Conditional Logic as Non-Monotonic Reasoning: Proof Systems and Semantics

Johannes Marti and Riccardo Pinosio

June 17, 2014

Crash course to conditionals

- Syntax
  - flat
  - nested
- Standard semantics
  - preorder
  - indexed preorder
- Applications
  - defeasible inference
  - conditionals

Other perspectives

- Selection functions
- Premise semantics
- Game semantics

A cry for help: proof theory?

	flat	nested
syntax	$A \mathrel{\vert\!\sim} C$	$\varphi \rightsquigarrow \psi$
semantics	$v \leq u$	$v \leq_w u$
applications	defeasible inference	conditionals

	flat	nested
syntax	$A \mathrel{\succ\!\!\sim} C$	$\varphi \rightsquigarrow \psi$
semantics	$v \leq u$	$v \leq_w u$
applications	defeasible inference	conditionals

$$\frac{A \mathrel{|\!\sim} C \qquad A \vdash B \qquad B \vdash A}{B \mathrel{|\!\sim} C}\ (LE) \qquad \frac{A \mathrel{|\!\sim} C \qquad C \vdash B}{A \mathrel{|\!\sim} B}\ (RW)$$

$$\frac{}{A \mathrel{|\!\sim} A}\ (Id) \qquad \frac{A \mathrel{|\!\sim} B \qquad A \mathrel{|\!\sim} C}{A \mathrel{|\!\sim} B \wedge C}\ (And)$$

$$\frac{A \mathrel{|\!\sim} B \qquad A \mathrel{|\!\sim} C}{A \wedge B \mathrel{|\!\sim} C}\ (CM) \qquad \frac{A \mathrel{|\!\sim} C \qquad B \mathrel{|\!\sim} C}{A \vee B \mathrel{|\!\sim} C}\ (Or)$$

$$\frac{A \mathrel{|\!\sim} B \qquad B \mathrel{|\!\sim} C}{A \vee B \mathrel{|\!\sim} C}\ (CC)$$

	flat	nested
syntax	$A \mathrel{\sim\!\!\!\mid} C$	$\varphi \rightsquigarrow \psi$
semantics	$v \leq u$	$v \leq_w u$
applications	defeasible inference	conditionals

- All instances of propositional tautologies
- Modus Ponens for $\to$
- If $\varphi \leftrightarrow \psi$ then $(\varphi \rightsquigarrow \chi) \leftrightarrow (\psi \rightsquigarrow \chi)$
- If $\varphi \leftrightarrow \psi$ then $(\chi \rightsquigarrow \varphi) \leftrightarrow (\chi \rightsquigarrow \psi)$
- $(\varphi \rightsquigarrow \psi) \to (\varphi \rightsquigarrow (\psi \vee \chi))$
- $\varphi \rightsquigarrow \varphi$
- $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow (\psi \wedge \chi))$
- $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \to ((\varphi \wedge \psi) \rightsquigarrow \chi)$
- $(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to ((\varphi \vee \psi) \rightsquigarrow \chi)$

	flat	nested
syntax	$A \mathrel{|\!\sim} B$	$\varphi \rightsquigarrow \psi$
semantics	$v \leq u$	$v \leq_w u$
applications	defeasible inference	conditionals

$\leq$ is a preorder.

$A \mathrel{\sim\!\!\!\!\mid} C$ iff for all $w \in [\![A]\!]$ there is a $v \in [\![A]\!]$ with $v \leq w$ such that $u \in [\![C]\!]$ for all $u \leq v$ with $u \in [\![A]\!]$

If $\leq$ is well-founded:

$A \mathrel{\sim\!\!\!\!\mid} C$ iff $w \in [\![C]\!]$ for all $w \in \mathrm{Min}(\leq, [\![A]\!])$

	flat	nested
syntax	$A \mid\!\sim C$	$\varphi \rightsquigarrow \psi$
semantics	$v \leq u$	$v \leq_w u$
applications	defeasible inference	conditionals

$\leq_w$ is a preorder for every world w.

$w \models \varphi \rightsquigarrow \psi$ iff for all $v \in [\![\varphi]\!]$ there is a $u \in [\![\varphi]\!]$ with $u \leq_w v$ such that $z \models \varphi \rightarrow \psi$ for all $z \leq_w u$

If $\leq_w$ is well-founded:

$$w \models \varphi \rightsquigarrow \psi \quad \text{iff} \quad v \models \psi \text{ for all } v \in \mathrm{Min}(\leq_w, [\![\varphi]\!])$$

	flat	nested
syntax	$A \mathrel{|\!\sim} C$	$\varphi \rightsquigarrow \psi$
semantics	$v \leq u$	$v \leq_w u$
applications	defeasible inference	conditionals

$$\text{Tweety is a bird} \mathrel{|\!\sim} \text{Tweety can fly}$$
$$\text{Tweety is a penguin} \vdash \text{Tweety is a bird}$$
$$\text{Tweety is a penguin} \not\mathrel{|\!\sim} \text{Tweety can fly}$$

Use the preorder $\leq$

Tweety is a bird Tweety can fly	$<$	Tweety is a penguin (and a bird) Tweety can not fly

$v \leq u$ means that v is more normal than u.

	flat	nested
syntax	$A \mathrel{|\!\sim} C$	$\varphi \rightsquigarrow \psi$
semantics	$v \leq u$	$v \leq_w u$
applications	defeasible inference	conditionals

$B^{\varphi}\psi$: The agent believes ψ conditionally on φ.

$v \leq_w u$ means that at w the agent considers v to be more plausible than u.

Introspection axioms and constraints on $\leq$

$\varphi \rightsquigarrow \psi$: If it had been φ then it would have been ψ.

$v \leq_w u$ means that v is more similar to w than u.

Centering: $w \leq_w v$ validates:
$(\varphi \rightsquigarrow \psi) \to (\varphi \to \psi)$

Totality: $u \leq_w v$ or $v \leq_w u$ validates:
$((\varphi \vee \psi) \rightsquigarrow \chi) \to ((\varphi \rightsquigarrow \chi) \vee (\psi \rightsquigarrow \chi))$

Totality and antisymmetry: $|\mathrm{Min}(\leq_w, U)| \leq 1$ validates:
$(\varphi \rightsquigarrow \psi) \vee (\varphi \rightsquigarrow \neg\psi)$

A selection function $M : \mathcal{P}W \to \mathcal{P}W$ satisfies

- $M(A) \subseteq A$
- $M(A) \subseteq B$ implies $M(A \cap B) \subseteq M(A)$
- $M(A \cup B) \subseteq M(A) \cup M(B)$

$A \mathrel{|\!\sim} C \quad \text{iff} \quad M(A) \subseteq C$

M is coherent if $M(A) \cap M(B) \subseteq M(A \cup B)$.

Coherent selection functions are well-founded posets.

Every selection function can be made coherent.

A premise set P is a set of propositions $P \subseteq \mathcal{P}W$.

$A \mathrel{|\!\sim} C$ iff for all A-consistent $\mathcal{X} \subseteq P$
there is a A-consistent $\mathcal{Y} \subseteq P$ with $\mathcal{X} \subseteq \mathcal{Y}$
such that $\bigcap \mathcal{Y} \cap A \subseteq C$

Consider the Alexandroff topological closure $\mathrm{Alex}(P)$ of P.

$A \mathrel{|\!\sim} C$ iff for all A-consistent $U \in \mathrm{Alex}(P)$
there is a A-consistent $V \in \mathrm{Alex}(P)$ with $V \subseteq U$
such that $V \cap A \subseteq C$.

Alexandroff topologies are the same as preorders.

For a well-founded $\leq$

$$\mathrm{Min}(\leq, A) \subseteq C \quad \text{iff} \quad A \subseteq \mathrm{cl}(A \cap C)$$

The validity game for

$$\frac{A_1 \mathrel{|\!\sim} C_1 \qquad \dots \qquad A_n \mathrel{|\!\sim} C_n}{A \mathrel{|\!\sim} C}$$

on a set of worlds W:

Position	P	Moves
initial	$\forall$	$\{(w, A \cap C) \mid w \in A \cap \overline{C}\}$
(w, F)	$\exists$	$\{(A_i \cap C_i, F \cup (A_i \cap \overline{C}_i)) \mid w \in A_i\}$
(R, F)	$\forall$	$\{(w, F) \mid w \in R \cap \overline{F}\}$

Infinite matches are won by $\forall$.

A proof in system P yields a winning strategy for $\exists$.

A winning strategy for $\exists$ yields a proof in system P.

A countermodel yields a winning strategy for $\forall$.

A winning strategy for $\forall$ yields a countermodel.

These entail completeness for the order semantics.

$$\frac{A \mid\!\sim C}{A \wedge B \mid\!\sim C}\ (M) \qquad \frac{A \mid\!\sim B \qquad A \mid\!\sim C}{A \wedge B \mid\!\sim C}\ (CM)$$

Position	P	Moves
initial	$\forall$	$\{(w, A \cap C) \mid w \in A \cap \overline{C}\}$
(w, F)	$\exists$	$\{(A_i \cap C_i, F \cup (A_i \cap \overline{C}_i)) \mid w \in A_i\}$
(R, F)	$\forall$	$\{(w, F) \mid w \in R \cap \overline{F}\}$

WANTED

Non-modal proof system for the nested case with turnstile $\mid\!\sim$