Filtrations in intermediate logics via locally finite reducts of Heyting algebras

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Applied Logic Seminar, Delft

Quick historic overview

Standard and selective flltrations

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If a model $\mathfrak{M}$ refutes a formula φ, then we wish to filter it out so that the resulting model $\mathfrak{N}$ is finite and still refutes φ.

The model $\mathfrak{N}$ can be constructed as a factor-model of $\mathfrak{M}$ (standard filtration) or as a submodel of $\mathfrak{M}$ (selective filtration).

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Algebraic approach to filtrations was revisited by Ghilardi (2010), van Alten, Conradie, Morton (2013), Bezhanishvili et al. (2014)

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Zakharyaschev (1980's) also applied the method to superintuitionistic logics.

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We will look at some consequences of this approach.

Outline

- Standard filtrations for modal logics
- Standard filtrations for superintuionistic logics
- Selective filtrations for superintuionistic logics
- Consequences

Standard filtrations model theoretically

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Let $W' = W/{\sim}$. Then $|W'| \leqslant 2^{|\Sigma|}$.

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Let $W' = W/{\sim}$. Then $|W'| \leqslant 2^{|\Sigma|}$.

Let R' be a relation on W' satisfying the following two conditions for all $w, v \in W$ and $\varphi \in \Sigma$:

$$wRv \;\Rightarrow\; [w]R'[v]. \tag{2}$$

$$[w]R'[v] \;\Rightarrow (\forall \Diamond\varphi \in \Sigma)\;\; (w \in V(\varphi) \;\Rightarrow\; v \in V(\Diamond\varphi)). \tag{3}$$

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It is easy to see that V' is well defined, so $\mathfrak{M}' = (\mathfrak{F}', V')$ is a finite model called a standard filtration (or simply a filtration) of the model $\mathfrak{M}$ through Σ.

Standard filtration model theoretically

It is well known that for each $\varphi \in \Sigma$ and $w \in W$, we have

Truth Lemma.

$$w \in V(\varphi) \ \text{ iff } \ [w] \in V'(\varphi), \tag{4}$$

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so if $\varphi \in \Sigma$ is refuted on $\mathfrak{M}$, then it is refuted on $\mathfrak{M}'$.

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This condition plays an important role in the study of stable logics, which we will discuss at the end of this talk.

Duality for modal algebras

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For a modal algebra $\mathfrak{A} = (A, \Diamond)$, let $\mathfrak{F}_A = (W_A, R_A)$ be the frame of ultrafilters of A,

where for each $x, y \in W_A$ we have

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where $\Diamond_R(S) = \{x \in W_A : R[x] \cap S \neq \emptyset\}$.

Standard filtration algebraically for modal logic

Stable homomorphisms and CDC

Lemma. Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras and let $h : A \to B$ be a Boolean homomorphism. We call h a stable homomorphism provided $\Diamond h(a) \leqslant h(\Diamond a)$ for each $a \in A$.

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It is easy to see that $h : A \to B$ is stable iff $h(\Box a) \leq \Box h(a)$ for each $a \in A$.

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Stable homomorphisms were considered under the name of semi-homomorphisms and under the name of continuous morphisms (Ghilardi, 2010).

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If V is a valuation on A, then by identifying A with the subsets of W, we can view V as a valuation on W.

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For a valuation V on A and a set of formulas Σ closed under subformulas, let A' be the Boolean subalgebra of A generated by $V(\Sigma) \subseteq A$ and let $D = \{V(\varphi) : \Diamond\varphi \in \Sigma\}$.

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For a modal operator $\Diamond'$ on A', the following two conditions are equivalent:

1. The inclusion $(A', \Diamond') \rightarrowtail (A, \Diamond)$ is a stable homomorphism satisfying CDC for D.
2. Viewing V as a valuation on W, there is a filtration $\mathfrak{M}' = (W', R', V')$ of $\mathfrak{M} = (W, R, V)$ through Σ such that (W', R') is the ultrafilter frame of $(A', \Diamond')$.

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Lemma (Filtration Lemma algebraically).
Let $\mathfrak{A}' = (A', \Diamond')$ be a filtration of $\mathfrak{A}$ through Σ and let V' be a valuation on A' that coincides with V on the propositional variables occurring in Σ. Then $V(\varphi) = V'(\varphi)$, for each $\varphi \in \Sigma$.

Least and greatest filtrations

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The least filtration of $\mathfrak{M} = (W, R, V)$ through Σ is $\mathfrak{M}^l = (W', R^l, V')$ and the greatest filtration is $\mathfrak{M}^g = (W, R^g, V')$, where

$$[x]R^l[y] \text{ iff } (\exists x', y' \in W)(x \sim x' \;\&\; y \sim y' \;\&\; x'Ry').$$

$$[x]R^g[y] \text{ iff } (\forall \Diamond\varphi \in \Sigma)(y \in V(\varphi) \Rightarrow x \in V(\Diamond\varphi)).$$

Least and greatest filtrations algebraically

Define $\Diamond^l$ and $\Diamond^g$ on A' by

$$\Diamond^l a = \bigwedge\{b \in A' : \Diamond a \leqslant b\} \text{ and } \Diamond^g a = \bigwedge\{\Diamond b : a \leqslant b \;\&\; b \in D^\vee\}.$$

where $D^\vee$ is the closure of D in A under finite joins.

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Theorem.

1. If $\mathfrak{A}' = (A', \Diamond')$ is a filtration of $\mathfrak{A}$ through Σ, then $\Diamond^l a \leqslant \Diamond' a \leqslant \Diamond^g a$ for each $a \in A'$.

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2. $(A', \Diamond^l)$ is the least filtration and $(A', \Diamond^g)$ is the greatest filtration of $\mathfrak{A}$ through the finite set of formulas Σ.

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2. $(A', \Diamond^l)$ is the least filtration and $(A', \Diamond^g)$ is the greatest filtration of $\mathfrak{A}$ through the finite set of formulas Σ.
3. Let (W, R) be the ultrafilter frame of $\mathfrak{A}$. Then (W', R^l) is the ultrafilter frame of $(A', \Diamond^l)$ and (W', R^g) is the ultrafilterframe of $(A', \Diamond^g)$.

Standard filtration for intuitionistic logic

Standard filtration model theoretically

Let Σ be a finite set of formulas closed under subformulas, and let $\mathfrak{M} = (\mathfrak{F}, \nu)$ be an intuitionistic model. Define an equivalence relation $\sim$ on W by

$$w \sim v \text{ if } (\forall \varphi \in \Sigma)(w \in \nu(\varphi) \text{ iff } v \in \nu(\varphi)). \tag{5}$$

Let $W' = W/\!\sim$. Then $|W'| \leqslant 2^{|\Sigma|}$.

Let $\leqslant'$ be a partial order on W' satisfying the following two conditions for all $w, v \in W$ and $\varphi \in \Sigma$:

$$w \leqslant v \;\text{ implies }\; [w] \leqslant' [v]. \tag{6}$$

$$[w] \leqslant' [v] \;\text{ and }\; w \in \nu(\varphi) \;\text{ imply }\; v \in \nu(\varphi). \tag{7}$$

Standard filtration model theoretically

Define a valuation ν' on $\mathfrak{F}' = (W', \leqslant')$ by

$$\nu'(p) = \{[w] : w \in \nu(p)\} \text{ for each } p \in \Sigma.$$

It is easy to see that ν' is well defined, so $\mathfrak{M}' = (\mathfrak{F}', \nu')$ is a finite model called a standard filtration (or simply a filtration) of the model $\mathfrak{M}$ through Σ.

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It is well known that for each $\varphi \in \Sigma$ and $w \in W$, we have

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$$w \in \nu(\varphi) \ \text{ iff } \ [w] \in \nu'(\varphi), \tag{8}$$

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so if $\varphi \in \Sigma$ is refuted on $\mathfrak{M}$, then it is refuted on $\mathfrak{M}'$.

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$$[w] \preceq [v] \text{ iff there exist } w' \sim w \text{ and } v' \sim v \text{ such that } w' \leqslant v', \quad (9)$$

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and let $\leqslant^l$ be the transitive closure of $\preceq$. The greatest filtration is given by

$$[w] \leqslant^g [v] \text{ iff } (\forall \varphi \in \Sigma)(w \in \nu(\varphi) \Rightarrow v \in \nu(\varphi)). \quad (10)$$

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Then A embeds into the Heyting algebra $\mathrm{Up}(\mathfrak{F}_A)$ of upward closed subsets of $\mathfrak{F}_A$ by $\alpha(a) = \{w \in W_A : a \in w\}$.

Standard filtration algebraically for superintuitionistic logics

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$$a \to_S b = \bigvee \{s \in S : a \wedge s \leqslant b\}$$

for each $a, b \in S$.

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Clearly $a \to_S b \leqslant a \to b$, and $a \to_S b = a \to b$ provided $a \to b \in S$.

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Proof. By induction on the complexity of $\varphi \in \Sigma$.

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Proof. By induction on the complexity of $\varphi \in \Sigma$. The case $\varphi = p$ follows from the definition of ν_S.

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Define a valuation ν_S on S by $\nu_S(p) = \nu(p)$ for each $p \in \Sigma$.

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Problem: This definition does not match the model theoretic one!

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Lemma. Let L be a finite bounded sublattice of A that contains S as a bounded sublattice.

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Lemma. Let L be a finite bounded sublattice of A that contains S as a bounded sublattice.

Then $w \cap L = v \cap L$ is equivalent to $w \cap S = v \cap S$ iff $\alpha[L]$ and $\alpha[S]$ generate the same Boolean subalgebra of the powerset of the prime filter frame of A.

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Free Boolean extension

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and for any Boolean algebra $\mathfrak{B}'$ and a bounded lattice homomorphism $g : L \to \mathfrak{B}'$, there is a unique Boolean homomorphism $h : \mathfrak{B} \to \mathfrak{B}'$ such that $h \circ f = g$.

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It follows that there is a 1-1 correspondence between filtrations $\mathfrak{M}'_A$ of $\mathfrak{M}_A$ and finite L such that $\nu[\Sigma] \subseteq L \trianglelefteq A$ and L and S have the same free Boolean extension.

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This motivates the following definition.

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Theorem. Let A be a Heyting algebra, ν be a valuation on A, and Σ be a finite set of formula closed under subformulas.

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Theorem. Let A be a Heyting algebra, ν be a valuation on A, and Σ be a finite set of formula closed under subformulas.

Define μ on the prime filter frame $\mathfrak{F}_A$ of A by $\mu = \alpha \circ \nu$. Then there is a 1-1 correspondence between the filtrations (L, ν_L) of (A, ν) through Σ and the filtrations $\mathfrak{M}'_A = (\mathfrak{F}'_A, \mu')$ of $\mathfrak{M}_A = (\mathfrak{F}_A, \mu)$ through Σ.

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Theorem. Let A be a Heyting algebra, ν be a valuation on A, and Σ be a finite set of formula closed under subformulas.

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Moreover, if (L, ν_L), (K, ν_K) are two filtrations of (A, ν) through Σ and $\mathfrak{M}'_A$, $\mathfrak{M}''_A$ are the corresponding filtrations of $\mathfrak{M}_A$ through Σ, then

$$L \trianglelefteq K \text{ iff } [w] \leqslant'' [v] \text{ implies } [w] \leqslant' [v].$$

Least and greatest filtrations

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Then (S, ν_S) corresponds to the greatest filtration $\mathfrak{M}^g = (W'_A, \leqslant^g, \mu')$ of $\mathfrak{M}_A$ through Σ,

while (T, ν_T) corresponds to the least filtration $\mathfrak{M}^l = (W'_A, \leqslant^l, \mu')$.

Selective filtration algebraically for superintuitionistic logics

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By Diego's Theorem, S is finite. Therefore, S is a Heyting algebra, where

$$a \vee_S b = \bigwedge \{s \in S : a, b \leqslant s\}$$

for each $a, b \in S$. It follows from the definition that $a \vee b \leqslant a \vee_S b$, and that $a \vee b = a \vee_S b$ provided $a \vee b \in S$.

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Consequences

For each (subdirectly irreducible) Heyting algebra A and $D \subseteq A^2$ we can write canonical formulas $\beta(A, D)$ and $\gamma(A, D)$ such that

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For each Heyting algebra B we have:

$B \not\models \beta(A, D)$ iff (A, D) is a selective filtration of B.

$B \not\models \gamma(A, D)$ iff (A, D) is a standard filtration of B.

These formulas axiomatize all superintuionistic logics.

Selective filtration

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We say that L admits filtration if for each non-theorem φ of L and some countermodel $\mathfrak{M} = (\mathfrak{F}, \nu)$ of φ, there is a filtration $\mathfrak{M}' = (\mathfrak{F}', \nu')$ of $\mathfrak{M}$ through some finite set Σ closed under subformulas and containing φ such that $\mathfrak{F}' \models L$.

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Every superintuitionistic logic admitting filtration has the finite model property.

This notion depends on at least three different parameters: formulas, models, frames.

Stable logics

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Every stable logic admits filtration, and hence has the finite model property.

Thus, stable logics in some way formalize the notion of admitting filtration by avoiding mentioning models and formulas.

Stable logics

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These logics are axiomatized by special types of canonical formulas, called stable formulas.

Subframe logics

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Fine (1985) formalizes this notion by defining subframe transitive modal logics,

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These logics are axiomatized by special types of canonical formulas, called subframe formulas and cofinal subframe formulas.

Conclusions

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We gave algebraic descriptions of filtrations for superintuitionistic logics via locally finite reducts of Heyting algebras.

We showed that the algebraic description of the standard filtration is based on the $\to$-free reduct of Heyting algebras, while that of selective filtration on the $\vee$-free reduct.

Thank you!