Deontic Fragments: Simple Syntactic Proofs

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Introduction

In his study of the deontic fragments of certain alethic modal systems, Lennart Åqvist wrote that “proof-theoretical methods seem to be less natural here” [2, p. 227]. I disagree. I show that some results in this area can easily be obtained by proof-theoretical methods. The proofs are at least as “natural” as Åqvist’s proofs.

OS4 is the deontic fragment of $S4_Q$

Deontic system *OS4*.

Language $\mathscr{L}(OS4)$: $F ::= p | \neg F | OF | F \wedge F | F \vee F | F \rightarrow F | F \leftrightarrow F$,
where p is an atomic formula.

Axiom schemata:

A1. All theorems of *PC*.
A2. $O(A \rightarrow B) \rightarrow (OA \rightarrow OB)$.
A3. $O(OA \rightarrow A)$.
A4. $OA \rightarrow OOA$.

Rules of inference:

R1. From A and $A \rightarrow B$ infer B.
R2. From A infer OA.

Alethic modal system *S4*.

Language $\mathscr{L}(S4)$: $F ::= p|\neg F|\Box F|F \wedge F|F \vee F|F \rightarrow F|F \leftrightarrow F$,
where p is an atomic formula.

Axiom schemata:

A1. All theorems of *PC*.
A5. $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$.
A6. $\Box A \rightarrow A$.
A7. $\Box A \rightarrow \Box\Box A$.

Rules of inference:

R1. From A and $A \rightarrow B$ infer B.
R3. From A infer $\Box A$.

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Mixed alethic-deontic system $\Box OS4$.

Language $\mathscr{L}(\Box OS4)$: $\mathscr{L}(\Box OS4) = \mathscr{L}(OS4) \cup \mathscr{L}(S4)$.

Axiom schemata: A1, . . . , A7 and

A8. $\Box(A \to B) \to (OA \to OB)$.

A9. $OA \to \Box OA$.

Rules of inference: R1, R2 and R3.

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Mixed alethic-deontic system $\Box OS4_Q$.

Language $\mathscr{L}(\Box OS4_Q)$:

$$F ::= p|Q|\neg F|OF|\Box F|F \wedge F|F \vee F|F \to F|F \leftrightarrow F,$$

where p is an atomic formula.

Axiom schemata: A1, ..., A9 and

A10. $OA \leftrightarrow \Box(Q \to A)$.

Rules of inference: R1, R2 and R3.

We refer to those formulas of $\Box OS4_Q$ in which Q occurs, if at all, only in contexts of the form $\Box(Q \to A)$, as Q-formulas of $\Box OS4_Q$. If A^Q is any Q-formula of $\Box OS4_Q$, then the O-transform of A^Q is the formula A^O got by replacing every part of A^Q of the form $\Box(Q \to A)$ by OA. Evidently, if A^Q is a Q-formula of $\Box OS4_Q$, then A^O will be a formula of $\Box OS4$.

Theorem (Theorem 1)

If A^Q is a Q-formula of $\Box OS4_Q$ and A^O is its O-transform, then $\Box OS4_Q \vdash A^Q$ iff $\Box OS4 \vdash A^O$.

PROOF: We first observe that in $\Box OS4_Q$ we have $\vdash OA \leftrightarrow \Box(Q \rightarrow A)$ and a derivable rule of substitution, so $\Box OS4_Q \vdash A^Q$ iff $\Box OS4_Q \vdash A^O$. This is half the battle. What remains to be proven is that $\Box OS4_Q$ is a conservative extension of $\Box OS4$, that is, that each Q-free formula of $\Box OS4_Q$ has a Q-free proof. Such a proof will also be a proof in $\Box OS4$, from which it will follow that if $\Box OS4_Q \vdash A^O$ then $\Box OS4 \vdash A^O$.

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The leading idea is that, although Q cannot be replaced by the same Q-free formula in every proof, it is still possible to find, for each proof of a Q-free formula, a particular Q-free formula that can replace Q throughout that proof. Let $A_1, \ldots, A_n$ ($A_n = A$) be a proof of A in $\Box OS4_Q$, and let $p_1, \ldots, p_m$ be a list of the propositional variables and constants occurring in $A_1, \ldots, A_n$. Then, for this proof of A, we define Q^* as $\bigwedge_{i=1}^{m}(Op_i \rightarrow p_i)$. Let A_i^* be the result of replacing Q throughout A_i by Q^*. We show inductively that each of $A_1^*, \ldots, A_n^*$ ($A_n^* = A^*$) has a Q-free proof in $\Box OS4_Q$, which is to say a proof in $\Box OS4$, as required.

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1. Base case: if A_i is one of the axioms A1, ..., A9 of $\Box OS4_Q$, then $\Box OS4 \vdash A_i^*$ by the same axiom.
2. If A_i is an axiom A10[$\rightarrow$] of $\Box OS4_Q$, then A_i^* has the form $OA \rightarrow \Box(Q^* \rightarrow A)$. We need to show that $\Box OS4 \vdash A_i^*$. Let q_1, ..., q_k be a list of the propositional variables and constants occurring in A. Then an easy induction on the length of A shows that $\bigwedge_{j=1}^{k}(Oq_j \rightarrow q_j) \rightarrow (OA \rightarrow A)$. Evidently, $Q^* \rightarrow \bigwedge_{j=1}^{k}(Oq_j \rightarrow q_j)$ since the q_j are all among the p_i, so

1.	$\Box OS4 \vdash Q^* \rightarrow (OA \rightarrow A)$	From the above.
2.	$\Box OS4 \vdash OA \rightarrow (Q^* \rightarrow A)$	From 1 by A1, R1.
3.	$\Box OS4 \vdash \Box(OA \rightarrow (Q^* \rightarrow A))$	From 2 by R3.
4.	$\Box OS4 \vdash \Box OA \rightarrow \Box(Q^* \rightarrow A)$	From 3 by A5, A1, R1.
5.	$\Box OS4 \vdash OA \rightarrow \Box(Q^* \rightarrow A)$	From 4 by A9, A1, R1.
6.	$\Box OS4 \vdash A_i^*$	From 5 by Def A_i^*.

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1. If A_i is an axiom A10[←] of $\Box OS4_Q$, then A_i^* has the form $\Box(Q^* \to A) \to OA$. We need to show that $\Box OS4 \vdash A_i^*$.

1.	$\Box OS4 \vdash \Box(Q^* \to A) \to (OQ^* \to OA)$	From A8.
2.	$\Box OS4 \vdash OQ^* \to (\Box(Q^* \to A) \to OA)$	From 1 by A1, R1.
3.	$\Box OS4 \vdash OQ^*$	From A3.
4.	$\Box OS4 \vdash \Box(Q^* \to A) \to OA$	From 2, 3 by R1.
5.	$\Box OS4 \vdash A_i^*$	From 4 by Def A_i^*.

2. If A_i is a conclusion from A_j and A_k by R1, R2 or R3, then $\Box OS4 \vdash A_j^*$ and $\Box OS4 \vdash A_k^*$ by the inductive hypothesis, and $\Box OS4 \vdash A_i^*$ by the same rule.

This completes the proof, which shows essentially that the addition of Q and axiom schema A10 to $\Box OS4$ is otiose, since $\Box OS4$ already contains an equivalent deontic theory.

Theorem 1 is also provable in systems based on the intuitionist propositional calculus, Fitch calculus and Johansson's minimal calculus [3, p. 223, p. 223, p. 299]. Since the proof of Theorem 1 does not depend on $A \to (B \to A)$, contraction, expansion, or distribution, it can also be used in the contexts of relevance and linear logic.

NOTE: If $A \to (B \to A)$ is available, then A8 $[\Box(A \to B) \to (OA \to OB)]$ can be replaced with $\Box A \to OA$:

1	$A \to (B \to A)$	Assumption
2	$A \to ((OB \to B) \to A)$	1
3	$\Box(A \to ((OB \to B) \to A))$	2, R3
4	$\Box A \to \Box((OB \to B) \to A)$	3, A5
5	$\Box((OB \to B) \to A) \to (O(OB \to B) \to OA)$	A8
6	$(O(OB \to B) \to OA) \to OA$	A3
7	$\Box A \to OA$	4, 5, 6
8	A8	7, A2

Alethic system $S4_Q$.

Language $\mathscr{L}(S4_Q)$: $F ::= p|Q|\neg F|\Box F|F \wedge F|F \vee F|F \rightarrow F|F \leftrightarrow F$,
where p is an atomic formula.

Axiom schemata: A1, A5, A6, A7.

Rules of inference: R1 and R3.

Theorem (Theorem 2)

$S4_Q \vdash A^Q$ iff $OS4_Q \vdash A^Q$.

Proof.

$OS4_Q$ is a conservative extension of $S4_Q$ because in $S4_Q$, OA can be defined as $\Box(Q \to A)$. □

Theorem (Theorem 3)

$OS4$ is the deontic fragment of $S4_Q$.

Proof.

From Theorems 1 and 2. □

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$OS5$ is the deontic fragment of $S5_Q$

1. $PA = \neg O \neg A$.
2. $\Diamond A = \neg \Box \neg A$.
3. $OS5 = OS4$ plus $POA \rightarrow OA$.
4. $S5_Q = S4_Q$ plus $\Diamond \Box A \rightarrow \Box A$.

Theorem (Theorem 4)

OS5 is the deontic fragment of $S5_Q$.

Proof.

From Theorems 1 and 2. □

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Let $\Sigma = S4, S5$.

1. $O\Sigma^{+} = O\Sigma$ plus $OA \to PA$.
2. $\Sigma_{Q}^{+} = \Sigma_{Q}$ plus $\Diamond Q$.

Theorem (Theorem 5)

Let $\Sigma = S4, S5$. $O\Sigma^{+}$ is the deontic fragment of Σ_{Q}^{+}.

Proof.

From Theorems 1 and 2. □

The proof of Theorem 4 is purely syntactic and considerably shorter than the semantical proof in [1], as described (but not reproduced) in [2]. Conclusion: at least some of Åqvist's results can easily be obtained by proof-theoretical methods. The resulting proofs are at least as "natural" as Åqvist's proofs.

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