Sequent Systems for Classical Modal Logics

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Joint work with D. Gilbert

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Introduction and motivation

- Proof theory for logics weaker than basic modal logic.
- Hilbert-style axioms and neighborhood semantics.
- Gentzen-rules and relational semantics.
- Analysis of formal derivations and proof search of theorems.
- Sequent system for logics with Kripke semantics.
- Simulation of non-normal logics by normal ones.

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Language $\mathcal{L}_1$

Let Var be a countable set of propositional variables.

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

for p in Var

$$\begin{array}{lll} \Diamond\varphi & := & \neg\Box\neg\varphi \\ \varphi \vee \psi & := & \neg(\neg\varphi \wedge \neg\psi) \\ \varphi \rightarrow \psi & := & \neg\varphi \vee \psi \\ \varphi \leftrightarrow \psi & := & \varphi \rightarrow \psi \wedge \psi \rightarrow \varphi \\ \top & := & \varphi \vee \neg\varphi \end{array}$$

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\varphi \leftrightarrow \psi & := & \varphi \to \psi \wedge \psi \to \varphi \\
\top & := & \varphi \vee \neg\varphi
\end{array}$$

Hilbert systems

E consists of

- Propositional tautologies
- Modus Ponens: From φ and $\varphi \to \psi$ follows ψ
- *RE*: from $\varphi \leftrightarrow \psi$ follows $\Box\varphi \leftrightarrow \Box\psi$

M consists of

- **E**
- M: $\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$

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C consists of

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- $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$

N consists of

- **E**
- N: $\Box\top$

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Semantics of $\mathcal{L}_1$

A neighborhood model $M = \langle W, n, V \rangle$ where

- W is a set
- $n : W \longrightarrow \wp(\wp(W))$
- $V : Var \longrightarrow \wp(W)$

$$M, w \models \Box\varphi \quad \text{iff} \quad [\![\varphi]\!] \in n(w)$$

where $[\![\varphi]\!] = \{w \in W \mid M, w \models \varphi\}$

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Characterization

Let $F = \langle W, n \rangle$ be a neighborhood frame.

- **E** is sound and complete w.r.t. all F
- **M** is sound and complete w.r.t. all F s.t.

$$a \in n(w) \text{ and } a \subseteq b \text{ implies } b \in n(w)$$

- **C** is sound and complete w.r.t. all F s.t.

$$a \in n(w) \text{ and } b \in n(w) \text{ implies } a \cap b \in n(w)$$

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Toward a Kripke Semantics for $\Box$

- $w \vDash \Box\varphi$ iff $[\![\varphi]\!] \in n(w)$
- $[\![\varphi]\!] \in n(w)$ means $a \in n(w)$ and $a = [\![\varphi]\!]$, for some a
- in turn, $a = [\![\varphi]\!]$ stands for $x \in a$ iff $x \vDash \varphi$, for all x
- $w \vDash \Box\varphi$ there is $a \in n(w)$ such that $x \vDash \varphi$ iff $x \in a$, for all x
- $w \vDash \Box\varphi$ iff **there is** a s.t. $a \in n(w)$ and **for all** x
  - $x \in a$ implies $x \vDash \varphi$
  - $x \notin a$ implies $x \nvDash \varphi$

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$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond_N \varphi \mid \Box_\ni \varphi \mid \Box_{\not\ni} \varphi \mid \sigma \mid \tau$$

for p in *Var*

- Three normal modalities: $\Diamond_N$, $\Box_\ni$ and $\Box_{\not\ni}$
- σ and τ are a nullary modalities (constant)
- σ is true at worlds, τ is true at neighborhoods

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Translation of $\mathcal{L}_1$ into $\mathcal{L}_5$

- Translation $*$ by Kracht & Wolter, *JSL*, 1999.

$$
\begin{array}{rcl}
p^* & := & p \\
(\neg\varphi)^* & := & \neg\varphi^* \\
(\varphi \wedge \psi)^* & := & (\varphi^* \wedge \psi^*) \\
(\Box\varphi)^* & := & \Diamond_N(\Box_{\ni}\varphi^* \wedge \Box_{\not\ni}\neg\varphi^*)
\end{array}
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Semantics of $\mathcal{L}_5$

Let M be neighborhood model. A relational model

$$M^\circ = \langle W^\circ, N, \ni, \not\ni, \sigma, \tau, V \rangle$$

- $W^\circ := W \cup \wp(W)$
- $N := \{\langle w, a \rangle \in W \times 2^W \mid a \in n(w)\}$
- $\ni := \{\langle a, w \rangle \in 2^W \times W \mid w \in a\}$
- $\not\ni := \{\langle a, w \rangle \in 2^W \times W \mid w \notin a\}$
- $\sigma := W$
- $\tau := \wp(W)$
- $V : Var \longrightarrow \wp(W)$

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Semantics of $\mathcal{L}_5$

Truth in a model M°

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\begin{array}{lll}
M^\circ, w \models \Box_{\ni}\varphi & \text{iff} & \text{for all } v,\ w \ni v \text{ implies } v \models \varphi \\
M^\circ, w \models \Box_{\not\ni}\varphi & \text{iff} & \text{for all } v,\ w \not\ni v \text{ implies } v \models \varphi \\
M^\circ, w \models \Diamond_N\varphi & \text{iff} & \text{for some } v,\ wNv \text{ and } v \models \varphi \\
M^\circ, w \models \sigma & \text{iff} & \sigma(w) \\
M^\circ, w \models \tau & \text{iff} & \tau(w)
\end{array}
$$

Theorem

Let M be a neighborhood model. For all φ in $\mathcal{L}_1$

$$M \models \varphi \quad \text{iff} \quad M^\circ \models \sigma \to \varphi^*$$

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From Kripke semantics to sequent calculus

The semantics is made explicit part of the calculus:

- Multisets of labelled formulas $w : \varphi$ or relations wRv;
- Logical rules for $w : \varphi$;
- Structural rules for wRv;
- Weakening, contraction and cut.

Our aim is to:

- Convert the definition of $\vDash$ into logical rules;
- Convert properties of F° into structural rules;
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From the definition of $M^\circ, w \vDash \Box_\ni \varphi$ (if direction)

$$\frac{w \ni v, \Gamma \Rightarrow \Delta, v : \varphi}{\Gamma \Rightarrow \Delta, w : \Box_\ni \varphi} \; R\Box_\ni$$

with v not in the conclusion. From the only-if direction

$$\frac{v : \varphi, w : \Box_\ni \varphi, w \ni v, \Gamma \Rightarrow \Delta}{w : \Box_\ni \varphi, w \ni v, \Gamma \Rightarrow \Delta} \; L\Box_\ni$$

The rules for $\Box_{\not\ni}$ are similar. Analogy with $\forall$ rules.

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$$\frac{w \ni v, \Gamma \Rightarrow \Delta, w : \Diamond_N \varphi, v : \varphi}{w \ni v, \Gamma \Rightarrow \Delta, w : \Diamond_N \varphi}$$

Analogy with $\exists$ rules.

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Analogy with $\exists$ rules.

From Kripke semantics to sequent calculus

From the definition of $M^\circ, w \vDash \Diamond_N \varphi$ (if direction)

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Cut elimination in presence of axioms

- Rules of accessibility relations
- Problem: cut-free Gentzen system with new rules, *i.e.*
- criteria for a new rule to be "good" w.r.t cut elimination.
- Example: $\sim$ is an equivalence relation
- Reflexivity and Euclideaness of $\sim$ as axioms

$$\Rightarrow x \sim x \qquad x \sim y, x \sim z \Rightarrow y \sim z$$

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- Nothing is both a world and a neighborhood

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- What (classes of) frame conditions are "inferentialializable"?

- Universal conditions (P_i atom, M_j conjunction of atoms):

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In **GE**

- Weakening is admissible;
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- Cut is admissible.

If $\vdash_{\mathbf{E}} \varphi$ then $\vdash_{\mathbf{GE}} \Rightarrow w : \sigma \to \varphi^*$ ✓

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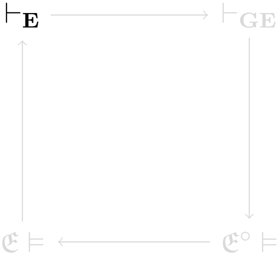

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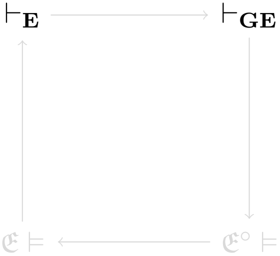

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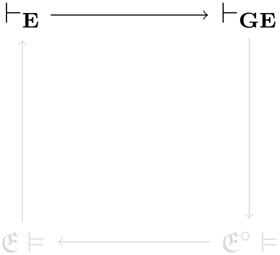

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$$W \in n(w)$$

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General Geometric Condition

- This corresponds to the system of rules

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- Condition on variable: a in common and not in Γ, Δ
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The system $\mathbf{GN}$

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- Derivability of the (translation of) axiom **N**, $\Box\top$, i.e.

$$\sigma \to \Diamond_N(\Box_{\ni}\top \wedge \Box_{\not\ni}\neg\top)$$

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wNa, \sigma(w) \Rightarrow w : \chi & R\Diamond_N \\
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  }{
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Monotonic Modal Logic

- Convert the neighborhood condition

$$a \in n(w) \text{ and } a \subseteq b \text{ implies } b \in n(w)$$

- Change the truth-condition of $\Box$

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Toward a Kripke Semantics for (Monotonic) $\Box$

- $w \models \Box\varphi$ iff $[\![\varphi]\!] \in n(w)$
- $[\![\varphi]\!] \in n(w)$ means $a \in n(w)$ and $a \subseteq [\![\varphi]\!]$, for some a
- in turn, $a \subseteq [\![\varphi]\!]$ stands for $x \in a$ implies $x \models \varphi$, for all x
- $w \models \Box\varphi$ there is $a \in n(w)$ such that $x \models \varphi$, for all $x \in a$
- $w \models \Box\varphi$ iff **there is** a s.t. $a \in n(w)$ and **for all** x
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Language $\mathcal{L}_2$

Let Var be a countable set of propositional variables.

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond_N \varphi \mid \Box_{\ni} \varphi$$

for p in Var

Translation of $\mathcal{L}_1$ into $\mathcal{L}_2$

- Translation $*$ by Kracht & Wolter, *JSL*, 1999.

$$\begin{array}{rcl} p^* & := & p \\ (\neg\varphi)^* & := & \neg\varphi^* \\ (\varphi \wedge \psi)^* & := & (\varphi^* \wedge \psi^*) \\ (\Box\varphi)^* & := & \Diamond_N \Box_{\ni} \varphi^* \end{array}$$

Semantics of $\mathcal{L}_2$

Let M be neighborhood model. A relational model

$$M^\circ = \langle W^\circ, N, \ni, V \rangle$$

- $W^\circ := W \cup \wp(W)$
- $N := \{\langle w, a \rangle \in W \times 2^W \mid a \in n(w)\}$
- $\ni := \{\langle a, w \rangle \in 2^W \times W \mid w \in a\}$
- $V : Var \longrightarrow \wp(W)$

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$$\frac{wNa, a : \varphi, \Gamma \Rightarrow \Delta}{w : \Diamond_N \varphi, \Gamma \Rightarrow \Delta} \; L\Diamond_N \qquad \frac{wNa, \Gamma \Rightarrow \Delta, w : \Diamond_N \varphi, a : \varphi}{wNa, \Gamma \Rightarrow \Delta, w : \Diamond_N \varphi} \; R\Diamond_N$$

$$\frac{x : \varphi, a : \Box_\ni \varphi, a \ni x, \Gamma \Rightarrow \Delta}{a : \Box_\ni \varphi, a \ni x, \Gamma \Rightarrow \Delta} \; L\Box_\ni \qquad \frac{a \ni x, \Gamma \Rightarrow \Delta, x : \varphi}{\Gamma \Rightarrow \Delta, a : \Box_\ni \varphi} \; R\Box_\ni$$

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Correspondence w.r.t axiomatic system

$$\begin{array}{c}
\Rightarrow w : \varphi \to \psi \\
\hline\hline
x : \psi \Rightarrow x : \varphi \\
\hline
x : \varphi, a \ni x, wNa, a : \Box_{\ni}\varphi \Rightarrow w : \Diamond_N \Box_{\ni}\psi, x : \psi \\
\hline
x : \varphi, a \ni x, wNa, a : \Box_{\ni}\varphi \Rightarrow w : \Diamond_N \Box_{\ni}\psi, x : \psi \\
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\hline
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\hline
w : \Diamond_N \Box_{\ni}\varphi \Rightarrow w : \Diamond_N \Box_{\ni}\psi \\
\hline
\Rightarrow w : \Diamond_N \Box_{\ni}\varphi \to \Diamond_N \Box_{\ni}\psi
\end{array}$$

Awareness and Local Reasoning

- Awareness is necessary condition for (explicit) knowledge.
- One cannot know something which (s)he is unaware of.
- Without awareness knowledge can only be implicit.
- K is the implicit-knowledge operator
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- $X\varphi \leftrightarrow A\varphi \wedge K\varphi$

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An awareness model $M = \langle W, R_{\mathsf{K}}, n_{\mathsf{A}}, V \rangle$ where

- W is a set
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$w \models \mathsf{K}\varphi$ iff for all w s.t. $wR_{\mathsf{K}}v$, $v \models \varphi$

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- People may have inconsistent knowledge
- if they receive contradictory information
- φ and $\neg\psi$ can be both known without knowing that φ and ψ are equivalent.
- Yet, contradictions are not known, i.e.
- $\Box\varphi \wedge \Box\neg\varphi$ is satisfiable,
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- In general, **C** fails.
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Conclusions

- Labelled sequent systems for various non-normal modal logic
- Cut-elimination and admissibility of the structural rules
- Monotonic modal logic
- Applications to logic of awareness and local reasoning

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