The Logic of Common Ignorance

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Introduction

Quote "Knowledge is a big subject. Ignorance is bigger... and it is more interesting."[1]

Claim Ignorance has some surprising properties.

Example Common ignorance.

[1] Stuart Firestein, Interview about S. Firestein, *Ignorance: How It Drives Science*, OUP 2012.

Question

- "Obama calls Trump ignorant about foreign affairs" (Google, August 16, 2016, 8 results).
- "Trump calls Obama ignorant about foreign affairs" (Google, August 16, 2016, about 135 results).
- Suppose that at least one of them were right. (Of course, both could be right.)
- Would this give the group of all humans *common ignorance* about foreign affairs?

Knowing that

To answer this question, we extend the (propositional) logic of individual, shared and common knowledge that A, $\boldsymbol{TEC}_{(m)}$, with a few uncontroversial definitions. $\boldsymbol{TEC}_{(m)}$ applies to a group having members $1, \ldots, m$. $\boldsymbol{TEC}_{(m)}$ is well-known and is axiomatized as follows.[2]

[2] J.-J. Ch. Meyer and W. van der Hoek, *Epistemic Logic for Computer Science and Artificial Intelligence* (Cambridge: Cambridge University Press, 1995), Ch. 2.1.

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Symbols

- Individual knowledge that A: $\boldsymbol{K}_i A$, where $1 \leq i \leq m$. $\boldsymbol{K}_i A$ is read as "i individually knows that A" or as "i has individual knowledge that A."
- Shared knowledge that A: $\boldsymbol{E}A$. $\boldsymbol{E}A$ is read as "everyone knows that A" or as "the group has shared knowledge that A."
- Common knowledge that A: $\boldsymbol{C}A$. $\boldsymbol{C}A$ is read as "it is commonly known that A" or as "the group has common knowledge that A."

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Axioms and derivation rules

A1 All instances of propositional tautologies.

A2 $\boldsymbol{K}_i(A \to B) \to (\boldsymbol{K}_i A \to \boldsymbol{K}_i B)$.

A3 $\boldsymbol{K}_i A \to A$.

A4 $\boldsymbol{E} A \leftrightarrow \bigwedge_{i=1}^{m} \boldsymbol{K}_i A$.

A5 $\boldsymbol{C} A \to A$.

A6 $\boldsymbol{C} A \to \boldsymbol{E}\boldsymbol{C} A$.

A7 $\boldsymbol{C}(A \to B) \to (\boldsymbol{C} A \to \boldsymbol{C} B)$.

A8 $\boldsymbol{C}(A \to \boldsymbol{E} A) \to (A \to \boldsymbol{C} A)$.

R1 From A and $A \to B$ infer B.

R2 From A infer $\boldsymbol{K}_i A$.

R3 From A infer $\boldsymbol{C} A$.

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Theorems

1.1 $\boldsymbol{C}A \rightarrow \boldsymbol{E}A$ (common knowledge that A implies shared knowledge that A).

1.2 $\boldsymbol{E}A \rightarrow \boldsymbol{K}_i A$ (shared knowledge that A implies individual knowledge that A).

1.3 $\boldsymbol{C}A \rightarrow \boldsymbol{K}_i A$ (common knowledge that A implies individual knowledge that A).

†1.4 $\boldsymbol{K}_i A \rightarrow \boldsymbol{C}A$ (individual knowledge that A implies common knowledge that A) is *invalid* [proof: by the semantics].

Intuitively, $\boldsymbol{C}A = \bigwedge_{i \geq 0} \boldsymbol{E}^i A$ (common knowledge that A is the conjunction of A, shared knowledge that A, shared knowledge that the group has shared knowledge that A, and so on).

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Knowledge whether/about

Symbols:[3]

- Individual knowledge about A: $\Delta_i A = \boldsymbol{K}_i A \vee \boldsymbol{K}_i \neg A$. $\Delta_i A$ is read as "i individually knows whether A" or as "i has individual knowledge about A."
- Common knowledge about A: $\boldsymbol{C}_\Delta A = \boldsymbol{C} A \vee \boldsymbol{C} \neg A$. $\boldsymbol{C}_\Delta A$ is read as "the group has common knowledge about A."

[3] See J. Fan, Y. Wang and H. van Ditmarsch, "Contingency and Knowing Whether," *The Review of Symbolic Logic*, 8:75–107, 2015.

Theorems

2.1 $\boldsymbol{C}_\Delta A \rightarrow \Delta_i A$ $[(\boldsymbol{C}A \vee \boldsymbol{C}\neg A) \rightarrow (\boldsymbol{K}_i A \vee \boldsymbol{K}_i \neg A)]$ (common knowledge about A implies individual knowledge about A) [from $\boldsymbol{C}A \rightarrow \boldsymbol{K}_i A$ (1.3) by propositional calculus].

†2.2 $\Delta_i A \rightarrow \boldsymbol{C}_\Delta A$ (individual knowledge about A implies common knowledge about A) is *invalid* [proof: by the semantics].

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Ignorance whether/about

Symbols:[4]

- Individual ignorance about A:
  $\nabla_i A = \neg\Delta_i A = \neg \boldsymbol{K}_i A \wedge \neg \boldsymbol{K}_i \neg A$ (individual ignorance about A is the negation of individual knowledge about A). $\nabla_i A$ is read as "i does not individually know whether A", as "i individually ignores whether A" or as "i has individual ignorance about A."
- Common ignorance about A:
  $\boldsymbol{C}_\nabla A = \neg \boldsymbol{C}_\Delta A = \neg \boldsymbol{C} A \wedge \neg \boldsymbol{C} \neg A$ (common ignorance about A is the negation of common knowledge about A). $\boldsymbol{C}_\nabla A$ is read as "the group has common ignorance about A."

[4] See Fan, Wang and Van Ditmarsch, "Contingency and Knowing Whether," op. cit.

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Theorems

3.1 $\nabla_i A \to \boldsymbol{C}_\nabla A$ $[\neg\Delta_i A \to \neg \boldsymbol{C}_\Delta A]$ (individual ignorance about A implies common ignorance about A) [from $\boldsymbol{C}_\Delta A \to \Delta_i A$ (2.1) by contraposition].

†3.2 $\boldsymbol{C}_\nabla A \to \nabla_i A$ (common ignorance about A implies individual ignorance about A) is *invalid* [proof: by the semantics].

Individual ignorance about A is therefore stronger than common ignorance about A. If agents have individual ignorance about A, all groups to which they belong have common ignorance about A.

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Answer to question

- Obama and Trump called each other ignorant about foreign affairs.
- Suppose that at least one of them were right.
- Question: would this give the group of all humans *common ignorance* about foreign affairs?
- Answer: yes, it would, by theorem $\nabla_i A \rightarrow \boldsymbol{C}_\nabla A$ (3.1).

Common ignorance about common ignorance

- $\boldsymbol{S5EC}_{(m)}$ is $\boldsymbol{TEC}_{(m)}$ plus $\neg \boldsymbol{K}_i A \rightarrow \boldsymbol{K}_i \neg \boldsymbol{K}_i A$ ("i does not know that A" implies "i knows that i does not know that A").
- $\boldsymbol{S5EC}_{(m)}$ has the following theorem.[5]
  4.1 $\neg \boldsymbol{C}_{\nabla} \boldsymbol{C}_{\nabla} A$ (there is no common ignorance about common ignorance about A).
- $\boldsymbol{TEC}_{(m)}$ does not have this theorem, as the semantics shows.
- The Obama/Trump case seems to show that 4.1 is false.
- We do have common ignorance about our common ignorance about foreign affairs.
- $\boldsymbol{TEC}_{(m)}$ is therefore preferable to $\boldsymbol{S5EC}_{(m)}$.

[5] H. Montgomery and R. Routley, "Contingency and Non-Contingency Bases for Normal Modal Logics," *Logique et Analyse*, 9:318–328, 1966.