The Universal Model for the negation-free fragment of IPC

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Universal Model

The n-universal model for IPC, $\mathcal{U}(n) = (U(n), R, V)$ is the "least" model of IPC that witnesses the failure of every unprovable formula of IPC.

Universal Model

- The first layer $\mathcal{U}(n)^1$ consists of 2^n nodes with the 2^n different n-colors under the discrete ordering.
- Under each element w in $\mathcal{U}(n)^k \setminus \mathcal{U}(n)^{k-1}$, for each color $s < col(w)$, we put a new node v in $\mathcal{U}(n)^{k+1}$ such that $v \prec w$ with $col(v) = s$, and we take the reflexive transitive closure of the ordering.
- Under any finite anti-chain X with at least one element in $\mathcal{U}(n)^k \setminus \mathcal{U}(n)^{k-1}$ and any color s with $s \leq col(w)$ for all $w \in X$, we put a new element v in $\mathcal{U}(n)^{k+1}$ such that $col(v) = s$ and $v \prec X$ and we take the reflexive transitive closure of the ordering.

The whole model $\mathcal{U}(n)$ is the union of its layers.

Properties of $\mathcal{U}(n)$

Lemma

For any finite rooted Kripke n-model $\mathfrak{M}$, there exists a unique $w \in \mathcal{U}(n)$ and a p-morphism of $\mathfrak{M}$ onto $\mathcal{U}(n)_w$.

Theorem

For any n-formula φ, $\mathcal{U}(n) \models \varphi$ iff $\vdash_{IPC} \varphi$.

de Jongh formulas for $\mathcal{U}(n)$

Proposition

For every $w \in \mathcal{U}(n)$ we have that

- $V(\varphi_w) = R(w)$, *where* $R(w) = \{w' \in \mathcal{U}(n) | wRw'\}$;
- $V(\psi_w) = \mathcal{U}(n) \backslash R^{-1}(w)$, *where* $R^{-1}(w) = \{w' \in \mathcal{U}(n) | w'Rw\}$.

de Jongh formulas for $\mathcal{U}(n)$

For any node w in an n-model $\mathfrak{M}$, if $w \prec \{w_1, \dots, w_m\}$, then we let
$\mathrm{prop}(w) := \{p_i | w \models p_i, 1 \leq i \leq n\}$,
$\mathrm{notprop}(w) := \{q_i | w \not\models q_i, 1 \leq i \leq n\}$,
$\mathrm{newprop}(w) := \{r_j | w \not\models r_j$ and $w_i \models r_j$ for each $1 \leq i \leq m$, for $1 \leq j \leq n\}$.

de Jongh formulas for $\mathcal{U}(n)$

If $d(w) = 1$, then let

$$\varphi_w := \bigwedge \mathrm{prop}(w) \wedge \bigwedge\{\neg p_k | p_k \in \mathrm{notprop}(w), 1 \leq k \leq n\},$$

and

$$\psi_w := \neg\varphi_w.$$

If $d(w) > 1$, and $\{w_1, \ldots, w_m\}$ is the set of all immediate successors of w, then define

$$\varphi_w := \bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{newprop}(w) \vee \bigvee_{i=1}^{m} \psi_{w_i} \to \bigvee_{i=1}^{m} \varphi_{w_i}),$$

and

$$\psi_w := \varphi_w \to \bigvee_{i=1}^{m} \varphi_{w_i}.$$

Universal Model for $[\vee, \wedge, \rightarrow]$-fragment

The n-universal model for the negation-free fragment of IPC, $\mathcal{U}^\star(n) = (U^\star(n), R^\star, V^\star)$, is a generated submodel of the universal model for IPC. It is (generated by):

$$\{u \in U(n) : \neg uRw_0\}$$

where w_0 is the maximal element of $\mathcal{U}(n)$ that satisfies all propositional atoms.

Universal Model for $[\vee, \wedge, \to]$-fragment

- The first layer $\mathcal{U}^\star(n)^1$ consists of $2^n - 1$ nodes with all the different n-colors – *excluding the color* $1 \ldots 1$ – under the discrete ordering.
- Under each element w in $\mathcal{U}^\star(n)^k \setminus \mathcal{U}^\star(n)^{k-1}$, for each color $s < col(w)$, we put a new node v in $\mathcal{U}^\star(n)^{k+1}$ such that $v \prec w$ with $col(v) = s$, and we take the reflexive transitive closure of the ordering.
- Under any finite anti-chain X with at least one element in $\mathcal{U}^\star(n)^k \setminus \mathcal{U}^\star(n)^{k-1}$ and any color s with $s \leq col(w)$ for all $w \in X$, we put a new element v in $\mathcal{U}^\star(n)^{k+1}$ such that $col(v) = s$ and $v \prec X$ and we take the reflexive transitive closure of the ordering.

The whole model $\mathcal{U}^\star(n)$ is the union of its layers.

Positive morphisms

Definition

A *positive morphism* is a partial function $f : (W, R, V) \to (W', R', V')$ such that:

1. $\mathrm{dom}(f) \supseteq \{w \in W : \exists p \in \mathrm{Prop}(w \notin V(p))\}$.
2. If $w, v \in \mathrm{dom}(f)$ and wRv then $f(w)R'f(v)$.
3. If $w \in \mathrm{dom}(f)$ and $f(w)R'v$ then there exists some $u \in \mathrm{dom}(f)$ such that $f(u) = v$ and wRu (**back**).
4. If $w \in \mathrm{dom}(f)$ and vRw, then $v \in \mathrm{dom}(f)$ (**downwards closed**).
5. For every $p \in \mathrm{Prop}$ we have $w \in V(p) \iff f(w) \in V'(p)$.

If the models are descriptive we furthermore require for every $Q \in \mathcal{Q}$ that $W \setminus R^{-1}(f^{-1}[W' \setminus Q]) \in \mathcal{P}$.

In the case of descriptive models these maps are also called strong partial Esakia morphisms.

$\mathcal{U}^\star(n)$ is universal

Theorem

For any finite rooted intuitionistic n-model $\mathfrak{M} = (M, R, V)$ *such that for some* $x \in M$ *and* $p \in \mathrm{Prop}$ *with* $x \notin V(p)$, *there exists unique* $w \in U^\star(n)$ *and positive morphism of* $\mathfrak{M}$ *onto* $\mathcal{U}^\star(n)_w$.

Theorem

For every n-formula $\varphi \in [\vee, \wedge, \to]$, $\mathcal{U}^\star(n) \models \varphi$ *if and only if* $\vdash_{IPC} \varphi$.

de Jongh formulas for $\mathcal{U}^\star(n)$

Proposition

For every $w \in \mathcal{U}^\star(n)$ we have that

- $V^\star(\varphi^\star_w) = R^\star(w)$
- $V^\star(\psi^\star_w) = \mathcal{U}^\star(n) \setminus (R^\star)^{-1}(w)$

de Jongh formulas for $\mathcal{U} \star (n)$

We have that $(\mathcal{U}^\star(n))^+$ is (isomorphic to) a generated submodel of $\mathcal{U}(n)$, whose domain consist of the elements of $U(n)$ whose only successor of depth 1 satisfies all propositional atoms. Let's call this generated submodel $\mathcal{M}$.

Definition

If $d(w) = 1$ then define

$$\varphi^\star_w = \bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{notprop}(w) \rightarrow \bigwedge \mathrm{notprop}(w))$$

and

$$\psi^\star_w = \varphi^\star_w \rightarrow \bigwedge_{i \in n} p_i.$$

de Jongh formulas for $\mathcal{U}\star(n)$

Definition

If $d(w) > 1$ then let $w \prec \{w_1, \ldots, w_r\}$ and define

$$\varphi^\star_w = \bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{newprop}(w) \vee \bigvee_{i \leq r} \psi^\star_{w_i} \to \bigvee_{i \leq r} \varphi^\star_{w_i})$$

and

$$\psi^\star_w = \varphi^\star_w \to \bigvee_{i \leq r} \varphi^\star_{w_i}.$$