Monotone Modal Logic and Display Calculi

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Contents

Monotone Modal Logic

Language of monotone modal logic:

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid [\exists\forall]\varphi \mid \langle\forall\exists\rangle\varphi$$

Axioms:
- Classical base: $\bot \mid \top \mid p \mid \neg\varphi$.
- Dual modalities: $\langle\forall\exists\rangle := \neg[\exists\forall]\neg$
- Monotone modalities:

$$[\exists\forall](A \wedge B) \longrightarrow [\exists\forall]A \wedge [\exists\forall]B$$

MML & Neighbourhood frames

Language: $\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid [\exists\forall]\varphi \mid \langle\forall\exists\rangle\varphi$

Neighbourhood frame:

$$M = (X, \sigma : X \longrightarrow \mathcal{P}\mathcal{P}X).$$

$N \in \sigma(x)$ is called a **neighbourhood** of x.

Monotone neighbourhood frame:

$$X \in \sigma(x) \text{ and } X \subseteq Y \text{ implies } Y \in \sigma(x)$$

Neighbourhood model: (X, σ, v)
(X, σ) a neighbourhood frame and $v : Prop \longrightarrow \mathcal{P}X$ a valuation

Semantics

Language: $\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid [\exists\forall]\varphi \mid \langle\forall\exists\rangle\varphi$

$\mathbb{M} = \langle X, \sigma : X \longrightarrow \mathcal{P}\mathcal{P}X, v\rangle$ a neighbourhood model

$w \in X$

semantics:

$$\mathbb{M}, w \Vdash [\exists\forall]\varphi \text{ iff } \exists C \in \sigma(w) \; : \; \forall c \in C, \; \mathbb{M}, c \Vdash \varphi$$

$$\mathbb{M}, w \Vdash \langle\forall\exists\rangle\varphi \text{ iff } \forall C \in \sigma(w) \; : \; \exists c \in C, \; \mathbb{M}, c \Vdash \varphi$$

Display Calculi: informally

Sequent calculus with:

- **Operational et Structural connectives**:

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \wedge B} \qquad \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X, Y}$$

- **Contextual structural connectives**:

$$\frac{X \vdash A}{\circ X \vdash \Diamond A} \qquad \frac{A \vdash X}{\Box A \vdash \circ X}$$

- **Display property**:

$$\frac{X \vdash \circ Y}{\bullet X \vdash Y}$$

Display property / Adjunction

Display property:

$$\frac{X \vdash \circ Y}{\bullet X \vdash Y}$$

Adjunction: $\Box$ in normal logic

$$\Box(A \wedge B) = \Box A \wedge \Box B$$

Property: $\Box$ has a left adjoint, that is, $\exists$ a monotone map $\blacklozenge$ such that:

$$A \leq \Box B \quad \text{iff} \quad \blacklozenge A \leq B$$

$\blacklozenge$ is monotone:

$$\frac{\bullet X, \bullet Y \vdash Z}{\bullet (X, Y) \vdash Z}$$

Display Calculi & Monotone Modalities

$[\exists\forall]$ is only **monotone**

$\Downarrow$

NO ADJUNCTION

$\Downarrow$

NO DISPLAY PROPERTY

$$\frac{X \vdash \circ Y}{\bullet X \vdash Y}$$

Virtual Adjoints: definition

Rewriting rule $\longrightarrow$ display property

$$\frac{X \vdash \circ Y}{\bullet X \vdash \ \ Y}$$

No structural or operational rules involving virtual adjoints: the following is NOT allowed

$$\frac{\bullet X, \bullet Y \vdash Z}{\bullet(X, Y) \vdash Z} \qquad \frac{\bullet X \vdash A \qquad \bullet\, Y \vdash B}{\bullet X, \bullet Y \vdash A \wedge B}$$

Problem: CUT ELIMINATION

$$\frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}$$

Basic Calculus for MML

Introduction rules

$$\frac{A \vdash X}{[\exists\forall]A \vdash \circ X} \qquad \frac{X \vdash \circ A}{X \vdash [\exists\forall]A}$$

Cut rule: Γ is monotone w.r.t. A

$$\frac{X \vdash A \qquad \Gamma[A] \vdash \Delta}{\Gamma[X] \vdash \Delta} \qquad \frac{\Delta \vdash \Gamma[A] \qquad A \vdash X}{\Delta \vdash \Gamma[X]}$$

Segregation property: A formula is always introduced in isolation. Example:

$$\frac{X \vdash A, B}{Y \vdash A \vee B} \text{ is OK} \qquad \frac{X \vdash \circ(A, B)}{Y \vdash \circ(A \vee B)} \text{ is NOT allowed}$$

Cut Elimination

$$
\dfrac{\dfrac{\dfrac{X' \vdash Y}{X' \vdash A}}{\vdots\,\pi} \quad}{}
$$

$$
\dfrac{\begin{array}{c}\dfrac{X' \vdash Y}{X' \vdash A} \\ \vdots\,\pi \\ X \vdash A\end{array} \qquad \Gamma[A] \vdash \Delta}{\Gamma[X] \vdash \Delta}
\quad \rightsquigarrow \quad
\begin{array}{c}\dfrac{\dfrac{X' \vdash Y}{X' \vdash A} \qquad \Gamma[A] \vdash \Delta}{\Gamma[X] \vdash \Delta} \\ \vdots\,\pi \\ X \vdash A\end{array}
$$

Conclusions and further research

Conclusion: Basic display calculus: perfect for MML

Aim:

- Understand how to control virtual adjoints in the context of display calculi
- Prove soundness and cut elimination for display calculi with virtual adjoints
- Generalization: To have a modular calculus that we can adapt for Game logic, coalition logic, ...