Relational Structures in Quantum Logic

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Seminar of Applied Logic TUDelft
June 16th, 2014

Outline

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Quantum Systems and Hilbert Spaces

A (closed) quantum system is described by a Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

Hilbert Space

A Hilbert space is a *vector space* over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ equipped with an *inner product* $(\cdot, \cdot)$ such that it is *complete* under the norm $\| \cdot \|$ defined as follows:

$$\| \left|\psi\right\rangle \| = \sqrt{(\left|\psi\right\rangle, \left|\psi\right\rangle)} = \sqrt{\langle\psi|\psi\rangle}, \text{ for every vector } \left|\psi\right\rangle.$$

Fact

Every finite-dimensional vector space over $\mathbb{C}$ equipped with an inner product is a Hilbert space.

States

(Pure) states of the quantum system are described by *one-dimensional subspaces* of $\mathcal{H}$.

One-Dimensional Subspace

A one-dimensional subspace of $\mathcal{H}$ is a set of the form

$$\mathbb{C}\,|\psi\rangle \stackrel{\text{def}}{=} \{c\,|\psi\rangle\,|c \in \mathbb{C}\}, \text{ for some } |\psi\rangle \in \mathcal{H} \setminus \{\mathbf{0}\}$$

$\Sigma(\mathcal{H})$: the set of all one-dimensional subspaces of $\mathcal{H}$

Closed Linear Subspaces

Closed Linear Subspace

A closed linear subspace of $\mathcal{H}$ is a set $V \subseteq \mathcal{H}$ such that:

- for any $n \in \mathbb{N}$, $|\psi_1\rangle, ..., |\psi_n\rangle \in V$ and $c_1, ..., c_n \in \mathbb{C}$, $\sum_{i=1,...,n} c_i |\psi_i\rangle \in V$;
- for every sequence $\{|\psi_i\rangle\}_{i \in \mathbb{N}}$ in V, $\lim_{i\to\infty} |\psi_i\rangle \in V$, if it exists in $\mathcal{H}$.

Fact

$V \subseteq \mathcal{H}$ is a closed linear subspace, if and only if $(V^\perp)^\perp = V$, where $V^\perp = \{|\psi\rangle \in \mathcal{H} \mid \langle\psi|\phi\rangle = 0$, for every $|\phi\rangle \in V\}$.

Projectors

Projector

A projector P is a linear operator on $\mathcal{H}$ such that:

- (Boundedness) there is a $c \in \mathbb{C}$ such that $\|\mathrm{P}\,|\psi\rangle\,\| \leq c\|\,|\psi\rangle\,\|$, for every $|\psi\rangle \in \mathcal{H}$;
- (Idempotence) $\mathrm{P} \circ \mathrm{P} = \mathrm{P}$;
- (Self-Adjointness) for any $|\phi\rangle\,, |\psi\rangle \in \mathcal{H}$,

$$(\mathrm{P}\,|\psi\rangle\,, |\phi\rangle) = (|\psi\rangle\,, \mathrm{P}\,|\phi\rangle)$$

Fact

Every projector P has exactly two eigenvalues 0 and 1 with both eigenspaces being closed linear subspaces of $\mathcal{H}$.

Testable Properties

Fact

There is a bijection between closed linear subspaces of $\mathcal{H}$ and projectors on $\mathcal{H}$.

Every testable property of the quantum system is described by:

- a *closed linear subspace* of $\mathcal{H}$; or equivalently,
- a *projector* of $\mathcal{H}$.

Tests of Properties

Do an experiment to test whether the system in state $\mathbb{C}\left|\psi\right\rangle$ has property P:

Result	State After the Test	Probability
'Yes'	$\mathbb{C}\mathsf{P}\left|\psi\right\rangle$	$\frac{\langle\psi\vert\mathsf{P}\vert\psi\rangle}{\langle\psi\vert\psi\rangle}$
'No'	$\mathbb{C}(\mathsf{I}-\mathsf{P})\left|\psi\right\rangle$	$\frac{\langle\psi\vert(\mathsf{I}-\mathsf{P})\vert\psi\rangle}{\langle\psi\vert\psi\rangle}$

Testing a property can change the state of the system!

Outline

The Logic of Quantum Mechanics

In [Birkhoff and von Neumann, 1936], it's shown that closed linear subspaces of $\mathcal{H}$ form an *orthocomplemented lattice*:

- Top: $\mathcal{H}$;
- Bottom: $\{\mathbf{0}\}$;
- Partial Order: set-theoretic inclusion;
- Meet: set-theoretic intersection;
- Join: closure of the linear span of the set-theoretic union;
- Orthocomplement: orthocomplement.

In [Birkhoff and von Neumann, 1936], it's shown that this lattice is non-distributive.

Piron's Theorem

Theorem [Piron, 1976]

- The lattice of bi-orthogonally closed subspaces of a generalized Hilbert space is always a Piron lattice.
- Every Piron lattice of height at least 4 is isomorphic to the lattice of bi-orthogonally closed subspaces of a generalized Hilbert space.

This theorem is significant, because:

- generalized Hilbert spaces resemble Hilbert spaces closely;
- Piron lattices are defined in purely lattice-theoretic terms.

Generalized Hilbert Spaces

Generalized Hilbert Space

A generalized Hilbert space is a *vector space* over a *division ring* K with an *involution*, equipped with an orthomodular *Hermitian form*.

Theorem [Amemiya and Araki, 1966]

1. Every Hilbert space is a generalized Hilbert space.
2. Every generalized Hilbert space, whose underlying division ring is $\mathbb{C}$ with *complex conjugate* being the involution, is a Hilbert space.
   Moreover, bi-orthogonally closed subspaces coincide with closed linear subspaces.

Piron Lattices

A Piron lattice $\mathfrak{L} = (L, \leq, -^{\perp})$ is a bounded lattice equipped with a unary operation satisfying the following 6 conditions:

1. **Orthocomplement:** The operation $-^{\perp} : L \to L$ satisfies:
    1. $p^{\perp\perp} = p$;
    2. $p \leq q$ implies $q^{\perp} \leq p^{\perp}$;
    3. $p \wedge p^{\perp} = O$ and $p \vee p^{\perp} = I$.
2. **Weak Modularity:** $q \leq p$ implies $q \vee (q^{\perp} \wedge p) = p$.
3. **Completeness:** For any $A \subseteq L$, $\bigwedge A$ and $\bigvee A$ are in L.
4. **Atomicity:** If $p \neq O$, there is an $a \in \mathrm{At}(\mathfrak{L})$ such that $a \leq p$.
5. **Covering Law:** If $a \in \mathrm{At}(\mathfrak{L})$ and $a \wedge p = O$, $a \vee p$ covers a.
6. **Superposition Principle:** For any two *distinct* $a, b \in \mathrm{At}(\mathfrak{L})$, there is a $c \in \mathrm{At}(\mathfrak{L}) \setminus \{a, b\}$ such that $a \vee c = b \vee c = a \vee b$.

Outline

Outline

Non-Orthogonality in Quantum Theory

Non-Orthogonality and Orthogonality

$$\mathbb{C}\,|\psi\rangle \to \mathbb{C}\,|\phi\rangle, \text{ if } \langle\psi|\phi\rangle \neq 0$$
$$\mathbb{C}\,|\psi\rangle \not\to \mathbb{C}\,|\phi\rangle, \text{ if } \langle\psi|\phi\rangle = 0$$

Some Terminologies of Kripke Frames

Kripke Frame

A Kripke frame $\mathfrak{F}$ is a tuple $(\Sigma, \to)$, where Σ is a non-empty set and $\to \subseteq \Sigma \times \Sigma$.

- Write $s \not\to t$ for $(s, t) \notin \to$.
- Given $P \subseteq \Sigma$, define the orthocomplement of P (w.r.t. $\to$):

$$\sim P \stackrel{\text{def}}{=} \{s \in \Sigma \mid s \not\to t, \text{ for every } t \in P\}$$

- P is bi-orthogonally closed, if $P = \sim\sim P$.
- $\mathcal{L}_{\mathfrak{F}} \stackrel{\text{def}}{=} \{P \subseteq \Sigma \mid P = \sim\sim P\}$.

Orthoframes

Orthoframe (modified from [Goldblatt, 1974])

An orthoframe $\mathfrak{F}$ is a Kripke frame $(\Sigma, \to)$ where the binary relation is reflexive and symmetric.

Theorem

- The tuple $(\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(-))$ is an ortho-lattice for any orthoframe $\mathfrak{F} = (\Sigma, \to)$. [Birkhoff, 1966]
- Every orthocomplemented lattice can be embedded into the ortho-lattice $(\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(-))$ for some orthoframe $\mathfrak{F} = (\Sigma, \to)$. [Goldblatt, 1974]

Paralleled to intuitionistic logic, orthoframes are axiomatized by a propositional logic called orthologic. [Goldblatt, 1974]

Orthomodular Frames

Orthomodular Frame (modified from [Goldblatt, 1974])

An orthomodular frame $\mathfrak{F}$ is a tuple $(\Sigma, \to, \Pi)$ in which $(\Sigma, \to)$ is an orthoframe and Π is a non-empty subset of $\wp(\Sigma)$ satisfying:

1. $P \in \Pi$ implies that $P = \sim\sim P$;
2. $P, Q \in \Pi$ implies that $P \cap Q, \sim P \in \Pi$;
3. $P \subseteq Q$ implies that $P = \sim(\sim P \cap Q) \cap Q$, for any $P, Q \in \Pi$.

- For every orthomodular frame $(\Sigma, \to, \Pi)$, $(\Pi, \subseteq, \sim(-))$ is an orthomodular lattice.
- Orthomodular frames are axiomatized by a propositional logic called orthomodular logic. [Goldblatt, 1974]

State Spaces

State Space (modified from [Moore, 1995])

A state space is a Kripke frame $(\Sigma, \to)$ in which $\to$ is *reflexive*, *symmetric* and *separated*, i.e.

> *there is a* $w \in \Sigma$ *such that* $w \not\to s$ *and* $w \to t$, *for any distinct* $s, t \in \Sigma$.

Property lattice

A property lattice is a *complete atomistic ortho-lattice*.

The main result in [Moore, 1995] is a duality between

- a category with *state spaces* as objects, and
- a category with *property lattices* as objects.

Outline

Tests in Quantum Theory

Quantum Test

For a testable property P and two states $\mathbb{C}|\psi\rangle$ and $\mathbb{C}|\phi\rangle$,

$$\mathbb{C}|\psi\rangle \xrightarrow{P?} \mathbb{C}|\phi\rangle, \text{ if } \mathbb{C}\mathsf{P}|\psi\rangle = \mathbb{C}|\phi\rangle.$$

The non-classical character of the "logic" of quantum-testable properties is not due to the fact that they are properties of a quantum system, but to the fact that we required them to be "testable" by quantum measurements. It is the non-classical nature of quantum actions (in particular, quantum tests) that explains the strangeness of quantum behaviour.
[Baltag and Smets, 2011]

Quantum Dynamic Frames [Baltag and Smets, 2005]

A quantum dynamic frame $\mathfrak{F}$ is a tuple $(\Sigma, \mathcal{L}, \{\stackrel{P?}{\to}\}_{P\in\mathcal{L}})$, in which

- Σ is a non-empty set;
- $\mathcal{L} \subseteq \wp(\Sigma)$;
- $\stackrel{P?}{\to} \subseteq \Sigma \times \Sigma$, for each $P \in \mathcal{L}$;

and it satisfies 7 conditions.

- Adequacy: If $s \in P$ and $P \in \mathcal{L}$, then $s \stackrel{P?}{\to} s$.
- Repeatability: If $P \in \mathcal{L}$ and $s \stackrel{P?}{\to} t$, then $t \in P$.
- ...

Non-Orthogonality in Quantum Dynamic Frames

Non-Orthogonality and Orthogonality

$s \to t \iff$ there is some $P \in \mathcal{L}$ such that $s \stackrel{P?}{\to} t$.

$s \not\to t \iff$ there is no $P \in \mathcal{L}$ such that $s \stackrel{P?}{\to} t$.

Quantum Dynamic Frames and Piron Lattices

Theorem 1 [Baltag and Smets, 2005],[Bergfeld et al., 2014]

$(\mathcal{L}, \subseteq, \sim(-))$ is a Piron lattice, where $\sim(-)$ is the orthocomplement operation (w.r.t. $\rightarrow$), for any quantum dynamic frame $\mathfrak{F} = (\Sigma, \mathcal{L}, \{\stackrel{P?}{\rightarrow}\}_{P \in \mathcal{L}})$.

Theorem 2 [Baltag and Smets, 2005],[Bergfeld et al., 2014]

Every Piron lattice $\mathfrak{L}$ is isomorphic to $(\mathcal{L}, \subseteq, \sim(-))$ for some quantum dynamic frame $\mathfrak{F} = (\Sigma, \mathcal{L}, \{\stackrel{P?}{\rightarrow}\}_{P \in \mathcal{L}})$.

Outline

Outline

Quantum Kripke Frames

A quantum Kripke frame $\mathfrak{F}$ is a Kripke frame $(\Sigma, \to)$ satisfying 4 conditions (following slides).

Conditions for Quantum Kripke Frames (1)

Reflexivity and Symmetry

- Reflexivity: $s \to s$, for every $s \in \Sigma$.
- Symmetry: $s \to t \Rightarrow t \to s$, for any $s, t \in \Sigma$.

Fact

- (Positive) Definiteness:
  $\langle\psi|\psi\rangle \neq 0$, for every $|\psi\rangle \in \mathcal{H} \setminus \{\mathbf{0}\}$.
- Conjugate Symmetry: $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$, for any $|\phi\rangle, |\psi\rangle \in \mathcal{H}$.

Conditions for Quantum Kripke Frames (2)

Separation

For any $s, t \in \Sigma$ satisfying $s \neq t$, there is a $w \in \Sigma$ such that $w \rightarrow s$ but $w \not\rightarrow t$.

Fact (Gram-Schmidt Trick)

For any linearly independent $|\phi\rangle, |\psi\rangle \in \mathcal{H} \setminus \{\mathbf{0}\}$, define

$$|\theta\rangle = |\phi\rangle - \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle} |\psi\rangle .$$

Then $\langle\theta|\phi\rangle \neq 0$ and $\langle\theta|\psi\rangle = 0$.

Conditions for Quantum Kripke Frames (3)

Existence of Good Approximation

For any $s \in \Sigma$ and $P \subseteq \Sigma$, if $\sim\sim P = P$ and $s \notin \sim P$, then there is a $t \in \Sigma$ such that

$$(\star)\ t \in P, \text{ and } s \to u \Leftrightarrow t \to u \text{ for each } u \in P.$$

Theorem (Orthogonal Decomposition)

For every $|\psi\rangle \in \mathcal{H}$ and closed linear subspace V of $\mathcal{H}$, there are $|\psi_0\rangle \in V$ and $|\psi_\perp\rangle \in V^\perp$ such that $|\psi\rangle = |\psi_0\rangle + |\psi_\perp\rangle$. Moreover, $\langle\psi|\phi\rangle = \langle\psi_0|\phi\rangle$, for every $|\phi\rangle \in V$.

Conditions for Quantum Kripke Frames (4)

Superposition

For any $s, t \in \Sigma$, there is a $w \in \Sigma$ such that $w \to s$ and $w \to t$.

Fact

For any $|\phi\rangle, |\psi\rangle \in \mathcal{H} \setminus \{\mathbf{0}\}$, there are $c_1, c_2 \in \mathbb{C}$ such that $|\theta\rangle = c_1 |\phi\rangle + c_2 |\psi\rangle$ satisfies $\langle\theta|\phi\rangle \neq 0$ and $\langle\theta|\psi\rangle \neq 0$.

Quantum Kripke Frames (Summary)

Quantum Kripke Frame

A quantum Kripke frame $\mathfrak{F}$ is a Kripke frame $(\Sigma, \to)$ such that:

- (i) $\to$ is reflexive and symmetric.
- (ii) (Separation) if $s \neq t$, then there is a $w \in \Sigma$ such that $w \to s$ and $w \not\to t$;
- (iii) (Existence of Good Approximation) if $s \notin {\sim}P$ and $P = {\sim}{\sim}P$, then there is a $t \in P$ such that $s \to u \Leftrightarrow t \to u$ for each $u \in P$;
- (iv) (Superposition) for any $s, t \in \Sigma$, there is a $w \in \Sigma$ such that $w \to s$ and $w \to t$.

Main Theorems

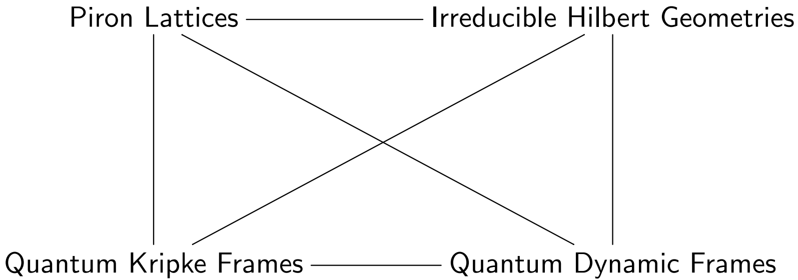
Theorem 1

For every quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, $(\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(-))$ is a Piron lattice, where $\sim(-)$ is the orthocomplement operation (w.r.t. $\to$).

Theorem 2

Every Piron lattice is isomorphic to $(\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(-))$ for some quantum Kripke frame $\mathfrak{F}$.

Correspondence among Quantum Structures



Good Approximations are the Best

Existence of Good Approximation

If $s \notin {\sim}P$ and ${\sim}{\sim}P = P$, then there is a $t \in \Sigma$ such that
$(\star)$ $t \in P$, and $s \to u \Leftrightarrow t \to u$ for each $u \in P$.

Separation guarantees that the t satisfying $(\star)$ is unique, which will be called the best approximation of s in P.

Given $P \in \mathcal{L}_{\mathfrak{F}}$, define a partial function $P?(-) : \Sigma \dashrightarrow \Sigma$ as follows:

$$P?(s) \stackrel{def}{=} \begin{cases} \text{the best approxiamtion of } s \text{ in } P, & \text{if } s \notin {\sim}P \\ \text{undefined}, & \text{otherwise} \end{cases}$$

Quantum Kripke Frames and Quantum Dynamic Frames

Proposition

Given a quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, for each $P \in \mathcal{L}_{\mathfrak{F}}$, define $\stackrel{P?}{\to} \subseteq \Sigma \times \Sigma$ such that:

$$s \stackrel{P?}{\to} t \iff s \notin \sim P \text{ and } t = P?(s).$$

Then $(\Sigma, \mathcal{L}_{\mathfrak{F}}, \{\stackrel{P?}{\to}\}_{P \in \mathcal{L}_{\mathfrak{F}}})$ is a quantum dynamic frame.

Proposition

Every quantum dynamic frame is isomorphic to $(\Sigma, \mathcal{L}_{\mathfrak{F}}, \{\stackrel{P?}{\to}\}_{P \in \mathcal{L}_{\mathfrak{F}}})$ for some quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$.

Outline

Probabilistic Kripke Frames from Hilbert Spaces

For a Hilbert space $\mathcal{H}$ over $\mathbb{C}$, the relational structure $(\Sigma(\mathcal{H}), \to)$ can be extended to a probabilistic one by adding a function $\rho_{\mathcal{H}} : \Sigma(\mathcal{H}) \times \Sigma(\mathcal{H}) \to [0, 1]$ called transition probability and defined as

$$\rho_{\mathcal{H}}(\mathbb{C}\left|\psi\right\rangle, \mathbb{C}\left|\phi\right\rangle) = \frac{\langle\psi|\phi\rangle\,\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\,\langle\phi|\phi\rangle}$$

Probabilistic Quantum Kripke Frames

A probabilistic quantum Kripke frame $\mathfrak{F}_P$ is a tuple $(\mathfrak{F}, \rho)$, where

- $\mathfrak{F} = (\Sigma, \to)$ is a quantum Kripke frame;
- ρ is a function from $\Sigma \times \Sigma$ to $[0, 1]$;

and it satisfies 4 conditions (following slides).

Conditions for Probabilistic Quantum Kripke Frames (1)

Condition 1

$\rho(s, t) = \rho(t, s)$, for any $s, t \in \Sigma$.

Fact

$$\rho_{\mathcal{H}}(\mathbb{C}\left|\psi\right\rangle, \mathbb{C}\left|\phi\right\rangle) = \frac{\langle\psi|\phi\rangle\,\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\,\langle\phi|\phi\rangle} = \frac{\langle\phi|\psi\rangle\,\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle\,\langle\phi|\phi\rangle} = \rho_{\mathcal{H}}(\mathbb{C}\left|\phi\right\rangle, \mathbb{C}\left|\psi\right\rangle),$$

for any $\mathbb{C}\left|\psi\right\rangle, \mathbb{C}\left|\phi\right\rangle \in \Sigma(\mathcal{H})$.

Conditions for Probabilistic Quantum Kripke Frames (2)

Condition 2

For any $s, t \in \Sigma$, $\rho(s, t) = 0$ if and only if $s \not\to t$.

Fact

$$\rho_{\mathcal{H}}(\mathbb{C}|\psi\rangle, \mathbb{C}|\phi\rangle) = \frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle} = 0 \Leftrightarrow \langle\psi|\phi\rangle = 0 \Leftrightarrow \langle\phi|\psi\rangle = 0$$

for any $\mathbb{C}|\psi\rangle, \mathbb{C}|\phi\rangle \in \Sigma(\mathcal{H})$.

Conditions for Probabilistic Quantum Kripke Frames (3)

Condition 3

If $\{t_i \mid i \in I\} \subseteq \Sigma$ satisfies that I is at most countable and $t_i \perp t_j$ whenever $i \neq j$, then $\sum_{i \in I} \rho(s, t_i) \leq 1$.
Moreover, equality holds if and only if $s \in \sim\sim\{t_i \mid i \in I\}$.

Fact

For any $\mathbb{C}\left|\psi\right\rangle \in \Sigma(\mathcal{H})$ and *orthogonal* set $\{\mathbb{C}\left|\phi_i\right\rangle \in \Sigma(\mathcal{H}) \mid i \in I\}$ such that I is at most countable,

$$\sum_{i \in I} \rho_{\mathcal{H}}(\mathbb{C}\left|\psi\right\rangle, \mathbb{C}\left|\phi_i\right\rangle) = \sum_{i \in I} \frac{\langle\psi|\phi_i\rangle \langle\phi_i|\psi\rangle}{\langle\psi|\psi\rangle \langle\phi_i|\phi_i\rangle} \leq 1$$

Moreover, equality holds if and only if $\mathbb{C}\left|\psi\right\rangle \in \{\mathbb{C}\left|\phi_i\right\rangle \mid i \in I\}^{\perp\perp}$.

Conditions for Probabilistic Quantum Kripke Frames (4)

Condition 4

If $P \in \mathcal{L}_{\mathfrak{F}}$, $s \notin \sim P$ and $t \in P$, then
$\rho(s, t) = \rho(s, P?(s)) \cdot \rho(P?(s), t)$.

Fact

For any $\mathbb{C}|\psi\rangle, \mathbb{C}|\phi\rangle \in \Sigma(\mathcal{H})$ and projector P on $\mathcal{H}$ such that $\mathrm{P}|\phi\rangle = |\phi\rangle$,

$$\begin{aligned}
\rho_{\mathcal{H}}(\mathbb{C}|\psi\rangle, \mathbb{C}|\phi\rangle) &= \frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle} \\
&= \frac{\langle\psi|\mathrm{P}|\psi\rangle\langle\psi|\mathrm{P}^\dagger|\psi\rangle}{\langle\psi|\psi\rangle\langle\psi|\mathrm{P}^\dagger\mathrm{P}|\psi\rangle} \cdot \frac{\langle\psi|\mathrm{P}^\dagger|\phi\rangle\langle\phi|\mathrm{P}|\psi\rangle}{\langle\phi|\phi\rangle\langle\psi|\mathrm{P}^\dagger\mathrm{P}|\psi\rangle} \\
&= \rho_{\mathcal{H}}(\mathbb{C}|\psi\rangle, \mathbb{C}\mathrm{P}|\psi\rangle) \cdot \rho_{\mathcal{H}}(\mathbb{C}\mathrm{P}|\psi\rangle, \mathbb{C}|\phi\rangle)
\end{aligned}$$

Probabilistic Quantum Kripke Frames (Summary)

Probabilistic Quantum Kripke Frame

A probabilistic quantum Kripke frame $\mathfrak{F}_P$ is a tuple $(\mathfrak{F}, \rho)$, where $\mathfrak{F} = (\Sigma, \rightarrow)$ is a quantum Kripke frame and ρ is a function from $\Sigma \times \Sigma$ to $[0, 1]$ satisfying the following:

1. $\rho(s, t) = \rho(t, s)$;
2. $\rho(s, t) = 0$, if and only if $(s, t) \notin \rightarrow$;
3. if $\{t_i \mid i \in I\} \subseteq \Sigma$ satisfies that I is at most countable and $t_i \perp t_j$ whenever $i \neq j$, then $\sum_{i \in I} \rho(s, t_i) \leq 1$; and equality holds if and only if $s \in \sim\sim\{t_i \mid i \in I\}$;
4. if $P \in \mathcal{L}_{\mathfrak{F}}$, $s \notin \sim P$ and $t \in P$, then $\rho(s, t) = \rho(s, P?(s)) \cdot \rho(P?(s), t)$.

Justification

Proposition

Given a Hilbert space $\mathcal{H}$ over $\mathbb{C}$, the tuple $(\Sigma(\mathcal{H}), \to, \rho_{\mathcal{H}})$ is a probabilistic quantum Kripke frame.

Proposition

Given a probabilistic quantum Kripke frame $(\mathfrak{F}, \rho)$, where $\mathfrak{F} = (\Sigma, \to)$, and $s \in \Sigma$, define a function $\mu_s : \mathcal{L}_{\mathfrak{F}} \to [0, 1]$ by

$$\mu_s(P) = \begin{cases} \rho(s, P?(s)), & \text{if } s \notin {\sim}P \\ 0, & \text{otherwise} \end{cases}$$

Then this function is a *quantum probability measure* on the Piron lattice $(\mathcal{L}_{\mathfrak{F}}, \subseteq, {\sim}(-))$.

Quantum Probability Measure

Quantum Probability Measure

A quantum probability measure is a function p from a Piron lattice $\mathfrak{L} = (L, \leq, (-)^{\perp})$ to $[0, 1]$ such that:

- $p(I) = 1$;
- $\sum_{i \in A} p(b_i)$ exists and is equal to $p(\bigvee_{i \in A} b_i)$, for every $\{b_i \mid i \in A\} \subseteq L$ with A at most countable and $b_i \leq b_j^{\perp}$ when $i \neq j$.
- $p(b) = p(c) = 0$ implies that $p(b \vee c) = 0$, for every $b, c \in L$.

This definition is adapted from Definition (4.38) on page 82 of [Piron, 1976].

Outline

Axiomatizing Quantum Logic

- Until now there is no adequate axiomatization of quantum logic.
- Orthomodular logic axiomatizes orthomodular lattices, but there are formulas which fail in some orthomodular lattice but hold in all Hilbert lattices.
- There have been some attempt to axiomatize quantum dynamic frames in PDL with tests.
- One of the challenges is that some conditions involve saying that a state can not access another state, which is a characteristic feature of undefinable properties of modal language.
- Axiomatizing quantum Kripke frames in the basic modal language faces similar challenges.

Describing Compound Quantum Systems

- In the standard Hilbert space formalism, if a quantum system consists of two subsystems described by two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, the system itself can be described by the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$.
- No construction in lattice theory have been found to match the power of tensor product of Hilbert spaces.
- It is interesting to see whether this problem can be solved from the perspective of relational structures.

Probing Probabilistic Quantum Kripke Frames

- Characterize quantum Kripke frames that are induced by Hilbert spaces over $\mathbb{C}$ with some conditions involving probability.
- Capture the notions of *quantum probability measure* (and thus *mixed states*) in this framework from a more local perspective.

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Thank you very much!

Quantum Dynamic Frame

1. **Closure Condition:** $\mathcal{L}$ is closed under arbitrary intersection and orthocomplement.
2. **Atomicity:** For any $s \in \Sigma$, $\{s\} \in \mathcal{L}$.
3. **Adequacy:** For any $s \in \Sigma$ and $P \in \mathcal{L}$, if $s \in P$, then $s \xrightarrow{P?} s$.
4. **Repeatability:** For any $s, t \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t$, then $t \in P$.
5. **Self-Adjointness:** For any $s, t, u \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t \rightarrow u$, then there is a $v \in \Sigma$ such that $u \xrightarrow{P?} v \rightarrow s$.
6. **Covering Property:** Suppose $s \xrightarrow{P?} t$ for $s, t \in \Sigma$ and $P \in \mathcal{L}$. Then, for any $u \in P$, if $u \neq t$ then $u \rightarrow v \not\rightarrow s$ for some $v \in P$.
7. **Proper Superposition:** For any $s, t \in \Sigma$ there is a $u \in \Sigma$ such that $u \rightarrow s$ and $u \rightarrow t$.

Quantum Kripke Frames and Classical Frames

Definition (Classical Frame)

A classical frame $\mathfrak{F}$ is a Kripke frame $(\Sigma, \to)$ in which $\to$ is the identity relation, i.e. $\to = \{(s, t) \in \Sigma \times \Sigma \mid s = t\}$.

For every classical frame $\mathfrak{F}$, $(\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(-))$ is a Boolean lattice.

Proposition

Let $\mathfrak{F} = (\Sigma, \to)$ be a Kripke frame satisfying conditions (i) to (iii) in the definition of quantum Kripke frames but not condition (iv), i.e. superposition. Then

- $\mathfrak{F}$ is a quantum Kripke frame, iff superposition holds;
- $\mathfrak{F}$ is a classical frame, iff $\to$ is transitive.

Moreover, if Σ has at least 2 elements, then superposition and transitivity of $\to$ can not hold simultaneously.