The proof theory of semi-De Morgan Algebras

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21th, Nov. 2016

Plan for talk

Part 1 Introduction to De Morgan and semi-De Morgan algebras

Part 2 Sequent calculus for semi-De Morgan algebras

Part 3 Display calculus for semi-De Morgan algebras

Part 4 Discussion about different non-classical negations

Part 5 Further work

The history of De Morgan Algebras

De Morgan algebras (also called “quasi-Boolean algebras”)

- were introduced by A. Bialynicki-Birula and H. Rasiowa, in "On the representation of quasi-Boolean algebras",1957.
- H.Rasiowa proposed a representation of De Morgan algebra in 1974
- In relevance logic, the logic of bilattices and pre-rough algebras, there are many applications of De Morgan algebra.

The history of Semi-De Morgan Algebras

semi-De Morgan algebras

- were originally introduced in "Semi-De Morgan algebra" , H. Sankappanavar 1987, as a common abstraction of De Morgan algebras and distributive pseudo-complemented lattices.
- D. Hobby presented a duality theory for semi-De Morgan algebras based on Priestly duality for distributive lattices in 1996.
- C. Palma and R. Santos investigated the Subvarieties of semi-De Morgan algebras in 2003.

De Morgan and Semi-De Morgan Algebras

Definition

If $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, then an algebra $\mathfrak{A} = (A, \vee, \wedge, \neg, 0, 1)$ is: for all $a, b \in A$:

De Morgan algebra	Semi-De Morgan algebra
$\neg(a \vee b) = \neg a \wedge \neg b$	$\neg(a \vee b) = \neg a \wedge \neg b$
$\neg(a \wedge b) = \neg a \vee \neg b$	$\neg\neg(a \wedge b) = \neg\neg a \wedge \neg\neg b$
$\neg\neg a = a$	$\neg\neg\neg a = \neg a$
$\neg 0 = 1, \neg 1 = 0$	$\neg 0 = 1$ and $\neg 1 = 0$

Notice that $a \wedge \neg a = 0$ and $a \vee \neg a = 1$ don't hold in both algebras!

De Morgan and Semi-De Morgan Algebras

The variety of all De Morgan algebras is denoted by dM, and the variety of all semi-De Morgan algebras is denoted by SdM.

Fact

A semi-De Morgan algebra $\mathfrak{A}$ is a De Morgan algebras if and only if $\mathfrak{A}$ satisfies the identity $a \vee b = \neg(\neg a \wedge \neg b)$.

Sequent calculus for semi-De Morgan algebras

- Language

$$\mathcal{T} \ni \varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi), \text{ where } p \in \Xi.$$

Define $\top := \neg\bot$. All terms are denoted by φ, ψ, χ etc. with or without subscripts.

Axioms

(Id)	$\varphi \vdash \varphi$	(D)	$\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
($\bot$)	$\bot \vdash \varphi$	($\neg\bot$)	$\varphi \vdash \neg\bot$
($\neg\neg\bot$)	$\neg\neg\bot \vdash \varphi$	($\neg\neg\neg$)	$\neg\neg\neg\varphi \dashv\vdash \neg\varphi$
($\neg\vee$)	$\neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi)$	($\neg\wedge$)	$\neg\neg\varphi \wedge \neg\neg\psi \vdash \neg\neg(\varphi \wedge \psi)$

Operation rules

- Rules for lattice

$$\frac{\varphi_i \vdash \psi}{\varphi_1 \wedge \varphi_2 \vdash \psi}\ (\wedge \vdash)(i = 1, 2) \qquad \frac{\varphi \vdash \psi \quad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}\ (\vdash \wedge)$$

$$\frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}\ (\vee \vdash) \qquad \frac{\varphi \vdash \psi_i}{\varphi \vdash \psi_1 \vee \psi_2}\ (\vdash \vee)(i = 1, 2)$$

- Cut rule:

$$\frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi}\ (\mathrm{Cut})$$

- Contraposition rule:

$$\frac{\varphi \vdash \psi}{\neg\psi \vdash \neg\varphi}\ (\mathrm{cp})$$

The basic sequent calculus for De Morgan algebras S_{dM} is obtained from S_{SdM} by adding the axiom $\varphi \vee \psi \dashv\vdash \neg(\neg\varphi \wedge \neg\psi)$.

Validity

Definition

Given a semi-De Morgan algebra $\mathfrak{A} = (A, \vee, \wedge, \neg, 0, 1)$, an *assignment* in $\mathfrak{A}$ is a function AtProp $\to A$. For any term $\varphi \in \mathcal{T}$ and assignment σ in $\mathfrak{A}$, define φ^σ inductively as follows:

$$p^\sigma = \sigma(p) \qquad \bot^\sigma = 0 \qquad (\neg\varphi)^\sigma = \neg\varphi^\sigma$$
$$(\varphi \wedge \psi)^\sigma = \varphi^\sigma \wedge \psi^\sigma \qquad (\varphi \vee \psi)^\sigma = \varphi^\sigma \vee \psi^\sigma$$

A sequent $\varphi \vdash \psi$ is said to be *valid* in a semi-De Morgan algebra $\mathfrak{A}$ if $\varphi^\sigma \leq \psi^\sigma$ for any assignment σ in $\mathfrak{A}$, where $\leq$ is the lattice order. For a class of semi-De Morgan algebras K, a sequent $\varphi \vdash \psi$ is *valid* in K if $\varphi \vdash \psi$ is valid in $\mathfrak{A}$ for all $\mathfrak{A} \in$ K.

Completeness

Theorem (Completeness)

For every sequent $\varphi \vdash \psi$,

1. $\varphi \vdash \psi$ *is derivable in* $\mathsf{S}_{\mathsf{SdM}}$ *if and only if* $\varphi \vdash \psi$ *is valid in* SdM*;*
2. $\varphi \vdash \psi$ *is derivable in* S_{dM} *if and only if* $\varphi \vdash \psi$ *is valid in* dM*.*

A G3-style Sequent Calculus for semi-De Morgan Algebras

See M. Ma and F. Liang. "Sequent calculi for semi-De Morgan and De Morgan algebras". Submitted. ArXiv preprint 1611.05231, 2016.

Definition

- Atomic G3SdM-structure
  φ or $*\varphi$ where φ is a term, denoted by α, β, γ etc.
- G3SdM-structure
  a multi-set of atomic structures, denoted by Γ, Δ,etc.
- Interpretation of structure

$*$		$,$	
$\neg$	$\neg$	$\wedge$	$\vee$

- G3SdM-sequent
  $\Gamma \vdash \alpha$, where Γ is an G3SdM-structure and α is an atomic G3SdM-structure.

Axioms

See O. Arieli and A. Avron. "The value of four values". *Artificial Intelligence*, 102:97-141, 1998.

(Id)	$p, \Gamma \vdash p$	$(\bot \vdash)$	$\bot, \Gamma \vdash \beta$
$(\vdash *\bot)$	$\Gamma \vdash *\bot$	$(*\neg\bot \vdash)$	$*\neg\bot, \Gamma \vdash \beta$

Operation rules

- operation rules

$$\frac{\varphi, \psi, \Gamma \vdash \beta}{\varphi \wedge \psi, \Gamma \vdash \beta} (\wedge \vdash) \quad \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} (\vdash \wedge)$$

$$\frac{\varphi, \Gamma \vdash \beta \qquad \psi, \Gamma \vdash \beta}{\varphi \vee \psi, \Gamma \vdash \beta} (\vee \vdash) \quad \frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_1 \vee \varphi_2} (\vdash \vee)(i \in \{1, 2\})$$

$$\frac{*\varphi, *\psi, \Gamma \vdash \beta}{*(\varphi \vee \psi), \Gamma \vdash \beta} (*\vee \vdash) \quad \frac{\Gamma \vdash *\varphi \qquad \Gamma \vdash *\psi}{\Gamma \vdash *(\varphi \vee \psi)} (\vdash *\vee)$$

$$\frac{*\neg\varphi, *\neg\psi, \Gamma \vdash \beta}{*\neg(\varphi \wedge \psi), \Gamma \vdash \beta} (*\neg\wedge \vdash) \quad \frac{\Gamma \vdash *\neg\varphi \qquad \Gamma \vdash *\neg\psi}{\Gamma \vdash *\neg(\varphi \wedge \psi)} (\vdash *\neg\wedge)$$

$$\frac{*\varphi, \Gamma \vdash \beta}{*\neg\neg\varphi, \Gamma \vdash \beta} (*\neg\neg \vdash) \quad \frac{\Gamma \vdash *\varphi}{\Gamma \vdash *\neg\neg\varphi} (\vdash *\neg\neg)$$

$$\frac{*\varphi, \Gamma \vdash \beta}{\neg\varphi, \Gamma \vdash \beta} (\neg \vdash) \quad \frac{\Gamma \vdash *\varphi}{\Gamma \vdash \neg\varphi} (\vdash \neg)$$

- structure rule

$$\frac{\varphi \vdash \psi}{*\psi, \Gamma \vdash *\varphi} (*)$$

Weakening admissible

Theorem

For any atomic G3SdM-structures α *and* β*, the weakening rule*

$$\frac{\Gamma \vdash \beta}{\alpha, \Gamma \vdash \beta}\,(Wk)$$

is height-preserving admissible in G3SdM*.*

Contraction admissible

Theorem

For any atomic G3SdM-structure α *and term* $\psi \in \mathcal{T}$*, the contraction rule*

$$\frac{\alpha, \alpha, \Gamma \vdash \psi}{\alpha, \Gamma \vdash \psi}\,(\mathit{Ctr})$$

is height-preserving derivable in G3SdM*.*

Cut admissible and decidability

Theorem

For any atomic G3SdM-structures α and β, the cut rule

$$\frac{\Gamma \vdash \alpha \quad \alpha, \Delta \vdash \beta}{\Gamma, \Delta \vdash \beta} \; (Cut)$$

is admissible in G3SdM.

Theorem (Decidability)

The derivability of an G3SdM-sequent in the calculus G3SdM *is decidable.*

Craig Interpolation

Definition

Given any G3SdM-sequent $\Gamma \vdash \beta$, we say that $(\Gamma_1; \emptyset)(\Gamma_2, \beta)$ is a *partition* of $\Gamma \vdash \beta$, if the multiset union of Γ_1 and Γ_2 is equal to Γ. An atomic G3SdM-structure α is called an *interpolant* of the partition $(\Gamma_1; \emptyset)(\Gamma_2, \beta)$ if the following conditions are satisfied:

1. G3SdM $\vdash \Gamma_1 \vdash \alpha$;
2. G3SdM $\vdash \alpha, \Gamma_2 \vdash \beta$;
3. $var(\alpha) \subseteq var(\Gamma_1) \cap var(\Gamma_2, \beta)$.

Let α be an interpolant of the partition $(\Gamma_1; \emptyset)(\Gamma_2, \beta)$. It is obvious that the term $t(\alpha)$ is also an interpolant of the partition.

Craig Interpolation

Theorem (Craig Interpolation)

For any G3SdM-sequent $\Gamma \vdash \beta$, if $\Gamma \vdash \beta$ is derivable in G3SdM, *then any partition of the sequent $\Gamma \vdash \beta$ has an interpolant.*

Display calculus for semi-De Morgan algebras

- The language of structure and operations in $\mathrm{D}_{\mathrm{SDL}}$ is defined as follows:

$$A ::= p \mid \top \mid \bot \mid\sim A \mid \neg A \mid A \wedge A \mid A \vee A$$
$$X ::= I \mid *X \mid \circledast X \mid X ; X \mid X > X$$

- Interpretation of structural $\mathrm{D}_{\mathrm{SDL}}$ connectives as their operational counterparts:

S connectives							
I		$*$		$;$		$>$	
$\top$	$\bot$	$\neg$	$\sim$	$\wedge$	$\vee$	$(\rightarrowtail)$	$(\rightarrow)$

$$\textit{Residuals}: \qquad \wedge \dashv \rightarrow \qquad \rightarrowtail \dashv \vee$$

Display structural rules

$$\mathsf{SN}\,\frac{*X \vdash Y}{\circledast Y \vdash X} \qquad \frac{X \vdash *Y}{Y \vdash \circledast X}\,\mathsf{SN}$$

$$\mathrm{S_D}\,\frac{X \,;\, Y \vdash Z}{Y \vdash X > Z} \qquad \frac{X \vdash Y \,;\, Z}{Y > X \vdash Z}\,\mathrm{S_D}$$

Structural rules

$$\mathrm{Id}\ \frac{}{p \vdash p} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \mathrm{Cut}$$

$$\mathrm{I}\ \begin{array}{c} X \vdash Y \\ \hline\hline X ; \mathrm{I} \vdash Y \end{array} \qquad \begin{array}{c} X \vdash Y \\ \hline\hline X \vdash Y ; \mathrm{I} \end{array}\ \mathrm{I}$$

$$\mathrm{E}\ \begin{array}{c} X ; Y \vdash Z \\ \hline\hline Y ; X \vdash Z \end{array} \qquad \begin{array}{c} X \vdash Y ; Z \\ \hline\hline X \vdash Z ; Y \end{array}\ \mathrm{E}$$

$$\mathrm{A}\ \begin{array}{c} (X ; Y) ; Z \vdash W \\ \hline\hline X ; (Y ; Z) \vdash Z \end{array} \qquad \begin{array}{c} X \vdash (Y ; Z) ; W \\ \hline\hline X \vdash Y ; (Z ; W) \end{array}\ \mathrm{A}$$

$$\mathrm{W}\ \frac{X \vdash Y}{X ; Z \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y ; Z}\ \mathrm{W}$$

$$\mathrm{C}\ \begin{array}{c} X ; X \vdash Y \\ \hline\hline X \vdash Y \end{array} \qquad \begin{array}{c} X \vdash Y ; Y \\ \hline\hline X \vdash Y \end{array}\ \mathrm{C}$$

$$\begin{array}{c} X \vdash *Y \\ \hline\hline X \vdash ***Y \end{array}\ *$$

Operational rules

$$\top\ \frac{\mathrm{I} \vdash X}{\top \vdash X} \qquad \frac{}{\mathrm{I} \vdash \top}\ \top$$

$$\bot\ \frac{}{\bot \vdash \mathrm{I}} \qquad \frac{X \vdash \mathrm{I}}{X \vdash \bot}\ \bot$$

$$\wedge\ \frac{A\,;B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X\,;Y \vdash A \wedge B}\ \wedge$$

$$\vee\ \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X\,;Y} \qquad \frac{X \vdash A\,;B}{X \vdash A \vee B}\ \vee$$

$$\neg\ \frac{*A \vdash X}{\neg A \vdash X} \qquad \frac{A \vdash X}{*X \vdash \neg A}\ \neg$$

$$\sim\ \frac{X \vdash A}{\sim A \vdash *X} \qquad \frac{X \vdash *A}{X \vdash \sim A}\ \sim$$

Translation functions

In order to translate sequents of the original language of semi-De Morgan logic into sequents in the Display semi-De Morgan logic, we will make use of the translation $\tau_1, \tau_2 : \mathrm{S_{SdM}} \to \mathrm{D_{SDL}}$ so that for all $A, B \in \mathrm{S_{SDM}}$ and $A \vdash B$, we write

$$
\begin{array}{lcl}
\tau_1(A) \vdash \tau_1(B) & \text{abbreviated as} & A^\tau \vdash B^\tau \\
\tau_2(A) \vdash \tau_2(B) & \text{abbreviated as} & A_\tau \vdash B_\tau
\end{array}
$$

The translation τ_1 and τ_2 are defined by simultaneous induction as follows:

$$
\begin{array}{rcl|rcl}
\top^\tau & ::= & \top & \top_\tau & ::= & \top \\
\bot^\tau & ::= & \bot & \bot_\tau & ::= & \bot \\
p^\tau & ::= & \mathrm{p} & p_\tau & ::= & \mathrm{p} \\
(A \wedge B)^\tau & ::= & A^\tau \wedge B^\tau & (A \wedge B)_\tau & ::= & A_\tau \wedge B_\tau \\
(A \vee B)^\tau & ::= & A^\tau \vee B^\tau & (A \vee B)_\tau & ::= & A_\tau \vee B_\tau \\
(\neg A)^\tau & ::= & \sim A_\tau & (\neg A)_\tau & ::= & \neg A^\tau
\end{array}
$$

Completeness

Lemma

$A \vdash B$ is derivable in $\mathsf{S}_{\mathsf{SdM}}$ iff $A^\tau \vdash B^\tau$ is derivable in $\mathsf{D}_{\mathsf{SDL}}$.

Theorem (Completeness)

$A^\tau \vdash B^\tau$ is valid in SdM iff $A^\tau \vdash B^\tau$ is derivable in $\mathsf{D}_{\mathsf{SDL}}$.

Theorem (Conservative extension)

$\mathsf{D}_{\mathsf{SDL}}$ is a conservative extension of $\mathsf{S}_{\mathsf{SdM}}$.

Cut elimination and Subformula property

Theorem (Cut elimination)

If $X \vdash Y$ is derivable in $\mathrm{D}_{\mathrm{SDL}}$, then it is derivable without Cut.

Theorem (Subformula property)

Any cut-free proof of the sequent $X \vdash Y$ in $\mathrm{D}_{\mathrm{SDL}}$ contains only structures over subformulas of formulas in X and Y.

Display Calculus for De Morgan Algebras

The language and the interpretation of the structural connectives of our calculus are defined as follows.

- Structural and operational language of Demorgan−Lattice:

$$
\mathsf{L}\begin{cases} A ::= \; p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid A \rightarrowtail A \mid \\ \\ X ::= \mathrm{I} \mid *X \mid X ; X \mid X > X \end{cases}
$$

Display Calculus for De Morgan Algebras

- Interpretation of structural D_{DM} connectives as their operational (i.e. logical) counterparts:

D connectives							
I		$*$		$;$		$>$	
$\top$	$\bot$	$\neg$	$\neg$	$\wedge$	$\vee$	$(\rightarrowtail)$	$(\rightarrow)$

Residuals : $\wedge \dashv \rightarrow$ $\quad \rightarrowtail \dashv \vee$

(Self)Adjoints : $\neg \dashv \neg$

Display structural rules

$$\text{SN}\ \frac{*X \vdash Y}{*Y \vdash X} \qquad \frac{X \vdash *Y}{Y \vdash *X}\ \text{SN}$$

$$\mathrm{S_D}\ \frac{X ; Y \vdash Z}{Y \vdash X > Z} \qquad \frac{X \vdash Y ; Z}{Y > X \vdash Z}\ \mathrm{S_D}$$

Structure rules

$$\mathrm{Id}\ \frac{}{p \vdash p} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \mathrm{Cut}$$

$$\mathrm{I}\ \frac{X \vdash Y}{X ; \mathrm{I} \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y ; \mathrm{I}}\ \mathrm{I}$$

$$\mathrm{E}\ \frac{X ; Y \vdash Z}{Y ; X \vdash Z} \qquad \frac{X \vdash Y ; Z}{X \vdash Z ; Y}\ \mathrm{E}$$

$$\mathrm{A}\ \frac{(X ; Y) ; Z \vdash W}{X ; (Y ; Z) \vdash Z} \qquad \frac{X \vdash (Y ; Z) ; W}{X \vdash Y ; (Z ; W)}\ \mathrm{A}$$

$$\mathrm{W}\ \frac{X \vdash Y}{X ; Z \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y ; Z}\ \mathrm{W}$$

$$\mathrm{C}\ \frac{X ; X \vdash Y}{X \vdash Y} \qquad \frac{X \vdash Y ; Y}{X \vdash Y}\ \mathrm{C}$$

$$\frac{X \vdash Y}{X \vdash * * Y}\ *$$

Operation rules

$$\top\ \frac{\mathrm{I} \vdash X}{\top \vdash X} \qquad \frac{}{\mathrm{I} \vdash \top}\ \top$$

$$\bot\ \frac{}{\bot \vdash \mathrm{I}} \qquad \frac{X \vdash \mathrm{I}}{X \vdash \bot}\ \bot$$

$$\wedge\ \frac{A ; B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X ; Y \vdash A \wedge B}\ \wedge$$

$$\vee\ \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X ; Y} \qquad \frac{X \vdash A ; B}{X \vdash A \vee B}\ \vee$$

$$\neg\ \frac{*A \vdash X}{\neg A \vdash X} \qquad \frac{X \vdash *A}{X \vdash \neg A}\ \neg$$

Completeness

Proposition

For every A in S_{dM}, $A \vdash A$ *is derivable in* D_{DM} .

Lemma

$A \vdash B$ *is derivable in* S_{dM} *iff* $A \vdash B$ *is derivable in* D_{DM} .

Theorem (Completeness)

$A \vdash B$ *is valid in* dM *iff* $A \vdash B$ *is derivable in* D_{DM} .

Cut elimination and Subformula property

Theorem (Cut elimination)

If $X \vdash Y$ is derivable in $\mathrm{D_{DM}}$, then it is derivable without Cut.

Theorem (Subformula property)

Any cut-free proof of the sequent $X \vdash Y$ in $\mathrm{D_{DM}}$ contains only structures over subformulas of formulas in X and Y.

Glivenko theorem

Theorem (Glivenko theorem)

For any DM sequent $A \vdash B$, $A \vdash B$ is derivable in De Morgan logic iff $\neg\neg A \vdash \neg\neg B$ is derivable in semi-De Morgan logic.

The relation between De Morgan and semi-De Morgan logic is very similar with the relation between Classical logic and Intuitionistic logic!

Discussions about different non-classical negations

- Some properties of negation:

Con	$A \vdash B / \neg B \vdash \neg A$		
$\neg\vee$	$\neg A \wedge \neg B \vdash \neg(A \vee B)$	$\neg\wedge$	$\neg(A \wedge B) \vdash \neg A \vee \neg B$
$\neg\neg\vee$	$\neg\neg(A \vee B) \vdash \neg\neg A \vee \neg\neg B$	$\neg\neg\wedge$	$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B)$
Nb	$\top \vdash \neg\bot$	Nt	$\neg\top \vdash \bot$
DNI	$A \vdash \neg\neg A$	DNE	$\neg\neg A \vdash A$
TNI	$\neg A \vdash \neg\neg\neg A$	TNE	$\neg\neg\neg A \vdash \neg A$
NA	$A \wedge \neg A \vdash \neg B$	AB	$A \wedge \neg A \vdash B$
NE	$\neg B \vdash A \vee \neg A$	EM	$B \vdash A \vee \neg A$

We talk about negations in bounded distributive lattice context!

Discussions about different non-classical negations

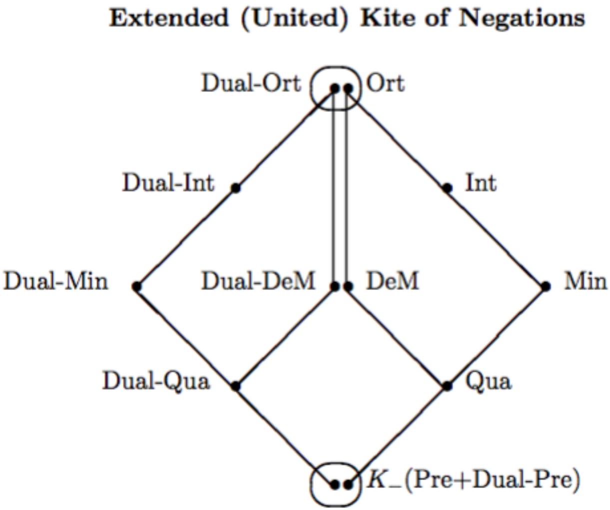
- Some derivations of difference properties

Con, DNI	$\vdash$ Nt	Con, DNE	$\vdash$ Nb
Con, DNI	$\vdash \neg\vee$	Con, DNE	$\vdash \neg\wedge$
Con, DNI, NA	$\vdash \neg\neg\wedge$	Con, DNI, Ab	$\vdash \neg\neg\wedge$
Con, DNE, NE	$\vdash \neg\neg\vee$	Con, DNE, EM	$\vdash \neg\neg\vee$
Con, DNI	$\vdash$ TNI,TNE	Con, DNE	$\vdash$ TNI, TNE
DNI, $\neg\vee, \neg\wedge$	$\vdash$ DNE	DNE,$\neg\vee, \neg\wedge$	$\vdash$ DNI
$\neg\vee, \neg\wedge$	$\vdash \neg\neg\vee, \neg\neg\wedge$		

Discussions about different non-classical negations

	$\neg\vee$	$\neg\wedge$	$\neg\neg\vee$	$\neg\neg\wedge$	Nt	DNI	DNE	TNI	TNE	NA	AB	NE	EM
PMN	√												
PMN^d		√											
QMN	√				√	√		√	√				
QMN^d		√			√	√		√	√				
SDM	√			√	√			√	√				
SDM^d		√	√		√			√	√				
QDM	√			√	√	√		√	√				
QDM^d		√	√		√		√	√	√				
MIN	√			√	√	√		√	√	√			
MIN^d		√	√		√	√		√	√			√	
OCM	√	√	√	√	√								
DMN	√	√	√	√	√	√	√	√	√				
INT	√			√	√	√		√	√	√	√		
INT^d		√	√		√	√		√	√			√	√
ORT	√	√	√	√	√	√	√	√	√	√	√	√	√

Michael Dunn's kite of negations



Further work

- Semantics: based on the compatibility frame, we can also give a compatibility semantics for semi-De Morgan logic by adding more frame conditions corresponds to the axioms.
- Applying to Justification logic (compatibility frame).
- Linear logic in semi-De Morgan context.

References

A. Anderson and N. Belnap. *Entailment: The Logic of Relevance and Necessity*, Vol. 1. Princeton University Press, 1975.

O. Arieli and A. Avron. Reasoning with logical bilattices. *Journal of Logic, Language and Information*, 5:25-63, 1996.

A. Avron. Negation: two points of view. In: D. Gabbay and H. Wansing (eds.). *What is Negation?*, pp. 3–22. Kluwer Academic Publishers, 1999.

A. Bialynicki-Birula and H. Rasiowa. On the representation of quasi-Boolean algebras. *Bulletin of the Polish Academy of Science*, Cl. III (5): 159–261, 1957.

R. Balbes and P. Dwinger. *Distributive Lattices*. Abstract Space Publishing, 2011.

F. Bou and U. Rivieccio. The logic of distributive bilattices. *Logic Journal of IGPL*, 19(1): 183–216, 2011.

M. Dunn. A relational representation of quasi-Boolean algebras. *Notre Dame Journal of Formal Logic*, 23(4): 353–357, 1982.

F. Liang, G.Greco and A. Palmigiano. Display calculi for Semi-De Morgan and De Morgan logic. In preparation.

D. Hobby. Semi-De Morgan algebras. *Studia Logica*, 56(1/2): 151–183, 1996.

References

N. Kamide. Notes on Craig interpolation for LJ with strong negation. *Mathematical Logic Quarterly*, 57(4): 395–399, 2011.

M. Ma and F. Liang. Sequent calculi for semi-De Morgan and De Morgan algebras. Submitted. ArXiv preprint 1611.05231, 2016.

S. Negri and J. von Plato. *Structural Proof Theory*. Cambridge University Press, 2001.

C. Palma and R. Santos. On a subvariety of semi-De Morgan algebras. *Acta Mathematica Hungarica*, 98(4): 323–328, 2003.

H. Rasiowa. *An Algebraic Approach to Non-Classical Logics*. North-Holland Co., Amsterdam, 1974.

H. Sankappanavar. Semi-De Morgan algebras. *The Journal of Symbolic Logic* 52(3):712–724, 1987.

A, Saha, J. Sen and M. K. Chakraborty. Algebraic structures in the vicinity of pre-rough algebra and their logics. *Information Sciences*, 282: 296-320, 2014.