Refinement Modal Logic: Algebraic Semantics

Zeinab Bakhtiari

LORIA, CNRS – Université de Lorraine, France

In collaboration with:

Hans van Ditmarsch (LORIA), Sabine Frittella (LIFO)

August 2016, TU Delft

Plan for talk

- Part 1: Logic
  - Introduction to Dynamic epistemic logic
  - Refinement modal logic
- Part 2: Algebra
  - Algebraic Semantics of action model logic
  - Algebraic Semantics of refinement modal logic

Introduction: Dynamic Epistemic Logic (DEL)

- Dynamic epistemic logics is a family of logics dealing with knowledge and information change.
  - **Epistemic** Describing knowledge and belief...
  - **Dynamic** Knowledge acquisition, belief updates...

Introduction: Dynamic Epistemic Logic (DEL)

- Dynamic epistemic logics is a family of logics dealing with knowledge and information change.
  - **Epistemic** Describing knowledge and belief...
  - **Dynamic** Knowledge acquisition, belief updates...
- Epistemic actions
  - **Examples**: Public announcements, private announcements, ...
- How can we make a formula true?

Introduction: Dynamic Epistemic Logic (DEL)

- Dynamic epistemic logics is a family of logics dealing with knowledge and information change.
  - **Epistemic** Describing knowledge and belief...
  - **Dynamic** Knowledge acquisition, belief updates...
- Epistemic actions
  - **Examples**: Public announcements, private announcements, ...
- How can we make a formula true?
- Quantifying over information change.

Different ways of quantifying over information change

- there is an announcement (by the agents in group G) after which φ;
  - In arbitrary public announcement logic (APAL) we quantify over announcements.
- there is an action model with precondition ψ after which φ;
  - In arbitrary action model logic (AAML) we quantify over action models.
- In these logics the quantification is over dynamic modalities for action execution . . .

Refinement quantifier

- Bozzelli, et al. in 2013 proposed a new form of quantification over information change, *independent* from the logical language.

Refinement quantifier

- Bozzelli, et al. in 2013 proposed a new form of quantification over information change, *independent* from the logical language.
- It is called refinement quantification, or just refinement.

Refinement quantifier

- Bozzelli, et al. in 2013 proposed a new form of quantification over information change, *independent* from the logical language.
- It is called refinement quantification, or just refinement.
- Refinement is the dual of simulation.

What is a refinement?

- A **refinement** of a model is a submodel of a bisimilar model:

What is a refinement?

- A **refinement** of a model is a submodel of a bisimilar model:

Consider this pointed model (epistemic state) M:

$$\circ \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

What is a refinement?

- A **refinement** of a model is a submodel of a bisimilar model:

Consider this pointed model (epistemic state) M:

$$\circ \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

M_1 is a bisimilar copy of the model M:

$$\bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \circ \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

- A **refinement** of a model is a submodel of a bisimilar model:

Consider this pointed model (epistemic state) M:

$$\circ \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

M_1 is a bisimilar copy of the model M:

$$\bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \circ \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

M_2 is a refinement of M: (M is a simulation of M_2:)

$$\bullet \longleftarrow \circ \longrightarrow \bullet \longrightarrow \bullet$$

What is a refinement?

- A **refinement** of a model is a submodel of a bisimilar model:

Consider this pointed model (epistemic state) M:

$$\circ \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

M_1 is a bisimilar copy of the model M:

$$\bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \circ \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

M_2 is a refinement of M: (M is a simulation of M_2:)

$$\bullet \longleftarrow \circ \longrightarrow \bullet \longrightarrow \bullet$$

Refinement Relation: Formal Definition

Let two models $M = (S, R, V)$ and $M' = (S', R', V')$ be given.
A non-empty relation $\mathfrak{R} \subseteq S \times S'$ is a refinement if for all $(s, s') \in \mathfrak{R}$, $p \in P$:

- **atoms** $s \in V(p)$ iff $s' \in V'(p)$;
- **back** if $R' s' t'$, there is a t such that Rst and $(t, t') \in \mathfrak{R}$.

$\underline{\leftrightarrow}$ bisimulation: atoms, forth, back

$\underline{\rightarrow}$ simulation: atoms, forth

$\underline{\leftarrow}$ refinement: atoms, back

Refinement Modal Logic — language and semantics

Language $\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \forall\varphi$

Structures pointed Kripke models

Semantics

$$(M,s) \models \forall\varphi \quad \text{iff} \quad \forall (M',s') : (M,s) \succeq (M',s') \text{ implies } (M',s') \models \varphi$$
$$(M,s) \models \exists\varphi \quad \text{iff} \quad \exists (M',s') : (M,s) \succeq (M',s') \text{ and } (M',s') \models \varphi$$

[Bozzelli, Laura, et al. "*Refinement modal logic.*" *Information and Computation 239 (2014): 303-339.*]

Arbitrary action model logic and refinement modal logic

- Action model execution is a refinement, and (surprisingly) vice versa (on finite models).

$$M_s \succeq (M \otimes \alpha)_{(s,\mathsf{u})}$$

Arbitrary action model logic and refinement modal logic

- Action model execution is a refinement, and (surprisingly) vice versa (on finite models).

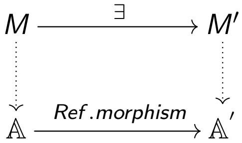
$$M_s \leftarrowtail (M \otimes \alpha)_{(s,\mathsf{u})}$$

- Refinement quantifier and action model quantifier:
  - $M_s \models \bar{\exists}\varphi$ iff there exists an action model α_{u} s.t. $M_s \models \langle \alpha_{\mathsf{u}} \rangle \varphi$.
- If $M_s \models \exists\varphi$ then we can find a multi-pointed action model α_{S} s.t. $M_s \models \langle \alpha_{\mathsf{S}} \rangle \varphi$.

As a result:

Refinement quantifier is equivalent to Action model quantifier!

[J. Hales. "Arbitrary action model logic and action model synthesis". 2013.]



Main Goal

- Dualize the notion of refinement on algebras,
- For any algebraic model $\mathcal{A} = (\mathbb{A}, V)$, we want to find a Boolean algebra with operator $\mathbb{U}_{\mathcal{A}}$ and a map $G : \mathbb{U}_{\mathcal{A}} \to \mathbb{A}$ such that for any $\varphi \in \mathcal{L}$,

$$[\![\exists\varphi]\!]_{\mathcal{A}} = G([\![\varphi]\!]_{\mathbb{U}_{\mathcal{A}}}).$$

Step 1: Dualize Refinement Relation

Refinement morphism

Let $\mathbb{A}$ and $\mathbb{A}'$ be two Boolean algebra with operators. A map

$$f : \mathbb{A} \to \mathbb{A}'$$

is a *refinement morphism* if

- it is monotone;
- preserves $\bot$ and $\vee$; and
- satisfies the following inequality

$$\blacklozenge^{\mathbb{A}'} \circ f \leq f \circ \blacklozenge^{\mathbb{A}}$$

where $\blacklozenge \dashv \Box$(adjoint operator).

Step 2: Epistemic update on algebras

For any algebraic model $\mathcal{A} = (\mathbb{A}, V)$ and any formula $\varphi \in \mathcal{L}$, we define

- Boolean algebra with operators $\mathbb{A}^\varphi$,
- A pair of maps $f^\varphi : \mathbb{A} \to \mathbb{A}^\varphi$, $g^\varphi : \mathbb{A}^\varphi \to \mathbb{A}$.

Step 2: Epistemic update on algebras

For any algebraic model $\mathcal{A} = (\mathbb{A}, V)$ and any formula $\varphi \in \mathcal{L}$, we define

- Boolean algebra with operators $\mathbb{A}^\varphi$,
- A pair of maps $f^\varphi : \mathbb{A} \to \mathbb{A}^\varphi$, $g^\varphi : \mathbb{A}^\varphi \to \mathbb{A}$.

For each formula φ, action model synthesis provides us with an action model $\alpha_{\mathsf{S}}^{\varphi} = (\mathsf{S}, \mathsf{R}, \mathsf{Pre})$, such that for every pointed model M_s we have

$$M_s \models \exists\varphi \quad \text{iff} \quad M \otimes \alpha_S^\varphi \models \varphi$$

Step 2: Epistemic update on algebras

For any algebraic model $\mathcal{A} = (\mathbb{A}, V)$ and any formula $\varphi \in \mathcal{L}$, we define

- Boolean algebra with operators $\mathbb{A}^\varphi$,
- A pair of maps $f^\varphi : \mathbb{A} \to \mathbb{A}^\varphi$, $g^\varphi : \mathbb{A}^\varphi \to \mathbb{A}$.

For each formula φ, action model synthesis provides us with an action model $\alpha^\varphi_{\mathsf{S}} = (\mathsf{S}, \mathsf{R}, \mathsf{Pre})$, such that for every pointed model M_s we have

$$M_s \models \exists\varphi \quad \text{iff} \quad M \otimes \alpha^\varphi_S \models \varphi$$

$$M \longrightarrow \coprod_{\alpha^\varphi} M \hookleftarrow M \otimes \alpha^\varphi$$

$$\Downarrow$$

$$\mathbb{A} \longleftarrow \prod_{a^\varphi} \mathbb{A} \twoheadrightarrow \mathbb{A}^\varphi$$

- Ma, Sadrzadeh and Palmigiano. *Algebraic semantics and model completeness for intuitionistic public announcement*.
- Kurz and Palmigiano. *Epistemic updates in algebras*.

Step 2: Epistemic updates on algebras

- $a = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_{a^\varphi})$: $\mathsf{Pre}_{a^\varphi} = V \circ \mathsf{Pre}_{\alpha^\varphi}$.
- $\prod_{a^\varphi_{\mathsf{S}}} \mathbb{A}$: $|\mathsf{S}|$-fold product of $\mathbb{A}$, which is set-isomorphic to the collection $\mathbb{A}^{\mathsf{S}}$ of the set maps $f : \mathsf{S} \to \mathbb{A}$.
- The equivalence relation $\equiv_{a^\varphi}$ on $\prod_a \mathbb{A}$ is defined as follows: for all $h, k \in \mathbb{A}^{\mathsf{S}}$,

$$h \equiv_{a^\varphi} k \quad \text{iff} \quad h \wedge \mathsf{Pre}_{a^\varphi} = k \wedge \mathsf{Pre}_{a^\varphi}.$$

Defining maps between $\mathbb{A}$ and $\mathbb{A}^\varphi$

Refinement morphism and its adjoint

$$f^\varphi : \mathbb{A} \to \mathbb{A}^\varphi \qquad\qquad g^\varphi : \mathbb{A}^\varphi \to \mathbb{A}$$

$$b \mapsto [h_b] \qquad\qquad [h] \mapsto \bigvee_{\mathrm{u} \in \mathrm{S}} (h(\mathrm{u}) \wedge \mathrm{Pre}_{a^\varphi}(\mathrm{u}))$$

where $h_b : \mathrm{S} \to \mathbb{A}$ is the map such that $h_b(\mathrm{u}) := b \wedge \mathrm{Pre}_{a^\varphi}(\mathrm{u})$ and $a = (\mathrm{S}, \mathrm{R}, \mathrm{Pre}_{a^\varphi})$ is the action model induced by $\alpha^\varphi_{\mathrm{S}}$ via V.

1. the map f^φ is a refinement morphism,
2. the map g^φ is monotone and preserves arbitrary joins,
3. $g^\varphi \dashv f^\varphi$.

Step 3: Constructing Big BAO

Refinement Algebra

For every algebraic model $\mathcal{A} = (\mathbb{A}, V)$, we define the following algebraic structure:

$$\mathbb{U}_{\mathcal{A}} := \prod_{\varphi \in \mathcal{L}} \mathbb{A}^{\varphi}.$$

Elements of $\mathbb{U}_{\mathcal{A}}$ are tuples $(b^{\varphi})_{\varphi \in \mathcal{L}}$ where $b^{\varphi} \in \mathbb{A}^{\varphi}$.

Step 3: Constructing Big BAO

Refinement Algebra

For every algebraic model $\mathcal{A} = (\mathbb{A}, V)$, we define the following algebraic structure:

$$\mathbb{U}_{\mathcal{A}} := \prod_{\varphi \in \mathcal{L}} \mathbb{A}^{\varphi}.$$

Elements of $\mathbb{U}_{\mathcal{A}}$ are tuples $(b^{\varphi})_{\varphi \in \mathcal{L}}$ where $b^{\varphi} \in \mathbb{A}^{\varphi}$.

Fact

The product of any family $\{\mathbb{A}_i\}_{i \in I}$ of normal Boolean algebra with operators, where I may be an uncountable set, is a normal Boolean algebra with operator, so Refinement algebra is a normal Boolean algebra with operator.

Step 4: Defining the map G

$$\mathbb{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{U}_{\mathcal{A}}$$

$$\begin{aligned} F : \mathbb{A} &\to \mathbb{U}_{\mathcal{A}} \\ a &\mapsto \prod_{\varphi \in \mathcal{L}} (f^{\varphi}(a)) \end{aligned} \qquad\qquad \begin{aligned} G : \quad \mathbb{U}_{\mathcal{A}} &\to \mathbb{A} \\ (b^{\varphi})_{\varphi} &\mapsto \bigvee_{\varphi \in \mathcal{L}} g^{\varphi}(b^{\varphi}) \end{aligned}$$

Properties of (F, G)

1. The map F is a refinement morphism,
2. the map G is monotone and preserves $\bot$, $\top$ and finite joins,
3. $G \dashv F$.

Algebraic Semantics of RML

Let $\mathcal{A} = (\mathbb{A}, V)$ be an algebraic model and $\mathbb{U}$ its refinement algebra. Let $\mathcal{A}'$ be the algebraic model $(\mathbb{U}, \mathcal{V})$ with $\mathcal{V}$: Atoms $\rightarrow \mathbb{U}$ and $\mathcal{V}(p) = (F \circ V)(p)$. The extension map $[\![.]\!]' : \mathcal{L} \rightarrow \mathbb{A}$ is defined as follows:

$$
\begin{aligned}
[\![p]\!]'_{\mathcal{A}} &:= V(p) && \\
[\![\bot]\!]'_{\mathcal{A}} &:= \bot^{\mathbb{A}} && \\
[\![\circ\varphi]\!]'_{\mathcal{A}} &:= \circ^{\mathbb{A}}[\![\varphi]\!]'_{\mathcal{A}} && \text{for } \circ \in \{\neg, \Diamond, \Box\} \\
[\![\varphi \bullet \psi]\!]'_{\mathcal{A}} &:= [\![\varphi]\!]'_{\mathcal{A}} \bullet^{\mathbb{A}} [\![\psi]\!]'_{\mathcal{A}} && \text{for } \bullet \in \{\vee, \wedge, \rightarrow\} \\
[\![\exists\varphi]\!]'_{\mathcal{A}} &:= G([\![\varphi]\!]'_{\mathcal{A}'}) &&
\end{aligned}
$$

Results and Future works

- Our Results
  - Algebraic semantics of Refinement modal logic
  - Soundness and Completeness
- Future research
  - weaken the classical propositional modal logical base to a non-classical propositional modal logical base,
  - develop multi-type calculi for such non-classical modal logics with refinement quantifiers, for example refinement intuitionistic (modal) logic.

Thank you!