [\boldsymbol{\mathcal{S}}]$, let $\mathbf{P}_\mathcal{S}$ be a probability structure on $\mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})$.
For every $\mathcal{X} \in [\boldsymbol{\mathcal{X}}]$, let $u_\mathcal{X} : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ be a $\boldsymbol{\mathcal{C}}$-compatible utility function.
The structure $[(\mathbf{P}_\mathcal{S})_{\mathcal{S} \in [\boldsymbol{\mathcal{S}}]}, (u_\mathcal{X})_{\mathcal{X} \in [\boldsymbol{\mathcal{X}}]}]$ is an *SEU structure* on $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$ if:

(PP) For all $\mathcal{S}_1, \mathcal{S}_2 \in [\boldsymbol{\mathcal{S}}]$, every measurable morphism in $\overrightarrow{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1, \mathcal{S}_2)$ is probability-preserving with respect to $\mathbf{P}_{\mathcal{S}_1}$ and $\mathbf{P}_{\mathcal{S}_2}$.

(UP) For all $\mathcal{X}_1, \mathcal{X}_2 \in [\boldsymbol{\mathcal{X}}]$, and every $\phi \in \overrightarrow{\boldsymbol{\mathcal{X}}}(\mathcal{X}_1, \mathcal{X}_2)$, the composition $u_{\mathcal{X}_2} \circ \widetilde{\phi}$ is a *positive affine transformation* of $u_{\mathcal{X}_1}$ —that is, there exist $A > 0$ and $B \in \mathbb{R}$ such that $u_{\mathcal{X}_2}[\widetilde{\phi}(x)] \ = \ A\, u_{\mathcal{X}_1}(x) + B$ for all $x \in \widetilde{\mathcal{X}}_1$.

This SEU structure *represents* a Savage structure $\mathfrak{S} = (\succeq^{\mathcal{S}}_{\mathcal{X}})^{\mathcal{S} \in [\boldsymbol{\mathcal{S}}]}_{\mathcal{X} \in [\boldsymbol{\mathcal{X}}]}$ if, for every $\mathcal{S} \in [\boldsymbol{\mathcal{S}}]$, and every $\mathcal{X} \in [\boldsymbol{\mathcal{X}}]$, and all $\alpha, \beta \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}, \mathcal{X})$, we have

$$\left(\alpha \succ^{\mathcal{S}}_{\mathcal{X}} \beta\right) \iff \left(\underline{\mathbb{E}}^u_{\mathbf{P}}[\alpha] > \overline{\mathbb{E}}^u_{\mathbf{P}}[\beta]\right),$$

In particular, if α and β are both $(u, \mathbf{P})$-integrable, then this implies:

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Subjective expected utility representations

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Part VI

Formal statement of axioms and main result

Structural conditions (S1)-(S3)

The decision context $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$ must satisfy three structural conditions:

(S1) Every pair of state places in $[\boldsymbol{\mathcal{S}}]$ have a product in the category $\boldsymbol{\mathcal{S}}$.

(S2) Every pair of outcome places in $[\boldsymbol{\mathcal{X}}]$ have a coproduct in $\boldsymbol{\mathcal{X}}$.

(S3) Consider a pullback diagram in the category $\boldsymbol{\mathcal{C}}$:

$$\begin{tikzcd} \mathcal{S}_\ulcorner \arrow[r, dashed, "\tau"] \arrow[d, dashed, "\lambda"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}_\urcorner \arrow[d, "\rho"] \\ \mathcal{S}_\llcorner \arrow[r, "\beta"'] & \mathcal{S}_\lrcorner \end{tikzcd}$$

If $\mathcal{S}_\urcorner$, $\mathcal{S}_\llcorner$, and $\mathcal{S}_\lrcorner$ are all in $[\boldsymbol{\mathcal{S}}]$, and ρ and β are $\boldsymbol{\mathcal{S}}$-morphisms, then $\mathcal{S}_\ulcorner$ is also in $[\boldsymbol{\mathcal{S}}]$, and τ and λ are also $\boldsymbol{\mathcal{S}}$-morphisms.

Interpretation: Given any two "random variables" (e.g. any two state places $\mathcal{S}_1$ and $\mathcal{S}_2$), (S1) says we can *couple* them into a single "random variable" (namely $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$) such that $\mathcal{S}_1$ and $\mathcal{S}_2$ are"marginals" of $\mathcal{S}$. ($\mathcal{S}_1$ and $\mathcal{S}_2$ might not be *independent* random variables in this coupling.)

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Structural condition (S3)

Suppose $\boldsymbol{\mathcal{C}}$ is pullback-complete. Then (S3) is equivalent to:

(S3′) For any $\mathcal{S}_{\urcorner}, \mathcal{S}_{\llcorner}, \mathcal{S}_{\lrcorner} \in [\boldsymbol{\mathcal{S}}]$, and any $\beta \in \overrightarrow{\boldsymbol{\mathcal{S}}}(\mathcal{S}_{\llcorner}, \mathcal{S}_{\lrcorner})$ and $\rho \in \overrightarrow{\boldsymbol{\mathcal{S}}}(\mathcal{S}_{\urcorner}, \mathcal{S}_{\lrcorner})$, there exists a fourth state place $\mathcal{S}_{\ulcorner}$, along with $\boldsymbol{\mathcal{S}}$-morphisms τ and λ yielding the following pullback diagram in the category $\boldsymbol{\mathcal{C}}$:

$$\begin{tikzcd}
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Solvability

Let $\boldsymbol{\mathcal{R}} = (\mathcal{R}; \iota_1, \ldots, \iota_N)$ be a coproduct of objects $\mathcal{R}_1, \ldots, \mathcal{R}_N \in [\boldsymbol{\mathcal{C}}]$. Let

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For any simple morphism $\sigma \in \Sigma(\boldsymbol{\mathcal{R}}, \mathcal{X})$, there exist quasielements $x_1, \ldots, x_N \in \widetilde{\mathcal{X}}$ such that $\sigma = [x_1 | \cdots | x_N]$.

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Axioms (A1) and (A2)

The Savage structure $\mathfrak{S} = (\succeq^{\mathcal{S}}_{\mathcal{X}})^{\mathcal{S}\in[\boldsymbol{\mathcal{S}}]}_{\mathcal{X}\in[\boldsymbol{\mathcal{X}}]}$ must satisfy five axioms.....

Recall: $\mathcal{K}(\mathcal{S},\mathcal{X})$ is the set of quasiconstant morphisms from $\mathcal{S}$ to $\mathcal{X}$. Heuristically, these represent "perfectly predictable" (i.e. "riskless") acts.

Notation: For any $\kappa \in \mathcal{K}(\mathcal{S},\mathcal{X})$, let $\overline{\kappa} \in \widetilde{\mathcal{X}}$ denote its $\sim$-equivalence class.

For any $\mathcal{X} \in [\boldsymbol{\mathcal{X}}]$, we require a quasipreference $[\succeq^{\mathrm{dom}}_{\mathcal{X}}]$ on $\mathcal{X}$ satisfying:

(A1) (*Ex post preferences*) Let $\succeq^{\mathrm{xp}}_{\mathcal{X}}$ be the preference order that $[\succeq^{\mathrm{dom}}_{\mathcal{X}}]$ induces on $\widetilde{\mathcal{X}}$. Then $\succeq^{\mathrm{xp}}_{\mathcal{X}}$ is nontrivial, and for any $\mathcal{S} \in [\boldsymbol{\mathcal{S}}]$ and any $\kappa_1, \kappa_2 \in \mathcal{K}(\mathcal{S},\mathcal{X})$, we have $\kappa_1 \succeq^{\mathcal{S}}_{\mathcal{X}} \kappa_2$ if and only if $\overline{\kappa}_1 \succeq^{\mathrm{xp}}_{\mathcal{X}} \overline{\kappa}_2$.

(A2) (*Statewise dominance*) For any $\mathcal{S} \in [\boldsymbol{\mathcal{S}}]$ and any $\alpha, \beta \in \overrightarrow{\mathcal{C}}(\mathcal{S},\mathcal{X})$, if $\alpha \succeq^{\mathrm{dom}}_{\mathcal{X}} \beta$, then $\alpha \succeq^{\mathcal{S}}_{\mathcal{X}} \beta$.

Interpretation: $\succeq^{\mathrm{xp}}_{\mathcal{X}}$ is the agent's "*ex post* preference relation" on $\widetilde{\mathcal{X}}$. (A1) says that $\succeq^{\mathrm{xp}}_{\mathcal{X}}$ governs the agent's preferences over "riskless" acts. If $\alpha \succeq^{\mathrm{dom}}_{\mathcal{X}} \beta$, then α delivers a better *ex post* outcome than β *in all circumstances*. Then (A2) says the agent should prefer α over β *ex ante*.

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Axiom (A3): Simple density

Recall. If $\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N; \mathcal{R}, \rho)$ is a partition of $\mathcal{S}$, then

$$\Sigma(\boldsymbol{\mathcal{R}}, \mathcal{X}) \quad := \quad \{\text{all simple morphisms from } \mathcal{R} \text{ to } \mathcal{X}\}.$$

(A3) (*Simple density*) For any $\mathcal{S} \in \boldsymbol{\mathcal{S}}$ and $\mathcal{X} \in \boldsymbol{\mathcal{X}}$, and any $\alpha, \beta \in \overrightarrow{\mathcal{C}}(\mathcal{S}, \mathcal{X})$, if $\alpha \succ_{\mathcal{X}}^{\mathcal{S}} \beta$, then there exists a partition $\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \iota_1; \ldots; \mathcal{R}_N, \iota_N;\ \mathcal{R}, \rho) \in \mathfrak{R}_{\boldsymbol{\mathcal{S}}}(\mathcal{S})$, and two simple acts $\alpha', \beta' \in \Sigma(\boldsymbol{\mathcal{R}}, \mathcal{X})$ such that $\alpha \circ \rho \succeq_{\mathcal{X}}^{\mathrm{dom}} \alpha' \succ_{\mathcal{X}}^{\mathcal{R}} \beta' \succeq_{\mathcal{X}}^{\mathrm{dom}} \beta \circ \rho$.

Idea: We can "approximate" any acts on $\mathcal{S}$ with simple acts.

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Let $\boldsymbol{\mathcal{R}} = (\mathcal{R}; \iota_1, \ldots, \iota_N)$ be a coproduct of some objects $\mathcal{R}_1, \ldots, \mathcal{R}_N \in [\boldsymbol{\mathcal{C}}]$ and suppose $\mathcal{R} \in \boldsymbol{\mathcal{S}}$. Let $\mathcal{X} \in \boldsymbol{\mathcal{X}}$.

For any $w, x, y, z \in \widetilde{\mathcal{X}}$, write $(w \rightsquigarrow x) \cong (y \rightsquigarrow z)$ if there exists $\sigma, \tau \in \Sigma(\boldsymbol{\mathcal{R}}, \mathcal{X})$ and $n \in [1 \ldots N]$ such that $(w_n|\sigma) \approx_{\mathcal{X}}^{\mathcal{R}} (x_n|\tau)$ and $(y_n|\sigma) \approx_{\mathcal{X}}^{\mathcal{R}} (z_n|\tau)$.

Idea: The gain in changing w to x on $\mathcal{R}_n$ is exactly equal to the gain in changing y to z on $\mathcal{R}_n$ (because both are exactly cancelled by the loss of changing σ to τ on the complement of $\mathcal{R}_n$).

Thus, the "value difference" between w and x should be the same as the "value difference" between y and z.

Example: If $\mathcal{R}$ and $\mathcal{X}$ were sets, and $\succeq_{\mathcal{X}}^{\mathcal{R}}$ had an SEU representation with utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, then

$$(w \rightsquigarrow x) \cong (y \rightsquigarrow z) \text{ if and only if } u(w) - u(x) = u(y) - u(z).$$

The next axiom is due to Köbberling and Wakker (2003).

(A4) (*Tradeoff Consistency*) Let $(\mathcal{R}; \iota_1, ..., \iota_N)$ be a coproduct of $\mathcal{R}_1, ..., \mathcal{R}_N$, with $\mathcal{R} \in \boldsymbol{\mathcal{S}}$. For any $\mathcal{X} \in \boldsymbol{\mathcal{X}}$, and any $w, w', x, y, z \in \widetilde{\mathcal{X}}$, if $(w \rightsquigarrow x) \cong (y \rightsquigarrow z)$ and $(w' \rightsquigarrow x) \cong (y \rightsquigarrow z)$, then $w \approx_{\mathcal{X}}^{\mathrm{xp}} w'$.

Axiom (A4): Tradeoff consistency

Let $\boldsymbol{\mathcal{R}} = (\mathcal{R}; \iota_1, \ldots, \iota_N)$ be a coproduct of some objects $\mathcal{R}_1, \ldots, \mathcal{R}_N \in [\boldsymbol{\mathcal{C}}]$ and suppose $\mathcal{R} \in \boldsymbol{\mathcal{S}}$. Let $\mathcal{X} \in \boldsymbol{\mathcal{X}}$.
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Let $\boldsymbol{\mathcal{R}} = (\mathcal{R}; \iota_1, \ldots, \iota_N)$ be a coproduct of some objects $\mathcal{R}_1, \ldots, \mathcal{R}_N \in [\boldsymbol{\mathcal{C}}]$ and suppose $\mathcal{R} \in \boldsymbol{\mathcal{S}}$. Let $\mathcal{X} \in \boldsymbol{\mathcal{X}}$.
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Let $\mathcal{C}$ be any biconnected category (e.g. Set, Meas, Top, Diff, etc.)

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Thank you.

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Savage's Theorem
Desiderata I
Desiderata II

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Decision Contexts
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Partition categories and common refinement
Probability structures
Partition preimages and measurability in a nutshell
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Spans

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Expected utility

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Structural conditions (S1)-(S3)

Solvability

Axioms (A1) and (A2)

Axiom (A3): Simple density

Axiom (A4): Tradeoff consistency

Axiom (A5): Archimedeanism

SEU characterization theorem (formal statement)

Thank you