# Craig Interpolation for PDL and its History. 

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Presenting a proof by Daniel Leivant.
Joint work with Yde Venema.

## Outline

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The Question

## Propositional Dynamic Logic

## Definition: Syntax

Atomic propositions $p$, atomic programs $a$.

- $\phi:=p|\neg \phi| \phi \vee \phi|\phi \wedge \phi| \phi \rightarrow \phi|\langle\alpha\rangle \phi|[\alpha] \phi$
- $\alpha:=a|\alpha ; \alpha| \alpha \cup \alpha\left|\alpha^{*}\right| \phi ?|1| 0$


## Definition: Models

A PDL-model $\mathcal{M}=(W, \mathcal{R}, V)$ consists of

- $W$ : set of worlds/states
- $\mathcal{R}=\left(R_{\xi}\right)_{\xi}$ : family of binary relations on $W$ such that
- $R_{\chi ; \xi}=R_{\chi} ; R_{\xi}$ (consecution)
- $R_{\chi \cup \xi}=R_{\chi} \cup R_{\xi}$ (union)
- $R_{\chi^{*}}=\left(R_{\chi}\right)^{*}$ (reflexive-transitive closure)
- $R_{\phi ?}=\{(w, w) \in W \times W \mid \mathcal{M}, w \vDash \phi\}$
- $R_{1}=\{(s, t) \in W \times W \mid s=t\}$ (identity on $W$ )
- $R_{0}=\varnothing$ (empty relation)
- $V$ : Prop $\rightarrow \mathcal{P}(W)$ : valuation function


## Propositional Dynamic Logic

Definition: Truth

- $\mathcal{M}, w \vDash p$ iff $w \in V(p)$
- $\mathcal{M}, w \vDash \neg \phi$ iff $\mathcal{M}, w \not \vDash \phi$
- $\mathcal{M}, w \vDash \phi \vee \psi$ iff $\mathcal{M}, w \vDash \phi$ or $\mathcal{M}, w \vDash \psi$
- $\mathcal{M}, w \vDash \phi \wedge \psi$ iff $\mathcal{M}, w \vDash \phi$ and $\mathcal{M}, w \vDash \psi$
- $\mathcal{M}, w \vDash \phi \rightarrow \psi$ iff $\mathcal{M}, w \not \vDash \phi$ or $\mathcal{M}, w \vDash \psi$
- $\mathcal{M}, w \vDash\langle\alpha\rangle \phi$ iff there is a $w^{\prime} \in W: w R_{\alpha} w^{\prime}$ and $\mathcal{M}, w^{\prime} \vDash \phi$.
- $\mathcal{M}, w \vDash[\alpha] \phi$ iff for all $w^{\prime} \in W: w R_{\alpha} w^{\prime}$ also $\mathcal{M}, w^{\prime} \vDash \phi$.


## Definition: Validity

A formula $\phi$ is valid iff it is true at all states in all models.
In this case we write $\vDash \phi$.

## Craig Interpolation

## Definition: Language

The language of a formula $\phi$ is the set $L(\phi)$ consisting of all atomic propositions and programs occurring in $\phi$.
Example: $L([a ; b] p \rightarrow\langle c\rangle q)=\{a, b, c, p, q\}$
Definition: Interpolation
A logic has Craig Interpolation iff for all formulas $\phi$ and $\psi$ such that $\vDash \phi \rightarrow \psi$ there is a formula $\theta$ called interpolant such that

- $\vDash \phi \rightarrow \theta$
- $\vDash \theta \rightarrow \psi$
- $L(\theta) \subseteq L(\phi) \cap L(\psi)$

Example: $q$ is an interpolant for $\vDash(p \wedge q) \rightarrow(q \vee r)$.
Propositional logic, first-order logic, intuitionistic logic, basic and multi-modal logic and the $\mu$-calculus have Craig Interpolation.

## Hall of Fame and Failure



Known proof attempts:

1. Daniel Leivant: Proof theoretic methodology for propositional dynamic logic. LNCS, 1981.
2. Manfred Borzechowski: Tableau-Kalkül für PDL und Interpolation. Diploma thesis, FU Berlin, 1988. Unpublished.
3. Tomasz Kowalski: PDL has interpolation. JSL, 2002. Revoked in 2004.

## Hall of Fame and Failure

Other notable references:

1. Marcus Kracht: Chapter The open question in Tools and techniques in modal logic, 1999.
2. D'Agostino \& Hollenberg: Logical questions concerning the $\mu$-calculus: interpolation, Lyndon and Łoś-Tarski. JSL, 2000.
3. Johan van Benthem: The many faces of Interpolation. Synthese, 2008.

## Honesty is a Virtue

The Journal of Symbolic Logic
Volume 69, Number 3, Sept. 2004

# RETRACTION NOTE FOR <br> "PDL HAS INTERPOLATION" 

TOMASZ KOWALSKI

In this journal I published a paper [1] entitled "PDL has interpolation" purporting to prove what the title announced. It has been pointed out to me by Yde Venema that my argument contains a serious error. As I have not been able to correct it, the problem of interpolation for Propositional Dynamic Logic is still open.

## Kracht VS. Leivant

"Chapter 10.6: The Unanswered Question
[...] the problem of interpolation for PDL. This is one of the major open problems in this area. Twice a solution has been announced, in [Leivant 1981] and [Borzechowski 1988], but in neither case was it possible to verify the argument."

Marcus Kracht: Tools and techniques in modal logic (1999)

It's a mess.
(It is an open question whether) ${ }^{2}$ PDL has Craig-Interpolation.

Leivant 1981, revised

## Leivant 1981

## Proof theoretic methodology for Propositional Dymanic logic

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#### Abstract

We relate by syntactic techniques finitary and infinitary axiomatizations for the iteratorconstruct * of Propositional Dynamic Logic PDL. This is applied to derive the Interpolation Theorem for PDL, and to provide a new proof of the semantic completeness of Segerberg's axionatic system for PDL.

Contrary to semantic techniques used to date, our proof of completeness is relatively insensitive to changes in the language and axioms used, provided some minimum syntactic closure properties hold. For instance, the presence of the test-operator adds no difficulty, and the proof also establishes the Interpolation Theorem and the closure under iteration of a constructive variant of PDL.


## Simplifying the question

Completeness of Segerberg's axioms is also shown by Leivant, but not our interest here.

In 2014 we know:

- PDL does not have uniform interpolation. [1]
- Test-free PDL has interpolation iff PDL has. [8]

Hence, reduce the syntax to:

- $\phi:=p|\neg \phi| \phi \rightarrow \phi \mid[\alpha] \phi$
- $\alpha:=a|\alpha ; \alpha| \alpha \cup \alpha\left|\alpha^{*}\right| 1 \mid 0$

Ignore tests.
Let $\vee, \wedge$ and $\langle\alpha\rangle$ be the appropriate abreviations.

## Steps of the proof

1. Define a sound and complete sequent calculus for PDL.
2. Use Maehara's method to show Partition-Interpolation.
2.1 Show that the calculus has the "step-by-step property".
2.2 For the $*$ case, find a repetitive scheme in long enough proofs.
2.3 Use linear transformations of programs to imply a $*$ formula.
3. Check that Partition-Interpolation implies Craig Interpolation.

## A sequent calculus for PDL

## Notation

$X, Y, Z$ : formulas
$f, g$ : sets of formulas
$\alpha, \beta$ : programs
Sequent example: $\quad f, X \vdash \phi$
Proof example

## A sequent calculus for PDL

Let $C D$ be the following proof system where $g$ is $\varnothing$ or a singleton.

$$
\begin{aligned}
& (\neg \mathrm{R}) \frac{f, X \vdash}{f, \vdash \neg X} \\
& (\neg \mathrm{~L}) \frac{f \vdash X}{f, \neg X \vdash} \\
& (\rightarrow \mathrm{R}) \frac{f, X \vdash Y}{f \vdash X \rightarrow Y} \\
& \text { (; R) } \frac{f \vdash[\alpha][\beta] X}{f \vdash[\alpha ; \beta] X} \\
& \text { (; L) } \frac{f,[\alpha][\beta] X \vdash g}{f,[\alpha ; \beta] X \vdash g} \\
& (\cup R) \frac{f \vdash[\alpha] X \quad f \vdash[\beta] X}{f \vdash[\alpha \cup \beta] X} \\
& (* L) \frac{f, X,[\alpha]\left[\alpha^{*}\right] X \vdash g}{f,\left[\alpha^{*}\right] X \vdash g} \\
& \text { (GEN) } \frac{f \vdash X}{[\alpha] f \vdash[\alpha] X}
\end{aligned}
$$

## A sequent calculus for PDL

Theorem (Leivant 1981)
$C D$ is a intuitionistic/constructive variant of $D$ which is a sound an complete system for PDL, i.e. we have:

$$
\vDash X \text { iff } \vdash_{\mathrm{D}} X \text { iff } \vdash_{\mathrm{CD}} X^{0}
$$

where $X^{0}$ is the result of inserting $\neg \neg$ in front of everything in $X$.
$\mathrm{NB}: C D$ is not sound and complete for intuitionistic/constructive PDL.
Remaining goal: Show that CD has interpolation.

## Maehara's method

## Idea

Find interpolants by going along the proof tree.
Given the previous interpolants, we define the next one.

## Example

Suppose the last step is $\cup R$ :

$$
\frac{\frac{\vdots}{f \vdash[\alpha] X} \frac{\vdots}{f \vdash[\beta] X}}{f \vdash[\alpha \cup \beta] X}(\cup \mathrm{R})
$$

Given any two interpolants $Z_{1}$ and $Z_{2}$ for $f \vdash[\alpha] X$ and $f \vdash[\beta] X$, let $Z:=Z_{1} \wedge Z_{2}=\neg\left(Z_{1} \rightarrow \neg Z_{2}\right)$. This interpolates $f \vdash[\alpha \cup \beta] X$.

## Partition-Interpolation

## Definition

Given a sequent $f \vdash X$ and a partition of $f$ into $f^{-} ; f^{+}$, we say that $K$ is an interpolant for $f^{-} ; f^{+} \vdash X$ iff

$$
L(K) \subseteq L\left(f^{-}\right) \cap L\left(f^{+}, X\right) \text { and } f^{-} \vdash K \text { and } f^{+}, K \vdash X
$$

Lemma 5.3.1 (Leivant 1981)
Let $f^{-} ; f^{+}$be any partition of $f$ and $q$ not occur in $f$.
(i) If $f \vdash_{\mathrm{CD}} X$, then there is an interpolant for $f^{-} ; f^{+} \vdash X$.
(ii) Suppose $P$ is a proof of $f \vdash[\alpha] q$ from $\left\{f_{i} \vdash q\right\}_{i<k}$ and let $f_{i}^{-} ; f_{i}^{+}$be the partitions of $f_{i}$ induced by $f^{-} ; f^{+}$for all $i<k$. If $K_{i}$ is an interpolant for $f_{i}^{-} ; f_{i}^{+} \vdash X$ for all $i<k$, then there is an interpolant of the form $\bigwedge_{i}\left[\beta_{i}\right] K_{i}$ for $f^{-} ; f^{+} \vdash[\alpha] X$.
Proof. By tree-induction on $P$, simultaneously for (i) and (ii).

## Partition-Interpolation

## An easy warm-up case

Suppose the last step is $\rightarrow \mathrm{L}$ :

$$
\frac{f \vdash X \quad f, Y \vdash Z}{f, X \rightarrow Y \vdash Z}(\rightarrow \mathrm{~L})
$$

Case a) partition $f^{-}, X \rightarrow Y ; f^{+}$. By induction hypothesis:

- $f^{+} ; f^{-} \vdash X$ (Note: flipped!) yields $K_{1}$ such that

$$
L\left(K_{1}\right) \subseteq L\left(f^{+}\right) \cap L\left(f^{-}, X\right) \text { and } f^{+} \vdash K_{1} \text { and } f^{-}, K_{1} \vdash X
$$

- $f^{-}, Y ; f^{+} \vdash Z$ yields $K_{2}$ such that

$$
L\left(K_{2}\right) \subseteq L\left(f^{-}, Y\right) \cap L\left(f^{+}, Z\right) \text { and } f^{-}, Y \vdash K_{2} \text { and } f^{+}, K_{2} \vdash Z
$$

Let $K:=K_{1} \rightarrow K_{2}$. This is interpolates $f^{-}, X \rightarrow Y ; f^{+} \vdash Z$.
Case b) partition $f^{-} ; X \rightarrow Y, f^{+}$. Then $K:=K_{1} \wedge K_{2}$ works.

## Partition-Interpolation

## The evil * case

Suppose the last step of $P$ is $(* \mathrm{R})$. For each $h=1 \leq M$ let $P_{h}$ be the proof of $f \vdash[\alpha]^{h} X$ occurring in $P$ above this premise:

$$
\frac{\frac{P_{0}}{f \vdash X} \quad \frac{P_{1}}{f \vdash[\alpha] X} \quad \ldots}{f \vdash[\alpha]^{M} X}(* \mathrm{R})
$$

Note: all active formulas on the right. Hence, only consider the given partition $f^{-}, f^{+}$without further manipulation.

Given: $M$ many interpolants. Goal: find a formula $K$ such that

$$
L(K) \subseteq L\left(f^{-}\right) \cap L\left(f^{+},\left[\alpha^{*}\right] X\right) \text { and } f^{-} \vdash K \text { and } f^{+}, K \vdash\left[\alpha^{*}\right] X
$$

How?!

## Positive Closure

## Definition

The positive closure of $f$, denoted by $\operatorname{PC}(f)$, is the smallest set $g \supseteq f$ such that:

- If $(X \rightarrow Y) \in g$, then $Y \in g$.
- If $[\alpha] X \in g$, then $X \in g$.
- If $[\alpha ; \beta] X \in g$, then $[\alpha][\beta] X \in g$.
- If $[\alpha \cup \beta] X \in g$, then $[\alpha] X \in g$ and $[\beta] X \in g$.
- If $\left[\alpha^{*}\right] X \in g$, then $[\alpha]\left[\alpha^{*}\right] X \in g$.

Note: Whenever $f$ is finite, $\mathrm{PC}(f)$ is also finite.

## Nice property 1

In certain proofs, $\mathrm{PC}(\cdot)$ is preserved in the following sense.
Lemma 4.2.1 (Leivant 1981, revision Venema 2014) If $P$ proves $f \vdash\left[\beta_{1}\right] \ldots\left[\beta_{k}\right][\alpha]^{m} q$ from $\left\{f_{i} \vdash q\right\}_{i}$ where $q \notin L(f)$, all $\beta_{i}$ s are subprograms of $\alpha, r<m$ and $f^{\prime} \vdash[\alpha]^{r} q$ is a sequent in $P$ (under a non-initial leaf) then $P C\left(f^{\prime}\right) \subseteq P C(f)$.

The case we need is $k=0$.

## Nice property 2

## Definition

Let $P[X / q]$ be the result of substituting $X$ for $q$ in $P$.
Lemma 4.2.2 (Leivant 1981)
Suppose $P$ proves $f \vdash[\alpha]^{r} X$ from $\left\{f_{i} \vdash X\right\}_{i}$ where $X \notin P C(f)$.
Then there is a proof $P^{\prime}$ of $f \vdash[\alpha]^{r} q$ from $\left\{f_{i}^{\prime} \vdash q\right\}_{i}$ such that $P=P^{\prime}[X / q]$.

Intuitively, this means that $P$ does not take $X$ apart:

$$
\frac{\left\{f_{i} \vdash X\right\}_{i}}{\vdots}=\binom{\frac{\left\{f_{i} \vdash q\right\}_{i}}{\vdots \vdash[\alpha]^{r} X}}{\frac{\vdots}{f \vdash[\alpha]^{r} q}}[X / q]
$$

## Nice property 3: Step by Step

Suppose $P$ is a CD-proof of $f \vdash[\alpha]^{n} X$. Then $P$ consists of proof parts $P_{0}, \ldots, P_{n}$ which build up the $[\alpha]$ s "step by step":

$$
\begin{gathered}
\frac{P_{0}}{\frac{\left\{f_{j} \vdash X\right\}_{j \in I_{0}}}{P_{1}}} \\
\frac{\frac{\left\{f_{j} \vdash[\alpha] X\right\}_{j \in I_{1}}}{P_{2}}}{\frac{\left\{f_{j} \vdash[\alpha]^{2} X\right\}_{j \in I_{2}}}{\vdots}} \\
\frac{\left.\vdots f_{j} \vdash[\alpha]^{n-1} X\right\}_{j \in I_{n-1}}}{P_{n}} \\
f \vdash[\alpha]^{n} X
\end{gathered}
$$

NB: This looks more linear than it actually is!

## Linear Transformations

Think of programs and formulas as a vector space:

$$
(\beta) \vec{Y}=\left(\begin{array}{ccc}
\beta_{1,1} & \cdots & \beta_{1, k} \\
\vdots & \ddots & \vdots \\
\beta_{k, 1} & \cdots & \beta_{k, k}
\end{array}\right)\left(\begin{array}{c}
Y_{1} \\
\vdots \\
Y_{k}
\end{array}\right):=\left(\begin{array}{c}
{\left[\beta_{1,1}\right] Y_{1} \wedge \cdots \wedge\left[\beta_{1, k}\right] Y_{k}} \\
\vdots \\
{\left[\beta_{k, 1}\right] Y_{1} \wedge \cdots \wedge\left[\beta_{k, k}\right] Y_{k}}
\end{array}\right)
$$

Lemma
For every $k \times k$ matrix $(\beta)$ there exists a $(\gamma)$ such that

$$
(\gamma) \equiv(\beta)^{*}=(\beta)(\beta)(\beta) \ldots
$$

## Linear Transformations

## Example

Let $\vec{Y}=\langle p, q\rangle$ and $(\beta)=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$. Then $(\beta) \vec{Y}=\binom{[a] p \wedge[b] q}{[c] p \wedge[d] q}$
and $(\beta)(\beta) \vec{Y}=\binom{[a]([a] p \wedge[b] q) \wedge[b]([c] p \wedge[d] q)}{[c]([a] p \wedge[b] q) \wedge[d]([c] q \wedge[d] q)} \ldots$
Let $\gamma:=\left(\begin{array}{cc}\left(a \cup\left(b ;\left(d^{*} ; c\right)\right)\right)^{*} & \left(a^{*} ; b\right)\left(\left(c ; a^{*} ; b\right) \cup d\right)^{*} \\ \left(d^{*} ; c\right)\left(a \cup\left(b ;\left(d^{*} ; c\right)\right)\right)^{*} & \left(\left(c ; a^{*} ; b\right) \cup d\right)^{*}\end{array}\right)$
Then $(\gamma) \equiv(\beta)^{*}$ and $(\beta)^{*} \vec{Y} \equiv(\gamma) \vec{Y}$.
This $\gamma$ can be found systematically. Moreover, it is useful:

$$
p \wedge[a] p \wedge([a] p \wedge[b] q) \wedge([a]([a] p \wedge[b] q) \wedge[b]([c] p \wedge[d] q)) \wedge \ldots
$$

$$
\equiv\left[\left(a \cup\left(b ;\left(d^{*} ; c\right)\right)\right)^{*}\right] p \wedge\left[\left(a^{*} ; b\right)\left(\left(c ; a^{*} ; b\right) \cup d\right)^{*}\right] q
$$

## Putting it all together

## Back to the evil * case

Now we can deal with this:

$$
\frac{\frac{P_{0}}{f \vdash X} \quad \frac{P_{1}}{f \vdash[\alpha] X} \quad \ldots}{\frac{P_{M}}{f \vdash[\alpha]^{M} X}}(* \mathrm{R})
$$

Fix a ridiculously large $h:=s+v+d$ where

- $d$ such that $[\alpha]^{d} X \notin \mathrm{PC}(f)$
- $v:=2^{|\mathrm{PC}(f)|} \cdot 2^{|f|}+1$
- $s:=1$ (for now).

Apply the step by step property to $P_{h}: \frac{\left\{f_{i}^{-} ; f_{i}^{+} \vdash[\alpha]^{d} X\right\}_{i \in I_{d}}}{\vdots} \frac{f^{-} ; f^{+} \vdash[\alpha]^{d+v+s} X}{}$

## Putting it all together

## Finding a repetitive pattern

Now $P_{h}$ has to look like this:

$$
\frac{\frac{Q_{j, i}}{\frac{\left\{f_{i}^{--} ; f_{i}^{+} \vdash[\alpha]^{d} X\right\}_{i \in I_{d}}}{R_{j}^{\prime}\left[[\alpha]^{d} X / q\right]}}}{\frac{\left\{f_{j}^{-} ; f_{j}^{+} \vdash[\alpha]^{d+v} X\right\}_{j \in I_{d+v}}}{U^{\prime}\left[[\alpha]^{d+v} X / q\right]}} \frac{f^{-} ; f^{+} \vdash[\alpha]^{d+v+s} X}{}
$$

For all $c \leq v, j \in I_{d+c}: f_{i} \subseteq \operatorname{PC}\left(f_{i}\right) \subseteq \operatorname{PC}(f)$ and $\left|\mathcal{P}\left(f_{j}\right)\right| \leq|\mathcal{P}(f)|$
Hence $\left|\cup\left\{\mathcal{P}\left(f_{j}\right) \mid c \leq v, j \in I_{d+c}\right\} \leq\right| \mathcal{P}(\operatorname{PC}(f)|\cdot| \mathcal{P}(f) \mid$
$=2^{|\mathrm{PC}(f)|} \cdot 2^{|f|}=v-1<v$.

## Repetitive Pattern

For some $m \neq n$ we have $\left\{f_{j}^{+} ; f_{j}^{-} \mid j \in I_{m}\right\}=\left\{f_{j}^{+} ; f_{j}^{-} \mid j \in I_{n}\right\}$.
Furthermore, we can assume $d<m<n<d+v$ and $I_{m}=I_{n}$.

## Putting it all together

## Applying the induction hypothesis

Let $r$ be such that $n=m+r$. Now $P_{h}$ can be divided as follows:
$\mathrm{IH}(\mathrm{i})$ yields $\vec{K}$ such that $K_{i}$ interpolates $f_{i}^{-} ; f_{i}^{+} \vdash[\alpha]^{m} X$. Using $\mathrm{IH}(\mathrm{ii}) r$ times: If $\vec{M}$ contains interpolants for $f_{i}^{-} ; f_{i}^{+} \vdash Y$ then there is a matrix $(\beta)$ such that $((\beta) M)_{i}$ interpolates $f_{i}^{--} f_{i}^{+} \vdash[\alpha]^{r} Y$.

Thus, for all $n$, by applying the latter to the former $n$ times:

$$
f_{i}^{-} \vdash\left((\beta)^{n} K\right)_{i} \text { and } f_{i}^{+},\left((\beta)^{n} K\right)_{i} \vdash[\alpha]^{m}[\alpha]^{r \times n} X
$$

## Putting it all together

## Done, repeat.

By linear transformations there is a $\gamma$ such that:

$$
f_{i}^{-} \vdash((\gamma) K)_{i} \text { and } f_{i}^{+},((\gamma) K)_{i} \vdash[\alpha]^{m}\left[\left(\alpha^{r}\right)^{*}\right] X
$$

Now apply $\mathrm{IH}(\mathrm{ii})$ to all the $((\gamma) K)_{i} \mathrm{~s}$ and $U^{\prime}$.
This yields an interpolant $H_{s}$ for $f^{-} ; f^{+} \vdash[\alpha]^{s}[\alpha]^{m}\left[\left(\alpha^{r}\right)^{*}\right] X$.
Repeat all of the above to obtain $H_{1}, \ldots, H_{v+d}$.
Finally, let $K:=\bigwedge_{s \leq v+d} H_{s}$. This interpolates $f^{-} ; f^{+} \vdash\left[\alpha^{*}\right] X$.

## Lemma

$\vdash \mathrm{CD} \bigwedge_{k<w}\left[\alpha^{k}\right]\left[\left(\alpha^{w}\right)^{*}\right] X \rightarrow\left[\alpha^{*}\right] X$.

## Putting it all together

## This is the end.

Theorem 5.3 .2 (i) (Leivant 1981)
PDL has Craig Interpolation.
Proof. Take any $\vDash X \rightarrow Y . D$ is complete, hence $\vdash_{D} X \rightarrow Y$.
Then $\vdash_{C D} X^{o} \rightarrow Y^{o}$ and thus $X^{o} \vdash_{C D} Y^{o}$.
Partition-interpolation of $X^{\circ} ; \varnothing \vdash Y^{\circ}$ yields $Z$ such that

- $L(Z) \subseteq L\left(X^{o}\right) \cap L\left(\varnothing, Y^{o}\right)$,
- $X^{o} \rightarrow Z \in \mathbf{P D L}$ and $Z \rightarrow Y^{o} \in \mathbf{P D L}$

By $X^{o} \equiv X, Y^{o} \equiv Y, L\left(X^{o}\right)=L(X)$ and $L\left(Y^{o}\right)=L(Y)$ :

- $L(Z) \subseteq L(X) \cap L(Y)$,
- $X \rightarrow Z \in \mathbf{P D L}$ and $Z \rightarrow Y \in \mathbf{P D L}$

Hence $Z$ is an interpolant for $X \rightarrow Y$.

## Criticism

## Criticism

Marcus Kracht: Tools and techniques in modal logic. (1999) Chapter 10.6. The Unanswered Question:
"[T]he problem of interpolation for PDL is one of the major open problems in this area. Twice a solution has been announced [...], but in neither case was it possible to verify the argument.
The argument of Leivant makes use of the fact that if $\phi \vdash_{\text {PDL }} \psi$ then we can bound the size of a possible countermodel so that the star $\alpha^{*}$ only needs to search up to a depth $d$ which depends on $\phi$ and $\psi . "[8, p$ 493]

## Criticism

Marcus Kracht (continued):
"The argument of Leivant makes use of the fact that if
$\phi \vdash_{\text {PDL }} \psi$ then we can bound the size of a possible countermodel so that the star $\alpha^{*}$ only needs to search up to a depth $d$ which depends on $\phi$ and $\psi$. Once that is done, we have reduced PDL to EPDL, which definitely has interpolation because it is a notational variant of polymodal K. However, this is tantamount to the following. Abbreviate by PDL ${ }^{n}$ the strengthening of PDL by axioms of the form $\left[a^{*}\right] p \leftrightarrow\left[a^{\leq n}\right] p$ for all $a$. Then, by the finite model property of PDL, PDL is the intersection of the logics $\mathbf{P D L}{ }^{n}$. Unfortunately, it is not so that interpolation is preserved under intersection."[8, p. 493]

## PDL and $\mathrm{PDL}^{n}$

## Definition

Semantic closure $\operatorname{SCL}(A):=\{\phi \mid A \vDash \phi\}$
$\left[\alpha^{\leq n}\right] \phi:=\phi \wedge[\alpha] \phi \wedge[\alpha ; \alpha] \phi \wedge \cdots \wedge\left[\alpha^{n}\right] \phi$
$\mathbf{P D L}^{n}:=\operatorname{SCL}\left(\mathbf{P D L} \cup\left\{\left[\alpha^{*}\right] p \leftrightarrow\left[\alpha^{\leq n}\right] p \mid \alpha \in P R O G, p \in \mathfrak{P}\right\}\right)$

Theorem

$$
\mathbf{P D L}^{0} \supseteq \mathbf{P D L}^{1} \supseteq \mathbf{P D L}^{2} \supseteq \cdots \supseteq \mathbf{P D L}=\bigcap_{n} \mathbf{P D L}^{n}
$$

Idea / Question
Is there an $n$, depending on $|\phi \rightarrow \psi|$ such that any
PDL ${ }^{n}$-interpolant for $\phi \rightarrow \psi$ is also a PDL-interpolant?

## Criticism

But this is not what Leivant is doing:

$$
\begin{aligned}
& (* \mathrm{R}) \frac{f \vdash \phi \quad f \vdash[\alpha] \phi \quad \cdots \quad f \vdash[\alpha]^{k} \phi}{f \vdash\left[\alpha^{*}\right] \phi} \\
& \quad \text { where } k=2^{|f|+|\phi|} \text { and therefore depends on } f \text { and } \phi .
\end{aligned}
$$

## Theorem: Finite-Model Property

If $\phi$ is satisfiable, then there is a model $\mathcal{M}=(W, \mathcal{R}, V)$ and a world $w \in W$ such that $\mathcal{M}, w \vDash \phi$ and $|W| \leq 2^{\operatorname{size}(\phi)}$.

Lemma
If $\vDash \wedge f \rightarrow[\alpha]^{n} \phi$ for all $n \leq k=2^{|f|+|\phi|}$, then $\vDash \Lambda f \rightarrow\left[\alpha^{*}\right] \phi$.

## Theorem

The finitary rule is admissible.

## Conclusion

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- There is a finitary sequent calculus for PDL. (In particular, Kracht's criticism does not apply.)
- This system has the "step by step" property.
- Therefore we can:
- find a repetitive pattern in long enough proofs.
- use linear transformations to build $*$ interpolants.
- This extends Maehara's method to show Craig Interpolation.

All this [ $\mathrm{c} \cup \mathrm{sh}$ ]ould have been known since 1981 .
Moreover, can this proof also be done in multi-type calculi?

## Epilogue

## Kracht: "Twice a solution has been announced ..."

Borzechowski 1988: unpublished, unknown and unread?



Ergebnis der Konstruktion ist also, daB $I_{0}=\left[A^{*}\right](p \vee[C] 0)$ ein Interpoland für die Formeln
$[(A ; A) *](p \wedge[A ;(B \cup C)] 0)$ und $\left[A A^{\prime}\right](p \vee[C] q)$ ist. Auf Grund der vorgenommenen überlegungen, einen nicht unnötig groben Interpolanden zu konstruieren, ist $I_{0}$ sogar frei von Tests. Dieses Ergebnis ist jedoch nicht immer erreichbar.

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# Thank you! 

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