Utrecht University

Propositional dependence logic

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Outline

1. Dependence logic
2. Structural completeness in logics of dependence
3. Future directions

Dependence logic

Motivating example

Let I be a subset of $\mathbb{R}$

Definition:

A function $f : I \to \mathbb{R}$ is said to be $\overset{uniformly}{\text{continuous}}$ on I if

for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x_0 \in I$ and any $x \in I$,

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

Continuity: $\forall x_0 \forall \epsilon \exists \delta \forall x \phi(x_0, \epsilon, \delta, x)$

Uniform continuity: $\forall \epsilon \exists \delta \forall x_0 \forall x \phi(x_0, \epsilon, \delta, x)$

First Order Quantifiers:

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \phi$$

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$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \phi$$

Henkin Quantifiers (Henkin, 1961):

$$\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix} \phi$$

meaning:

$$\exists f \exists g \forall x_1 \forall x_2 \phi(x_1, x_2, f(x_1), g(x_2))$$

Theorem (Enderton, Walkoe, 1970)

FO + *Henkin quantifiers* $\equiv \Sigma^1_1$ *(existential second-order logic).*

Independence Friendly Logic (Hintikka and Sandu, 1989):

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2/\{x_1\}\phi$$

- (Non-compositional) game theoretical semantics
- (Compositional) team semantics (Hodges 1997)

	x	y	z
s	a	b	c

$M \models_s \phi(x,y,z)$

X

	x	y	z
s_1	a	b	c
s_2	a	b	d
s_3	b	c	a
s_4	d	a	c

$M \models_X \phi(x,y,z)$

Theorem

IF-logic $\equiv \Sigma^1_1$.

First-order dependence Logic (Väänänen 2007):

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 (=(x_2, y_2) \wedge \phi)$$

First-order logic $+ \; =(\vec{x}, y)$

The value of *y* is
functionally determined
by the values of $\vec{x}$.

Theorem

$$\begin{aligned} \textit{First-order dependence logic} &\equiv \Sigma^1_1 \\ &\equiv \textit{IF-logic} \\ &\equiv \textbf{FO} + \textit{Henkin quantifiers} \end{aligned}$$

Team Semantics (Hodges, 1997)

A *team* X

	name	cloth	muddy
s_1	Abelard	white	no
s_2	Bill	blue	yes
s_3	Cath	white	no
s_4	Danny	white	no
s_5	Eloise	blue	yes
s_6	Father	blue	no

Y

- Does $M \models_{s_1} {=}(c, m)\,{=}(x, y)$, or does m depend on c under s_1?
- On the *team* X, m depends on c, or $M \models_X {=}(c, m)$.
- $M \not\models_Y {=}(c, m)$.
- In general, define $M \models_X {=}(\vec{x}, y)$ iff for any $s, s' \in X$,

$$s(\vec{x}) = s'(\vec{x}) \implies s(y) = s'(y).$$

This type of dependence corresponds precisely to *functional dependency* widely investigated in Database Theory (Armstrong 1974, etc.).

First-order dependence Logic = **FO** + $=(x_1, \ldots, x_n, y)$

Propositional dependence Logic (**PD**) = **CPC**+ $=(p_1, \ldots, p_n, q)$

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		happy	rainy	**d**ark cloth	**m**uddy
X	v_1	0	1	1	1
	v_2	1	1	0	0
	v_3	0	0	1	1
	v_4	1	0	0	0

- $X \models {=}(d, m)$: Whether Abelard is muddy depends completely on whether he wears dark cloth or not.
- $X \models {=}(h, d)$: Whether Abelard wears dark cloth depends entirely on whether he is happy or not.
- Therefore, whether Abelard is muddy depends on his mood (and his cloth color).

Armstrong axioms: $=(p, q), =(q, r) \vdash =(p, r),$
$=(q, r) \vdash =(p, q, r), \ldots$

Propositional dependence Logic (**PD**) $=$ **CPC**$+$ $=\!(p_1, \ldots, p_n, q)$

- Syntax of **PD**:

$$\phi ::= p \mid \neg p \mid \bot \mid =\!(\vec{p}, q) \mid \phi \wedge \phi \mid \phi \vee \otimes \phi$$

- A valuation is a function $v : \mathrm{Prop} \to \{0, 1\}$.
- A *team* is a set of valuations.

	happy	rainy	dark cloth	muddy
v_1	0	1	1	1
v_2	1	1	0	0
v_3	0	0	1	1
v_4	1	0	0	0

Team Semantics: Let X be a team.

- $X \models =\!(\vec{p}, q)$ iff for all $v, v' \in X$,
$$v(\vec{p}) = v'(\vec{p}) \implies v(q) = v'(q).$$
- $X \models p$ iff for all $v \in X$, $v(p) = 1$.
- $X \models \neg p$ iff for all $v \in X$, $v(p) = 0$.
- $X \models \phi \wedge \psi$ iff $X \models \phi$ and $X \models \psi$.
- $X \models \phi \otimes \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t.
$$Y \models \phi \text{ and } Z \models \psi.$$
- $X \models \bot$ iff $X = \emptyset$.

Fix $N = \{p_1, \ldots, p_n\}$, the set

$$\llbracket \phi(p_1, \ldots, p_n) \rrbracket := \{X \subseteq 2^N \mid X \models \phi\}.$$

- is downwards closed, that is, $Y \subseteq X \in \llbracket \phi \rrbracket \implies Y \in \llbracket \phi \rrbracket$;
- and nonempty, since $\emptyset \in \llbracket \phi \rrbracket$.

An algebraic view

Write $\mathcal{L}(\wp(2^N))$ for the set of all nonempty downwards closed subsets of $\wp(2^N)$.

Abramsky and Väänänen (2009):

Consider the algebra $(\mathcal{L}(\wp(2^N)), \otimes, \cap, \cup, \{\emptyset\}, \subseteq)$, where $A \otimes B = \downarrow \{X \cup Y \mid X \in A \text{ and } Y \in B\}$.

- $(\mathcal{L}(\wp(2^N)), \otimes, \{\emptyset\}, \subseteq)$ is a commutative quantale, in particular, $A \otimes B \leq C \iff A \leq B \multimap C$;
- $(\mathcal{L}(\wp(2^N)), \cap, \cup, \{\emptyset\})$ is a complete Heyting algebra, in particular, $A \cap B \leq C \iff A \leq B \to C$.

In logic terms, we can define

- $X \models \phi \otimes \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t.
  $$Y \models \phi \text{ and } Z \models \psi.$$
- $X \models \phi \multimap \psi$ iff for all Y if $Y \models \phi$, then $X \cup Y \models \psi$.
- $X \models \phi \rightarrow \psi$ iff for all $Y \subseteq X$: $\quad Y \models \phi \Longrightarrow Y \models \psi$.
- $X \models \phi \vee \psi$ iff $X \models \phi$ or $X \models \psi$.

Theorem (Y. 2013)

First-order dependence logic with intuitionistic connectives has the same expressive power as full second-order logic.

Propositional intuitionistic dependence logic (**PID**):

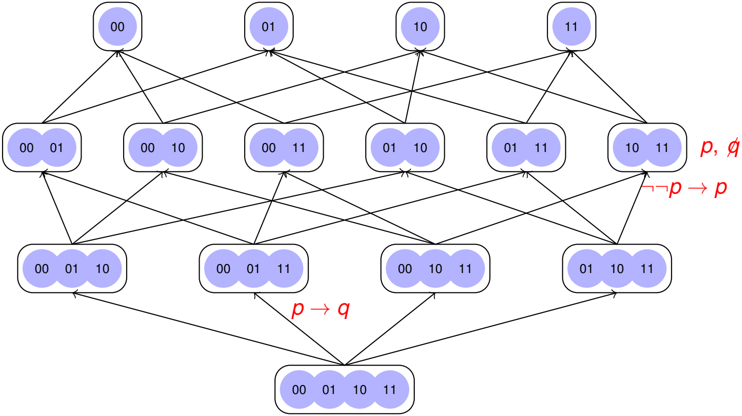
$$\phi ::= p \mid \bot \mid =(\vec{p}, q) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi$$

Observation (Y. 2014)

PID *is essentially equivalent to Inquisitive Logic*, **InqL** *(Groenendijk, Ciardelli and Roelofsen).*

The same semantics (team semantics), almost the same syntax.

A Medvedev frame: $(\wp(\mathbf{2^N}) \setminus \{\emptyset\}, \supseteq)$



Ciardelli and Roelofsen (2011):

$$\mathbf{PID}^{-} = \mathbf{InqL} = \mathbf{ML}^{\neg} = \{\phi \mid \tau(\phi) \in \mathbf{ML},\ \text{where } \tau(p) = \neg p\}$$

$$= \mathbf{KP}^{\neg} = \mathbf{KP} \oplus \neg\neg p \to p$$

Theorem (ess. Ciardelli and Roelofsen, 2011)

PID *is complete w.r.t. the following Hilbert style deductive system:*

Axioms:

- *all substitution instances of* **IPC** *axioms*
- *all substitution instances of*

$$\text{(KP)} \qquad \big(\neg p \to (q \vee r)\big) \to \big((\neg p \to q) \vee (\neg p \to r)\big).$$

- $\neg\neg p \to p$ *for all propositional variables* p
- $=\!(p_1, \cdots, p_n, q) \equiv \bigwedge_{i=1}^{n}(p_i \vee \neg p_i) \to (q \vee \neg q)$

Rules:

- *Modus Ponens*

Theorem (Y. and Väänänen, 2014)

PD *is sound and complete w.r.t. its natural deduction system.*

Fix $N = \{p_1, \dots, p_n\}$. Clearly, for each formula $\phi(p_1, \dots, p_n)$, $\{X \subseteq 2^N \mid X \models \phi\} = [\![\phi]\!] \in \mathcal{L}(\wp(2^N))$.

Theorem (Ciardelli, Huuskonen, Y.)

PD, $\mathbf{PD}^{\vee}$, **PID**, **InqL** *are maximal downwards closed logics, i.e., if* L *is one of these logics, then*

$$\mathcal{L}(\wp(2^N)) = \{[\![\phi]\!] \mid \phi(p_1, \dots, p_n) \textit{ is a formula of } \mathsf{L}\}.$$

In particular, $\mathbf{PD} \equiv \mathbf{PD}^{\vee} \equiv \mathbf{PID} \equiv \mathbf{InqL}$.

Theorem (Y.)

Every instance of $\vee$ *and* $\to$ *is definable in* **PD**, *but* $\vee$ *and* $\to$ *are not uniformly definable in* **PD**.

Theorem (Ciardelli, Huuskonen, Y.)

PD, **PD**$^\vee$, **PID**, **InqL** *are maximal downwards closed logics, i.e., if* L *is one of these logics, then*

$$\mathcal{L}(\wp(2^N)) = \{[\![\phi]\!] \mid \phi(p_1,\dots,p_n) \text{ is a formula of L}\}.$$

In particular, **PD** $\equiv$ **PD**$^\vee$ $\equiv$ **PID** $\equiv$ **InqL**.

Proof. We only treat **PD**$^\vee$ and **PID**. First, consider a team on N.

		p	q
	v_1	1	1
X	v_2	1	0
	v_3	0	1

Let

$$\Theta_X := \begin{cases} \bigotimes_{v\in X}(p_{i_1}^{\dot{v}(i_1)} \wedge \cdots \wedge p_{i_n}^{\dot{v}(i_n)}), & \text{for } \mathbf{PD}^\vee; \\ \neg\neg \bigvee_{v\in X}(p_{i_1}^{\dot{v}(i_1)} \wedge \cdots \wedge p_{i_n}^{\dot{v}(i_n)}), & \text{for } \mathbf{PID}. \end{cases}$$

Then $Y \models \Theta_X \iff Y \subseteq X$, for any team Y on N.

For each $\mathcal{K} \in \mathcal{L}(\wp(2^N))$, consider $\bigvee_{X\in\mathcal{K}} \Theta_X$. For any team Y on N,

$$Y \models \bigvee_{X\in\mathcal{K}} \Theta_X \iff \exists X \in \mathcal{K}(Y \subseteq X) \iff Y \in \mathcal{K}.$$

Hence $[\![\bigvee_{X\in\mathcal{K}} \Theta_X]\!] = \mathcal{K}$. □

Definition

A formula ϕ is said to be flat if

$$X \models \phi \iff \forall v \in X : \{v\} \models \phi.$$

Example:

- Formulas without any occurrences of $=(\vec{p}, q)$ or $\vee$ are flat.
- Negated formulas of **PID** and **InqL** are flat, i.e., $\neg\phi$ is always flat.

Lemma

For flat formulas ϕ of $\mathsf{L} \in \{\mathbf{PD}, \mathbf{PID}, \mathbf{InqL}\}$*,*

$$\vdash_{\mathbf{CPC}} \phi \iff \vdash_{\mathsf{L}} \phi$$

Structural completeness in logics of dependence

Joint work with Rosalie Iemhoff

Definition

Let $\vdash_{\mathsf{L}}$ be a consequence relation of a logic L. A substitution $\sigma : \mathrm{Prop} \to \mathrm{Form}_{\mathsf{L}}$ is called an L-*substitution* if $\vdash_{\mathsf{L}}$ is *closed under* σ, i.e., for every formulas ϕ, ψ of L,

$$\phi \vdash_{\mathsf{L}} \psi \Longrightarrow \sigma(\phi) \vdash_{\mathsf{L}} \sigma(\psi).$$

Fact: None of the logics **PD**, **PID**, **InqL** is closed under uniform substitution. E.g., for **PID**, $\vdash \neg\neg p \to p$, but $\nvdash \neg\neg(p \vee \neg p) \to (p \vee \neg p)$.

Lemma

Flat substitutions are L*-substitutions, for* L $\in \{$**PD**, **PID**, **InqL**$\}$.

Proof. For **InqL** and **PID**, it follows from (Ciardelli and Roelofsen, 2011). For **PD**, non-trivial. □

Let L be a logic, and $\mathcal{S}$ a class of L-substitutions.

Definition

A rule ϕ/ψ of L is said to be *$\mathcal{S}$-admissible*, in symbols $\phi \mid\!\sim^{\mathcal{S}}_{\mathsf{L}} \psi$, if

$$\forall \sigma \in \mathcal{S}: \vdash_{\mathsf{L}} \sigma(\phi) \implies \vdash_{\mathsf{L}} \sigma(\psi).$$

Definition

A logic L is said to be *$\mathcal{S}$-structurally complete* if every $\mathcal{S}$-admissible rule of L is derivable in L, i.e., $\phi \mid\!\sim^{\mathcal{S}}_{\mathsf{L}} \psi \iff \phi \vdash_{\mathsf{L}} \psi$.

Example:

- KP rule is admissible in all intermediate logics, but KP rule is not derivable in **IPC**.
- **KP** is not structurally complete, **ML** is structurally complete.
- **CPC** is structurally complete.

Theorem

For L $\in \{$**PD**, **PID**, **InqL**$\}$, L *is $\mathcal{F}$-structurally complete, where $\mathcal{F}$ is the class of all flat substitutions of the logic.*

Recall: For L $\in \{\mathbf{PD}, \mathbf{PID}, \mathbf{InqL}\}$, every formula $\phi(p_1, \ldots, p_n)$ of L is (semantically or/and provably) equivalent to a formula in the normal form $\bigvee_{i\in I} \Theta_{X_i}$, where

$$\Theta_{X_i} = \begin{cases} \bigotimes_{v\in X_i} (p_1^{v(1)} \wedge \cdots \wedge p_n^{v(n)}), & \text{for } \mathbf{PD}; \\ \neg\neg \bigvee_{v\in X_i} (p_1^{v(1)} \wedge \cdots \wedge p_n^{v(n)}), & \text{for } \mathbf{PID}, \mathbf{InqL}. \end{cases}$$

Definition (Projective formula)

Let L be a logic, and $\mathcal{S}$ a set of L-substitutions. A consistent L-formula ϕ is said to be *$\mathcal{S}$-projective* in L if there exists $\sigma \in \mathcal{S}$ such that

(1) $\vdash_{\mathsf{L}} \sigma(\phi)$

(2) $\phi, \sigma(\psi) \vdash_{\mathsf{L}} \psi$ and $\phi, \psi \vdash_{\mathsf{L}} \sigma(\psi)$ for all L-formulas ψ.

Such σ is called an projective unifier of ϕ.

Example:

- Every consistent formula is projective in **CPC**.
- Every consistent negated formula (i.e. $\neg\phi$) is projective in every intermediate logic.

Let $\mathsf{L} \in \{\mathbf{PD}, \mathbf{PID}, \mathbf{InqL}\}$.

Lemma

If $X \neq \emptyset$, then Θ_X is $\mathcal{F}$-projective in L.

Theorem

L *is $\mathcal{F}$-structurally complete, i.e.,* $\phi \mathrel{|\!\!\sim}^{\mathcal{F}}_{\mathsf{L}} \psi \iff \phi \vdash_{\mathsf{L}} \psi$.

Proof. It suffices to prove "$\Longrightarrow$". We only treat **PID**. Suppose $\phi \mathrel{|\!\!\sim}^{\mathcal{F}} \psi$ and ϕ is consistent. We have that $\vdash \phi \leftrightarrow \bigvee_{i \in I} \Theta_{X_i}$, where each $X_i \neq \emptyset$.

By the lemma, each Θ_{X_i} is $\mathcal{F}$-projective in **PID**. Let $\sigma_i \in \mathcal{F}$ be a projective unifier of Θ_{X_i}. Then $\vdash \sigma_i(\Theta_{X_i})$, which implies that $\vdash \sigma_i(\phi)$. Now, since $\phi \mathrel{|\!\!\sim}^{\mathcal{F}} \psi$, we obtain that $\vdash \sigma_i(\psi)$.

On the other hand, as σ_i is a projective unifier of Θ_{X_i}, we have that $\Theta_{X_i}, \sigma_i(\psi) \vdash \psi$, thus $\Theta_{X_i} \vdash \psi$ for all $i \in I$. It then follows that $\bigvee_{i \in I} \Theta_{X_i} \vdash \psi$, which implies that $\phi \vdash \psi$, as desired. ☐

Future directions

proof theory

- First-order dependence logic is not axiomatizable (since it is equivalent to Σ_1^1).
- Propositional logics of dependence (**PD**, **PID**, **InqL**) have Hilbert style deductive systems, natural deduction systems and labelled tableau calculi (Ciardelli, Roelofsen, 2011), (Y., Väänänen, 2014), (Sano, Virtema, 2014).
- Gentzen-style calculi for propositional logics of dependence?

algebraic approach

Abramsky and Väänänen (2009):

Consider the algebra $(\mathcal{L}(\wp(2^N)), \otimes, \cap, \cup, \{\emptyset\}, \subseteq)$.

- $(\mathcal{L}(\wp(2^N)), \otimes, \{\emptyset\}, \subseteq)$ is a commutative quantale, in particular, $A \otimes B \leq C \iff A \leq B \multimap C$;
- $(\mathcal{L}(\wp(2^N)), \cap, \cup, \{\emptyset\})$ is a complete Heyting algebra, in particular, $A \wedge B \leq C \iff A \leq B \to C$.

- $\mathcal{L}(\wp(2^N))$ is an algebra of the Logic of Bunched Implications (Pym, O'Hearn)
- For example, $\mathcal{L}(\wp(2^N)) = \{[\![\phi]\!] \mid \phi(p_1, \ldots, p_n)$ is a formula of **PID**$\}$.
- $\vdash_{\textbf{PID}} \phi \overset{?}{\iff} \mathcal{L}(\wp(2^N)) \models \alpha\phi \approx \mathbf{1}$ for all negative assignments α.

database theory

	name	mood	cloth	muddy
s_1	Abelard	happy	white	no
s_2	Bill	unhappy	blue	yes
s_3	Cath	happy	red	no
s_4	Danny	happy	green	no

- (Grädel and Väänänen, 2013): Independence logic (**Ind**)
  Ind $= \mathbf{FO} + \vec{y} \perp_{\vec{x}} \vec{z}$ (multivalued dependency)
  Ind is equivalent to Σ^1_1 (Galliani, 2012), thus captures NP over finite structures.
- (Galliani, 2012): Inclusion logic (**Inc**)
  Inc $= \mathbf{FO} + \vec{x} \subseteq \vec{y}$ (inclusion dependency)
  Inc is equivalent to the Least Fixed Point Logic (Galliani and Hella, 2014) over finite structures, thus captures PTIME over ordered finite structures.

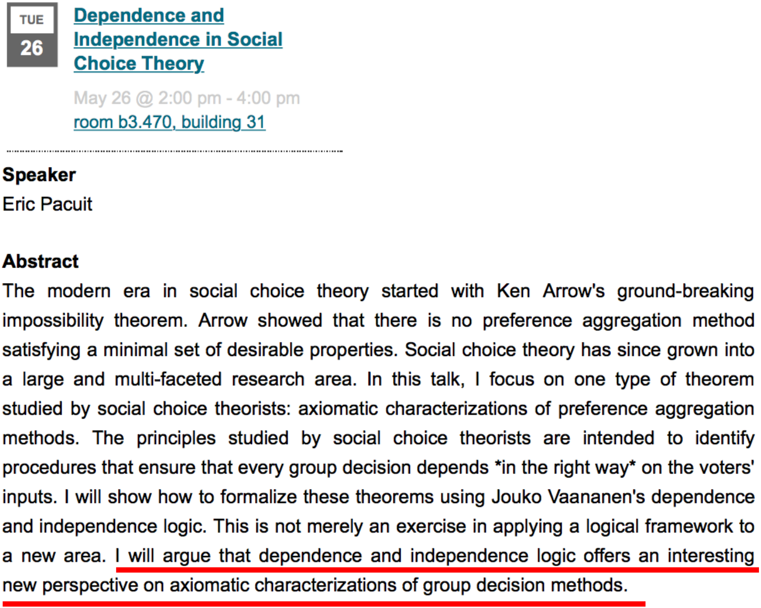
A logical formalism for reasoning about dependency in Big Data?

(Kontinen, Link and Väänänen, *Independence in Database Relations*, 2013;
Kontinen, Hannula and Link, *On Independence Atoms and Keys*, 2014)

modal and dynamic epistemic logic with team semantics

- (Väänänen, 2008): Modal dependence logic.
- (Kontinen, Müller, Schnoor and Vollmer, 2014): A van Benthem theorem for modal team logic.
- (Y., 2014): Modal intuitionistic dependence logic is complete w.r.t. a certain class of bi-relation Kripke models (closely related to the Kripke models of Fischer Servi's intuitionistic modal logic **IK**).
- (Galliani, 2013): Public announcement operator for dependence logic. In particular, $=(\vec{p}, q)$ can be read as "when the values of $\vec{p}$ are publicly announced, the value of q is determined".
- (Ciardelli and Roelofsen, 2014): Inquisitive dynamic epistemic logic.

Social choice theory



TUE
26

Dependence and Independence in Social Choice Theory

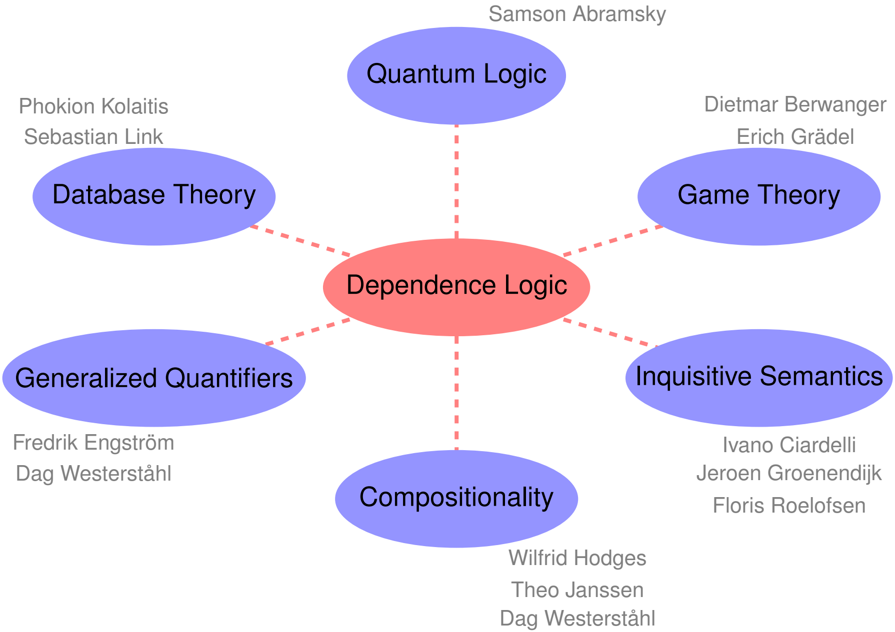
May 26 @ 2:00 pm - 4:00 pm
room b3.470, building 31

Speaker

Eric Pacuit

Abstract

The modern era in social choice theory started with Ken Arrow's ground-breaking impossibility theorem. Arrow showed that there is no preference aggregation method satisfying a minimal set of desirable properties. Social choice theory has since grown into a large and multi-faceted research area. In this talk, I focus on one type of theorem studied by social choice theorists: axiomatic characterizations of preference aggregation methods. The principles studied by social choice theorists are intended to identify procedures that ensure that every group decision depends *in the right way* on the voters' inputs. I will show how to formalize these theorems using Jouko Vaananen's dependence and independence logic. This is not merely an exercise in applying a logical framework to a new area. I will argue that dependence and independence logic offers an interesting new perspective on axiomatic characterizations of group decision methods.



Samson Abramsky

Quantum Logic

Phokion Kolaitis
Sebastian Link

Database Theory

Dietmar Berwanger
Erich Grädel

Game Theory

Dependence Logic

Generalized Quantifiers

Fredrik Engström
Dag Westerståhl

Inquisitive Semantics

Ivano Ciardelli
Jeroen Groenendijk
Floris Roelofsen

Compositionality

Wilfrid Hodges
Theo Janssen
Dag Westerståhl