Subordinations, closed relations and compact Hausdorff spaces

Sumit Sourabh

ILLC, University of Amsterdam

Applied Logic Seminar
TU Delft

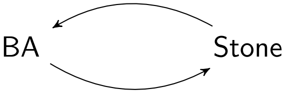
This talk is in two parts:

Part I: Duality results for Boolean algebra with a relation

(based on *Subordinations, closed relations and compact Hausdorff spaces*. Guram Bezhanishvili, Nick Bezhanishvili, Sumit Sourabh, Yde Venema. Submitted, December 2014.)

Part II: Canonicity results for Boolean algebra with a relation

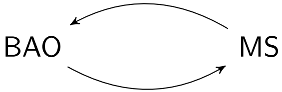
(work in progress)



Stone duality (1936)

A Stone space is a compact Hausdorff and zero-dimensional space.

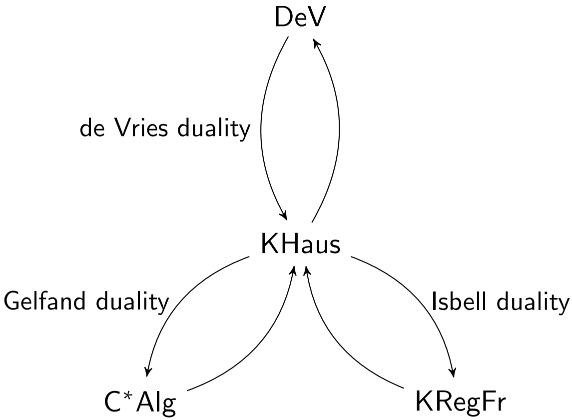
Jóhnsson-Tarski Duality (1951-52)



Jóhnsson-Tarski duality

A modal space is a Stone space W with a relation R which satisfies:
(i) $R[w]$ is a closed set
(ii) $R^{-1}(C)$ is a clopen set for each clopen $C \subseteq W$.

Dualities for Compact Hausdorff spaces



Dualities for KHaus

de Vries algebra

de Vries algebra [de Vries (1962)]

A de Vries algebra is a pair $(A, \prec)$ consisting of a complete Boolean algebra A and a binary relation $\prec$ on A satisfying the following

(S1) $0 \prec 0$ and $1 \prec 1$;

(S2) $a \prec b, c$ implies $a \prec b \wedge c$;

(S3) $a, b \prec c$ implies $a \vee b \prec c$;

(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

(S5) $a \prec b$ implies $a \leq b$;

(S6) $a \prec b$ implies $\neg b \prec \neg a$;

(S7) $a \prec b$ implies there is $c \in B$ with $a \prec c \prec b$;

(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

de Vries algebra

Example 1 The set of regular open sets ($U = \mathbf{IC}U$) of a compact Hausdorff space X form a complete Boolean algebra.

For $U, V \in \mathrm{RegOp}(X)$ define $U \prec V$ if $\mathbf{C}U \subseteq V$. Then $(\mathrm{RegOp}(X), \prec)$ is a de Vries algebra.

Example 2 For B a complete Boolean algebra, $(B, \leq)$ is de Vries.

Example 3 Let $B = \mathcal{P}\mathbb{N}$ be the power set of the natural numbers and define $S \prec T$ iff $S \subseteq T$ and at least one of S, T is finite or cofinite.

de Vries duality

For a de Vries algebra $(B, \prec)$ and $A \subset B$, define

$$\Uparrow A = \{b : a \prec b \text{ for some } a \in A\}$$

A filter F of a de Vries algebra B is round if $F = \Uparrow F$. The maximal round filters are called ends. The set $\mathcal{E}B$ of ends of B is topologized by the basis of sets $\varphi(b) = \{E : b \in E\}$.

Theorem

$\mathcal{E}B$ is a compact Hausdorff space whose de Vries algebra of regular open sets is isomorphic to B.

Subordinations

Definition

A subordination on a Boolean algebra B is a binary relation $\prec$ satisfying:

(S1) $0 \prec 0$ and $1 \prec 1$;

(S2) $a \prec b, c$ implies $a \prec b \wedge c$;

(S3) $a, b \prec c$ implies $a \vee b \prec c$;

(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

Let Sub be the category whose objects are pairs $(B, \prec)$, where B is a BA and $\prec$ is a subordination on B, and whose morphisms are Boolean homomorphisms h satisfying $a \prec b$ implies $h(a) \prec h(b)$.

Duality for Subordinations

Let StR be the category whose objects are pairs (X, R), where X is a Stone space and R is a closed relation on X, and whose morphisms are continuous stable morphisms[1].

For $(B, \prec) \in \text{Sub}$, let $(B, \prec)_* = (X, R)$, where X is the Stone space of B and xRy iff $\Uparrow x \subseteq y$. Then $(X, R) \in \text{StR}$

For $(X, R) \in \text{StR}$, let $(X, R)^* = (\text{Clop}(X), \prec)$, where $U \prec V$ iff $R[U] \subseteq V$. Then $(Clop(X), \prec) \in \text{Sub}$.

Theorem

The categories Sub *and* StR *are dually equivalent.*

[1] We say $f : X_1 \to X_2$ is stable if xR_1y implies $f(x)R_2f(y)$

Characteristic function of the relation

A map $\to : B \times B \to \mathbf{2}$ a strict implication if it satisfies

(I1) $0 \to a = a \to 1 = 1$.

(I2) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$.

(I3) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

Example If $(B, \prec) \in \mathrm{Sub}$, then $\prec_R : B \times B \to \mathbf{2}$ as defined below is a strict implication.

$$\to_{\prec} (x, y) := \begin{cases} 1 & \text{if } x \prec y \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, if $\to : B \times B \to \mathbf{2}$ is a strict implication, then $\prec_{\to} \subseteq B \times B$ as defined below is a subordination.

$$a \prec_{\to} b \text{ iff } a \to b = 1$$

Generalized Jónsson-Tarski duality

By the generalized Jónsson-Tarski duality the dual ternary relation $S \subseteq X \times Y \times Z$ of a dual operator map $f : A \times B \to C$ is given by

$(x, y, z) \in S$ iff $(\forall a \in A)(\forall b \in B)(f(a, b) \in z$ implies $a \notin x$ or $b \in y)$;

The Stone space of $\mathbf{2}$ is the singleton discrete space $\{z\}$, where $z = \{1\}$ is the only ultrafilter of $\mathbf{2}$.

Therefore, the dual ternary relation $S \subseteq X \times X \times \{z\}$ of $\to: B \times B \to \mathbf{2}$ is given by

$$(x, y, z) \in S \text{ iff } (\forall a, b \in B)(a \to b = 1 \text{ implies } a \notin x \text{ or } b \in y).$$

Generalized Jónsson-Tarski duality

The ternary relation S reduces to a binary relation $R \subseteq X \times X$ by

$$xRy \text{ iff } (x, y, 1) \in S.$$

Using equivalence between strict implications and subordinations,

$$xRy \text{ iff } (\forall a, b \in B)(a \prec b \text{ implies } a \notin x \text{ or } b \in y) \text{ iff } \twoheaduparrow x \subseteq y.$$

From Jónsson-Tarski duality, the dual ternary relation $S \subseteq X \times X \times \{z\}$ satisfies: $S^{-1}(\{z\})$ is closed. Hence, $R = S^{-1}(\{1\})$ is a closed relation.

Precontact algebra [Düntsch, Vakarelov (2003)]

A precontact algebra is a pair (A, C) where A is a BA and C is a binary relation on A satisfying:
$(C0)$ aCb implies $a, b \neq 0$.
$(C+)$ $aC(b \vee c)$ implies aCb or aCc; $(a \vee b)Cc$ implies aCb or aCc.

Precontact algebra and their subvarieties are used in the algebraic analysis of theory of regions.

Proximity lattices

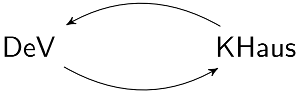
Proximity lattice [Jung, Sünderhauf (1996)]

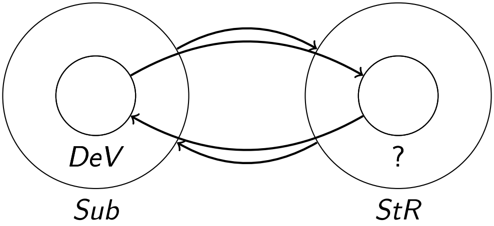
A proximity lattice is a pair $(\mathbb{L}, R)$, where L is a lattice and $R \subseteq L \times L$ is a relation satisfying the following axioms:

1. $R \circ R = R$.
2. For any finite set $A \subseteq L$ and $b \in L$, $\bigvee A R b \Leftrightarrow \forall a \in A \; aRb$.
3. For any finite set $B \subseteq L$ and $b \in L$, $aR \bigwedge B \Leftrightarrow \forall b \in B \; aRb$.

Strong proximity lattices are the algebraic structures dual to stably compact spaces.

A "modal" de Vries duality?





Elementary conditions

Let $(B, \prec)$ be a subordination, which satisfies the following axioms.

(S5) $a \prec b$ implies $a \leq b$;

(S6) $a \prec b$ implies $\neg b \prec \neg a$;

(S7) $a \prec b$ implies there is $c \in B$ with $a \prec c \prec b$;

Lemma

Let $(X, R) \in \mathrm{StR}$ be the dual space of $(B, \prec)$.

1. *R is reflexive iff $\prec$ satisfies (S5).*
2. *R is symmetric iff $\prec$ satisfies (S6).*
3. *R is transitive iff $\prec$ satisfies (S7).*

Lattice subordination [G. Bezhanishvili (2013)]

A lattice subordination is a subordination $(A, \prec)$ where $\prec$ additionally satisfies:
(S9) $a \prec b$ implies that there exists $c \in B$ with $c \prec c$ and $a \leq c \leq b$.

A quasi-order on a Stone space X is a Priestly quasi-order if $x \not\leq y$ implies that there exists a clopen up-set U of X with $x \in U$ and $y \notin U$.

Lemma

R is a Priestley quasi-order iff $\prec$ satisfies (S9).

A "modal" de Vries duality

A continuous map $f : X \to Y$ between compact Hausdorff spaces is irreducible provided the f-image of each proper closed subset of X is a proper subset of Y.

We call a closed equivalence relation R on a compact Hausdorff space X irreducible if the factor-map $\pi : X \to X/R$ is irreducible.

A closed equivalence relation R is irreducible iff for each proper closed subset F of X, we have $R[F]$ is a proper subset of X (non-elementary!).

(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

Lemma

Let $(B, \prec) \in \mathrm{Sub}$ *and let* (X, R) *be the dual of* $(B, \prec)$. *Then the closed equivalence relation* R *is irreducible iff* $\prec$ *satisfies (S8).*

We call a pair (X, R) a *Gleason space* if X is an extremely disconnected space and R is an irreducible equivalence relation on X.

Theorem

Gle *is dually equivalent to* DeV, *hence* Gle *is equivalent to* KHaus.

Categories of algebras

Category	Objects
Sub	Boolean algebras with a subordination
PCon	Boolean algebras with a precontact relation
MSub	Boolean algebras with a modally definable subordination
SubK4	Sub satisfying (S7)
SubS4	Sub satisfying (S5) and (S7)
SubS5	Sub satisfying (S5), (S6), and (S7)
LSub	Boolean algebras with a lattice subordination
Com	Sub satisfying (S5), (S6), (S7) and (S8)
DeV	De vries algebras

Categories of Boolean algebras with subordination

Categories of spaces

Category	Objects
StR	Stone spaces with a closed relation
MS	Modal spaces
$\mathsf{StR}^{\mathsf{tr}}$	Stone spaces with a closed transitive relation
$\mathsf{StR}^{\mathsf{qo}}$	Stone spaces with a closed reflexive and transitive relation
$\mathsf{StR}^{\mathsf{eq}}$	Stone spaces with a closed equivalence relation
QPS	Quasi-ordered Priestley spaces
$\mathsf{StR}^{\mathsf{ieq}}$	Stone space with an irreducible closed relation
KHaus	Compact Hausdorff spaces
Gle	Gleason spaces

Categories of spaces

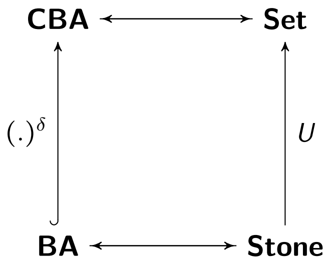
Main results

$$
\begin{array}{|lllll|}
\hline
\mathsf{PCon} & \cong & \mathsf{Sub} & \sim^{d} & \mathsf{StR} \\
\hline
 & & & & \\
\mathsf{MSub} & \sim^{d} & \mathsf{MS}^{\mathsf{st}} & & \\
\mathsf{MA} & \cong & \mathsf{MSub}^{\mathsf{m}} & \sim^{d} & \mathsf{MS} \\
\hline
 & & & & \\
\mathsf{SubK4} & \sim^{d} & \mathsf{StR}^{\mathsf{tr}} & & \\
\mathsf{SubS4} & \sim^{d} & \mathsf{StR}^{\mathsf{qo}} & & \\
\mathsf{SubS5} & \sim^{d} & \mathsf{StR}^{\mathsf{eq}} & & \\
\hline
 & & & & \\
\mathsf{LSub} & \sim^{d} & \mathsf{QPS} & & \\
\hline
\mathsf{Com} & \sim^{d} & \mathsf{StR}^{\mathsf{ieq}} & & \\
\hline
\mathsf{DeV} & \sim^{d} & \mathsf{Gle} & \sim & \mathsf{KHaus} \\
\hline
\end{array}
$$

Main isomorphisms, equivalences, and dual equivalences

Canonical extensions

Canonical extension of a BA provides an algebraic characterization of its double dual.



Canonical extensions

Canonical extension of a BA

The *canonical extension* of a BA A is a complete BA A^δ containing A as a subalgebra, such that

- (*denseness*) Every element of A^δ can be expressed both as a join of meets and as a meet of joins of elements from A;
- (*compactness*) For all $S, T \subseteq A$ with $\bigwedge S \leq \bigvee T$ in A^δ, there exist finite sets $F \subseteq S$ and $G \subseteq T$ such that $\bigwedge F \leq \bigvee G$.

Theorem [Jónsson, Tarksi (1951)] The canonical extension of a BA exists and is unique.

Extension for maps

- An element $x \in A^\delta$ is closed (resp. open) if it is the meet (resp. join) of some subset of A.

- A monotone map $f : A \to B$ can be extended to a map $: A^\delta \to B^\delta$ in two canonical ways. For all $u \in \mathbb{A}^\delta$, define

$$f^\sigma(u) = \bigvee\{\bigwedge\{f(a) : x \leq a \in A\} : u \geq x \in K(A^\delta)\}$$

$$f^\pi(u) = \bigwedge\{\bigvee\{f(a) : y \geq a \in A\} : u \leq y \in O(A^\delta)\}$$

  The map f is smooth if $f^\sigma = f^\pi$.

- Lemma [Gehrke, Jónsson (1994)]
  1. The σ-extension of an operator is a complete operator.
  2. The π-extension of a dual operator is a complete dual operator.

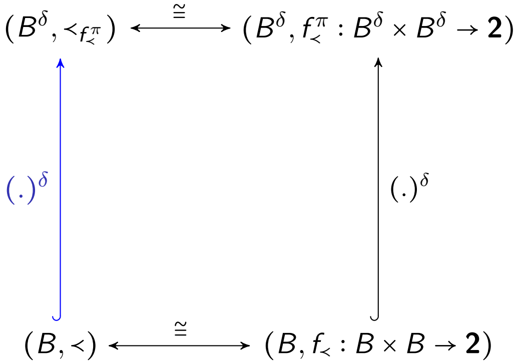
Canonical extension for Sub

$$
\begin{array}{ccc}
(B^\delta, \prec_{f_\prec^\pi}) & \xleftrightarrow{\;\cong\;} & (B^\delta, f_\prec^\pi : B^\delta \times B^\delta \to \mathbf{2}) \\
 & & \uparrow (.)^\delta \\
(B, \prec) & \xleftrightarrow{\;\cong\;} & (B, f_\prec : B \times B \to \mathbf{2})
\end{array}
$$

Canonical extension for Sub

$$
\begin{array}{ccc}
(B^{\delta}, \prec_{f_{\prec}^{\pi}}) & \xleftrightarrow{\;\cong\;} & (B^{\delta}, f_{\prec}^{\pi} : B^{\delta} \times B^{\delta} \to \mathbf{2}) \\
\Big\uparrow (.)^{\delta} & & \Big\uparrow (.)^{\delta} \\
(B, \prec) & \xleftrightarrow{\;\cong\;} & (B, f_{\prec} : B \times B \to \mathbf{2})
\end{array}
$$

Canonical extension for Sub



Theorem

The canonical extension of a Sub *exists and is unique.*

Using $(B^\delta, R_{f^\pi_R}) \cong (\mathcal{P}(PrI(B)), \prec_{[R]})$.

Applications

Recall, the axioms $(S5)$, $(S6)$, $(S7)$, $(S8)$ and $(S9)$ define sub-(quasi)varieties of Boolean algebra with a subordination.

Proposition

The axioms $(S5)$, $(S6)$, $(S7)$, $(S8)$ and $(S9)$ are preserved under taking canonical extension of Sub.

Hence, the existence and uniqueness of the canonical extensions for sub-(quasi)varieties of a Sub follows from the above proposition.

Jónsson-style canonicity [Jónsson (1994), Gehrke, Nagahashi, Venema (2005)]

Canonicity $A \vDash \phi \leq \psi \Rightarrow A^\delta \vDash \varphi \leq \psi$

$$\mathbb{A} \vDash \varphi \leq \psi$$

$$\Updownarrow$$

$$\varphi^{\mathbb{A}} \leq \psi^{\mathbb{A}}$$

$$\Downarrow$$

$$\varphi^{\mathbb{A}^\delta} \leq (\varphi^{\mathbb{A}})^\sigma \leq (\psi^{\mathbb{A}})^\sigma \leq \psi^{\mathbb{A}^\delta}$$

σ-expanding σ-contracting

$$\mathbb{A}^\delta \vDash \varphi \leq \psi$$

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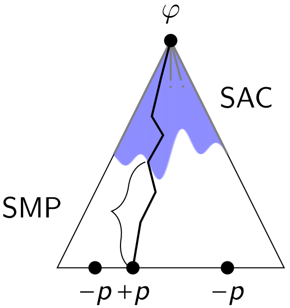
σ-expanding σ-contracting

$$\mathbb{A}^{\delta} \vDash \varphi \leq \psi$$

add. coord.				mult. prod.	
$+$	$\vee$	$\wedge$	g	$+$	$\wedge$
$-$	$\wedge$	$\vee$	f	$-$	$\vee$



Sahlqvist antecedent

Future work

- Characterize the classes of Kripke frames dual to lattice subordinations, de Vries algebras (Correspondence theory).
- Finitary calculus for (modal) compact Hausdorff spaces.
- Generalize this approach to (distributive) lattice setting and compare it to the notion of canonical extension for stably compact spaces in [van Gool 2012].

Thank you!