Determinacy, measurable cardinals and more

Apostolos Tzimoulis

July 26, 2014

Motivation

Theorem (Woodin)

Assume $V = \mathbf{L}(\mathbb{R}) + \mathsf{AD}$. *Then if* $\Theta = \delta$ *we have that*

$$\mathrm{HOD} \models \delta \textit{ is a Woodin cardinal.}$$

Goal

Assuming the axiom of determinacy, we want to present an abstract method for taking a combinatorial object (an *ultrafilter*) from $\mathbf{L}(\mathbb{R})$ and turn in into a stronger large cardinal property (an *elementary embedding*) in HOD.

Reasons for inner models

In the absence of the axiom of choice the existence of ultrafilters and elementary embeddings are not equivalent.

Lemma

Assume that $j : V \to M$ is an elementary embedding, where M is a transitive model, with critical point κ. Then $V_\kappa \models \mathrm{ZF}$. If AD is true, it is also true in V_κ.

Proof Sketch.

Let $f : x \to V_\kappa$, where $x \in V_\kappa$. Since κ is the critical point $j(x) = x$. So $\mathrm{dom}(j(f)) = x$, by the elementarity of j. Furthermore for $y \in x$, $j(y) = y$ and $j(f(y)) = f(y)$. Hence $j(f) = f$. Therefore $\mathrm{ran}(f) \subseteq V_\alpha$ for $\alpha < \kappa$. □

Details I

Theorem (Martin)

Assume AD*. Then the filter containing the Turing cones,* M_{T}*, is an ultrafilter.*

Theorem (Kunen)

Assume $\mathrm{AD}+\mathrm{DC}$ *and let* $\kappa < \Theta$*. Then every* ω_1*-complete ultrafilter over* κ *is ordinal definable.*

Proof Sketch.

We can define a function $f : \mathbb{R} \to \kappa$ such that $f^*[M_{\mathrm{T}}] = \mathcal{U}$. From DC we have that $V^{\mathbb{R}}/M_{\mathrm{T}}$ is well-founded and thus its transitive collapse is a standard class M (and f is represented by γ in M). Hence we can define an embedding $j : V \to M$. Then, since M_{T} is OD, we have that so is j. Finally $\mathcal{U}$ can be extracted in a definable way from j and γ:

$$X \in \mathcal{U} \iff f^{-1}[X] \in M_{\mathrm{T}} \iff \gamma \in j(X).$$

Details II

Proposition

If $\mathcal{U}$ is an OD*, λ-complete (normal) ultrafilter over κ, then so is $\mathcal{U} \cap \mathrm{HOD}$ in* HOD*.*

Proof Sketch.

If f is regressive in HOD, it is also regressive in V and the normality of $\mathcal{U}$ implies that f is constant with value γ in $\mathcal{U}$. Then this set is definable from f and γ, hence it is OD. □

Theorem

Assume DC*. If $\mathcal{U}$ is an* OD*, ω_1-complete ultrafilter over κ, then if M is the transitive collapse of $\mathrm{HOD}^\kappa/\mathcal{U}$, we have that $M \subseteq \mathrm{HOD}$.*

Proof Sketch.

By Łoś Theorem, we define a well-ordering of M, $<$, such that $\{x : x < y\}$ is a set for all $y \in M$. This yields a function $F : M \to \mathrm{On}$. That is, $M \subseteq \mathrm{HOD}$. □

Details III

Corollary

Assume that φ is a property such that

$$(\forall \xi \in \kappa)[\varphi(\xi) \implies \varphi^{\mathrm{HOD}}(\xi)].$$

If $\mathcal{U}$ is an OD ultrafilter over κ such that $\{\xi \in \kappa : \varphi(\xi)\} \in \mathcal{U}$ then

$$\mathrm{HOD} \models \{\xi \in \kappa : \varphi(\xi)\} \in \mathcal{U} \cap \mathrm{HOD}.$$

Examples:

- "*ξ is a cardinal*"
- "*ξ is an inaccessible cardinal*"
- "*ξ is a Mahlo cardinal*"
- "*ξ is measurable*"
- "*ξ is 1-measurable*"
- $\ldots$

Application

Definition

A *Spector pointclass* is an ω-parametrised normed pointclass, closed under trivial substitution, $\wedge$, $\vee$, $\exists^{\leq}$, $\forall^{\leq}$, $\exists^{\omega}$, $\forall^{\omega}$ and it contains all semirecursive sets.

Lemma (Bounding Lemma)

Let Γ be a Spector pointclass closed under $\forall^{\mathbb{R}}$. If $\varphi : S \twoheadrightarrow \delta$ is a Γ-norm with $S \in \mathbf{\Gamma} \setminus \mathbf{\Delta}$ and $Q \in \check{\mathbf{\Gamma}}$ with $Q \subseteq S$, then there exists some $\xi \in \delta$ such that $\varphi[Q] \subseteq \xi$.

Theorem

Assume AD*. If Γ is a Spector pointclass closed under $\forall^{\mathbb{R}}$ then $o(\mathbf{\Delta})$ is a regular cardinal.*

Application

Theorem

Assume AD*. Let* Γ *be a Spector pointclass closed under* $\forall^{\mathbb{R}}$*. If* $\kappa = o(\mathbf{\Delta})$ *then* $\kappa \to (\kappa)_2^{\omega+\omega}$*.*

The game.

Let $f : [\kappa]^{\omega+\omega} \to 2$ and let $\varphi : S \twoheadrightarrow \kappa$, be a Γ-norm. Players plays $\omega \cdot (\omega + \omega)$ many elements of S (x_ξ and y_ξ). For $\delta \in \omega + \omega$ we define

$$\alpha_\delta = \bigcup(\{\varphi(x_{\omega\cdot\delta+n})n \in \omega\} \cup \{\varphi(y_{\omega\cdot\delta+n})n \in \omega\}).$$

Player I wins exactly when $f(\{\alpha_\delta : \delta \in \omega + \omega\}) = 0$. □

Application

Proof Sketch.

Assuming that I has a winning strategy σ, II can take control of the game: Using the closure of Γ under $\forall^{\mathbb{R}}$ we have that for every $\nu \in \kappa$

$$B_\nu = \{((\sigma \star y)_I)_\xi : \forall \zeta < \xi : \varphi(y_\zeta) < \nu\}$$

is in $\check{\Gamma}$. By the Bounding lemma it is bounded. Hence for every $\nu \in \kappa$ we can define the bound $\rho(\nu)$. The set

$$C = \{\beta \in \kappa : (\forall \gamma < \beta)[\rho(\gamma) < \beta]\}$$

is a club set. Then $C \cap E^\kappa_\omega$ is homogeneous for f. □

Theorem (Kleinberg)

If $\kappa \to (\kappa)_2^{\lambda+\lambda}$ then $\mathcal{C}_\kappa^\lambda$ is a normal κ-complete ultrafilter over κ.

Application

Theorem (Kleinberg)

If $\kappa \to (\kappa)_2^{\lambda+\lambda}$ then $\mathcal{C}_\kappa^\lambda$ is a normal κ-complete ultrafilter over κ.

Theorem (Kleinberg)

Let κ be a cardinal. If for all $\mu \in \kappa$ we have that $\kappa \to (\kappa)_\mu^\kappa$ then for every stationary set S there is a normal κ-complete ultrafilter $\mathcal{U}$ over κ such that $S \in \mathcal{U}$.

Application

Definition

A cardinal κ is called 1-embedding if there is an elementary embedding $j : V \to M$ with critical point κ such that κ is measurable in M.

Corollary

Assume $V = \mathbf{L}(\mathbb{R})$ *and* AD *and let* $\delta = \Theta$. *Then*

$$\mathrm{HOD} \models \delta \textit{ is an inaccessible limit of } 1\textit{-embedding cardinals.}$$