Fuzzy eubouliatic logic

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Introduction

Alan Ross Anderson [1] was interested in *eubouliatic logic*, the logic of prudence and related concepts, such as safety and risk. "Eubouliatic" comes from the Greek *euboulos*, meaning "prudent."

Eubouliatic logic

Anderson defined eubouliatic logic $\mathbf{E_R}$ as relevant system $\mathbf{R}$ plus:

- Constant $\mathcal{G}$ ("the good thing").
- Operator $\mathcal{S}$ ("it is prudent (safe) that").
- Definition $\mathcal{S}A \stackrel{\text{def}}{=} A \to \mathcal{G}$ ("A guarantees the good thing").

Anderson himself wrote $\mathcal{R}_w A$ ("A is without risk") instead of $\mathcal{S}A$.

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Eubouliatic fragment

The eubouliatic fragment of $\mathbf{E_R}$ ($\mathbf{E_R}$ without $\mathcal{G}$) can be axiomatized as **R** plus the following axioms [5]:

(E1) $(A \to B) \to (\mathcal{S}B \to \mathcal{S}A)$.

(E2) $A \to \mathcal{S}\mathcal{S}A$.

Proof: for each derivation $A_1, \ldots, A_n$ define $\mathcal{G}$ as $\mathcal{S}t$, where $t \stackrel{\text{def}}{=} \bigwedge_{i=1}^{m}(p_i \to p_i)$ and $p_1, \ldots, p_m$ is a list of the propositional variables occurring in $A_1, \ldots, A_n$.

Additional notions

Some additional notions were defined as follows:

1. $\mathcal{R}A$ ("it is risky that A"): $\mathcal{R}A \stackrel{\text{def}}{=} \neg\mathcal{S}A$.
2. $\mathcal{H}A$ ("it is heedless that A"): $\mathcal{H}A \stackrel{\text{def}}{=} \mathcal{S}\neg A$.
3. $\mathcal{C}A$ ("it is cautious that A"): $\mathcal{C}A \stackrel{\text{def}}{=} \neg\mathcal{H}A$.

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Logic of additional notions

The logic of these additional notions can be axiomatized as follows [5]:

1. **R** plus $(A \to B) \to (\mathcal{R}A \to \mathcal{R}B)$ and $\mathcal{R}(A \circ B) \to (A \circ \mathcal{R}B)$.
2. **R** plus $(A \to B) \to (\mathcal{H}A \to \mathcal{H}B)$ and $\mathcal{H}(\mathcal{H}A \to A)$.
3. **R** plus $(A \to B) \to (\mathcal{C}B \to \mathcal{C}A)$ and $\mathcal{C}\mathcal{C}A \to A$.

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Definitions

As usual,

$$(\neg)\quad \neg A \stackrel{\text{def}}{=} A \to f,$$

$$(\circ)\quad A \circ B \stackrel{\text{def}}{=} \neg(A \to \neg B),$$

$$(\leftrightarrow)\quad A \leftrightarrow B \stackrel{\text{def}}{=} (A \to B) \wedge (B \to A).$$

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Square of opposition

$\mathcal{S}A$ and $\mathcal{R}A$ are contradictories. $\mathcal{H}A$ and $\mathcal{C}A$ are contradictories. The so-called "axiom of avoidance" says that $\mathcal{S}A \to \neg\mathcal{S}\neg A$. Hence:

1. $\vdash \mathcal{S}A \to \mathcal{C}A$: $\mathcal{S}A$ and $\mathcal{C}A$ are subalterns.
2. $\vdash \mathcal{H}A \to \mathcal{R}A$: $\mathcal{H}A$ and $\mathcal{R}A$ are subalterns.
3. $\vdash \neg(\mathcal{S}A \wedge \mathcal{H}A)$: $\mathcal{S}A$ and $\mathcal{H}A$ are contraries.
4. $\vdash \mathcal{C}A \vee \mathcal{R}A$: $\mathcal{C}A$ and $\mathcal{R}A$ are subcontraries.

These notions can be depicted in a square of opposition (Fig. 1).

Illustration

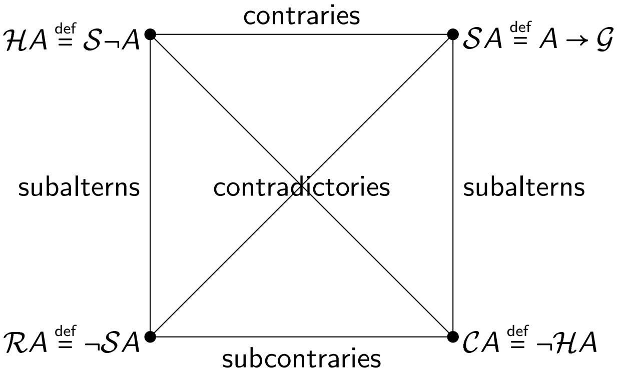

Figure: Four eubouliatic concepts [1, Fig. 8].

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Objection

- $(A \wedge B) \to A$ is a theorem, therefore $\mathcal{S}A \to \mathcal{S}(A \wedge B)$ is also a theorem, by axiom (E1).
- Let A stand for "x drinks a cup of tea" and let B stand for "x detonates a bomb."
- (E1) says that if it is safe to drink a cup of tea, then it is also safe to drink a cup of tea and detonate a bomb.
- But it is not.

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Refutation of objection

- "It is safe to drink a cup of tea and detonate a bomb" should not be formalized as $\mathcal{S}(A \wedge B)$ but as $\mathcal{S}(A \circ B)$.
- $\mathcal{S}(A \circ B)$ is equivalent with $B \rightarrow \mathcal{S}A$.
- Both $\mathcal{S}A \rightarrow \mathcal{S}(A \circ B)$ and $\mathcal{S}A \rightarrow (B \rightarrow \mathcal{S}A)$ are invalid.

Shortcomings

We claim that:

- The square of opposition drawn in Fig. 1 is an acceptable representation of the relations obtaining between these concepts.

However ... This square is not good enough.

Fuzzification

1. The operator "it is safe that..." is *incomplete*. Safe *for whom*? Example: the safer an SUV is for its occupants, the less safe it is for the pedestrians who happen to be around.
2. Safety has *many aspects*, which may lead to several judgments. OED: "safety" has at least eleven different senses. Causal, epistemic, modal, probabilistic and temporal notions all seem to play some role.
3. Therefore: safety is a *fuzzy concept*. This is the contemporary *consensus*. Thousands of references can be given.

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Hedges

1. This is further illustrated by the concept of "hedges". The operator "safe" is typically used with linguistic "hedges": sort of, kind of, loosely speaking, more or less, on the ... side, roughly, pretty (much), relatively, somewhat, rather, mostly, technically, strictly speaking, essentially, in essence, basically, particularly, par excellence, largely, for the most part, very, highly, especially, exceptionally, quintessentially, literally, often, almost, typically/typical, as it were, in a sense, in one sense, in a real sense, in an important sense, in a way, details aside, so to say, practically, anything but, nominally, in name only, actually, really, ... [4].
2. This implies: "safe" *itself* is a fuzzy operator. Hedges for crisp, black/white concepts simply do not make sense. This explains why there are so many jokes about ladies who are "a little bit pregnant."

We are therefore going to propose a fuzzy system of eubouliatic logic.

Fuzzy relevant logic

A fuzzy logic is a logic in which there are not just two extreme degrees of truth—truth (1) and falsity (0)—but various intermediate degrees of truth, i.e., degrees of truth between 1 and 0. These degrees of truth are linearly ordered, so that every two values are comparable, i.e., for any A, B either $v(A) \leq v(B)$ or $v(A) \geq v(B)$ [3]. The oldest example of a fuzzy logic is Łukasiewicz's logic **Ł3**, in which sentences have values 1, $\frac{1}{2}$ or 0 and $0 < \frac{1}{2} < 1$.

A relevant logic is a logic in which $A \to B$ is a theorem if and only if A and B share a propositional variable or a (meta-definable) propositional constant.

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RM

There is one well-known logic that is fuzzy and "semi-relevant": **RM**, which is **R** plus axiom scheme $A \to (A \to A)$. **RM** has theorem $(A \to B) \vee (B \to A)$, so it is fuzzy. However, **RM** also has theorems $\neg(A \to A) \to (B \to B)$ and $(A \wedge \neg A) \to (B \vee \neg B)$, so **RM** is not relevant.

FR

Fuzzy relevant logic **FR** [6] [9] [10] is **R** plus:

(Lin) $(A \to B) \vee (B \to A)$.

Theorem

$\mathbf{R} \subset \mathbf{FR} \subset \mathbf{RM}$.

FR is weaker than **RM** because **FR** does not prove $A \to (A \to A)$, as Fig. 2 shows.

Matrix

Generated by `MaGIC` [7].

Logic: **R**. Extra: $(A \to B) \vee (B \to A)$.
Fragment: $\to$, $\wedge$, $\vee$, $\neg$, $\circ$, t, f, T, F.
Fail: $A \to (A \to A)$. Negation table:

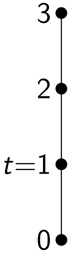
a	0	1	2	3
$\neg a$	3	2	1	0

Order: $0 \leq 1 \leq 2 \leq 3$. Choice of t: 1.

Implication matrix:

$\to$	0	1	2	3
0	3	3	3	3
1	0	1	2	3
2	0	0	1	3
3	0	0	0	3

Failure: $2 \to (2 \to 2)$.



R ⊂ FR

R is weaker than **FR** because the following formulas are provable in **FR** but not in **R**.

1. $(A \wedge B) \rightarrow (A \circ B)$ [9, (2)]
2. $((A \rightarrow (B \vee C)) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$ [9, (5)].

FR: algebraic semantics

Eunsuk Yang [9] [10].

An **FR**-algebra is a structure $\mathbf{A} = (A, t, f, \wedge, \vee, *, \rightarrow)$, where

1. $(A, \wedge, \vee)$ is a distributive lattice,
2. $(A, *, t)$ is a commutative monoid,
3. $y \leq x \rightarrow z$ iff $x * y \leq x$, for all $x, y, z \in A$ (residuation),
4. $x \leq x * x$ (contraction),
5. $((x \rightarrow f) \rightarrow f) \leq x$ (double negation elimination),
6. $x \leq y$ or $y \leq x$ (linear order).

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$\mathcal{A}$-evaluation

Let $\mathcal{A}$ be an algebra. An $\mathcal{A}$-evaluation is a function $v\colon \mathrm{WFF} \to \mathcal{A}$ satisfying

1. $v(A \to B) = v(A) \to v(B)$,
2. $v(A \wedge B) = v(A) \wedge v(B)$,
3. $v(A \vee B) = v(A) \vee v(B)$,
4. $v(A \circ B) = v(A) * v(B)$,
5. $v(f) = f$.

A is $\mathcal{A}$-valid iff $t \leq v(A)$ for all $\mathcal{A}$-evaluations v. An $\mathcal{A}$-model of T is an $\mathcal{A}$-evaluation such that $t \leq v(A)$ for all $A \in T$. $\mathrm{Mod}(T, \mathcal{A})$ is the class of $\mathcal{A}$-models of T. A is a semantic consequence of T w.r.t. $\mathcal{K}$ iff $\mathrm{Mod}(T, \mathcal{A}) = \mathrm{Mod}(T \cup \{A\}, \mathcal{A})$ for all $\mathcal{A} \in \mathcal{K}$.

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Soundness and completeness

We write $T \vdash A$ for A is derivable from T. Note that $T \cup \{A\} \vdash B$ iff $T \vdash (A \wedge t) \to B$. Let T be a theory over **FR**. Let $[A]_T = \{B \in \text{WFF} \colon T \vdash A \leftrightarrow B\}$. A_T is the set of all classes $[A]_T$. A_T is an **FR** algebra.

Theorem

A is derivable from T iff A is a semantic consequence of T.

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FR: algebraic Kripke-style semantics

Eunsuk Yang [9] [10].
An **FR**-frame is an algebraic Kripke frame $\mathbf{X} = \langle X, t, f, \leq, *, \rightarrow \rangle$, where $\langle X, t, f, \leq, *, \rightarrow \rangle$ is a linearly ordered residuated pointed commutative monoid satisfying $x = (x \rightarrow f) \rightarrow f$ and $x \leq x * x$. The members of X are called nodes.

Forcing

A forcing is a relation between nodes and propositional variables such that:

1. if $p \in \mathrm{AT}$ then if $x \vDash p$ and $y \leq x$, then $y \vDash p$ (backward heredity),
2. $x \vDash t$ iff $x \leq t$,
3. $x \vDash f$ iff $x \leq f$,
4. $x \vDash A \wedge B$ iff $x \vDash A$ and $x \vDash B$,
5. $x \vDash A \vee B$ iff $x \vDash A$ or $x \vDash B$,
6. $x \vDash A \circ B$ iff there are $y, z \in X$ such that $y \vDash A$, $z \vDash B$ and $x \leq y * z$,
7. $x \vDash A \rightarrow B$ iff for all $y \in X$, if $y \vDash A$, then $x * y \vDash B$.

Soundness

An **FR**-model is a pair $(\mathbf{X}, \vDash)$ where $\mathbf{X}$ is an **FR**-frame and $\vDash$ is a forcing on X. A is true in $(\mathbf{X}, \vDash)$ iff $t \vDash A$. A is valid in $\mathbf{X}$ ($\mathbf{X} \vDash A$) iff A is true in $(\mathbf{X}, \vDash)$ for every forcing $\vDash$ on $\mathbf{X}$.

Theorem
If $\vdash A$, then A is valid in every **FR***-frame.*

Completeness

Let $\mathbf{X} = \langle X, t, f, \leq, *, \rightarrow \rangle$ be an **FR**-frame. Then

$$\mathbf{A} = \langle X, t, f, \max, \min, *, \rightarrow \rangle$$

is an **FR**-algebra. Let $\mathbf{X}$ be the $\{t, f, \leq, *, \rightarrow\}$-reduct of an **FR**-algebra $\mathbf{A}$ and let v be a forcing in $\mathbf{A}$. Let $x \vDash p$ iff $x \leq v(p)$ for all $x \in \mathbf{A}$ and all $p \in \mathrm{AT}$. Then $(\mathbf{X}, \vDash)$ is an **FR**-model and $x \vDash A$ iff $x \leq v(A)$.

Theorem

FR *is strongly complete with respect to the class of all* **FR**-*frames.*

Fuzzy eubouliatic logic

Fuzzy eubouliatic logic $\mathbf{E_{FR}}$ is **FR** with the four operators $\mathcal{S}$, $\mathcal{H}$, $\mathcal{R}$, $\mathcal{C}$, defined as in $\mathbf{E_R}$.
$\mathbf{E_{FR}}$ extends $\mathbf{E_R}$, so all results obtained above for $\mathcal{S}$, $\mathcal{H}$, $\mathcal{R}$, $\mathcal{C}$ in $\mathbf{E_R}$ also hold in $\mathbf{E_{FR}}$. Note that $\mathbf{E_{FR}}$ provides $\mathcal{S}(A \circ B) \to \mathcal{S}(A \wedge B)$:

1	$(A \wedge B) \to (A \circ B)$	**FR**
2	$(A \to B) \to (\mathcal{S}B \to \mathcal{S}A)$	(E1)
3	$\mathcal{S}(A \circ B) \to \mathcal{S}(A \wedge B)$	1,2.

In the bomb-example, $\mathcal{S}(A \circ B)$ was false, even though $\mathcal{S}(A \wedge B)$ was true.

Conclusion

Anderson's eubouliatic logic can be extended to a system of fuzzy eubouliatic logic. The square of opposition does not change.

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