Bilattice Modal Logic

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Introduction

Our starting point

- Belnap-Dunn logic seems to us indeed a useful four-valued logic, as Belnap (1977) suggested.
- From a computer science perspective, Belnap-Dunn logic would be even more useful if it had modalities.
- How to define (and axiomatize) such a logic?

Introduction

Inspiration

- Bou, Esteva, Godo & Rodríguez (2011) provide a recipe for studying the minimum many-valued modal logic over a finite (integral) residuated lattice, assuming we have axiomatized the non-modal logic of the lattice.
- If we add an implication to the four-element Belnap lattice **Four**, then we can view it as a (non-integral) residuated lattice, and we know its logic.
- We can then adapt the recipe to obtain the minimum **Four**-valued modal logic over the Belnap lattice.
- Initially, this is what we thought we would do, but ...

Kripke semantics

The recipe

A **Four**-valued Kripke model is a structure $\langle W, R, v\rangle$ such that W is a set of 'worlds' and both the accessibility relation R and the valuation v are four-valued, i.e.,

- $R\colon W \times W \to \textbf{Four}$
- $v\colon Fm \times W \to \textbf{Four}$
- $v(\varphi \circ \psi, w) = v(\varphi, w) \circ v(\psi, w)$
  for each non-modal connective $\circ$

The semantics of the modal (necessity) operator is given by

$$v(\Box\varphi, w) := \bigwedge\{R(w, w') \to v(\varphi, w') : w' \in W\}.$$

Kripke semantics

Remarks

- Alternative definitions are available, e.g. (Odintsov & Wansing, 2010):

$$v(\boxdot\varphi, w) := \bigwedge\{R(w, w') \supset v(\varphi, w') : w' \in W\}.$$

- However, $\boxdot$ does not take advantage of four-valuedness of R:

$$v(\boxdot\varphi, w) = \bigwedge\{v(\varphi, w') : R(w, w') \in \{\top, \mathrm{t}\}\}.$$

- Moreover, $\boxdot$ can be recovered as

$$\boxdot\varphi := \Box(\varphi \vee \bot) \oplus (\Box\varphi \wedge \bot)$$

but not the other way around.

Kripke semantics

Remarks

- It seems that other alternatives (e.g., replacing $\bigwedge$ by $\bigotimes$) are not technically feasible.
- Moreover, our $\Box$ has a dual $\Diamond = \neg\Box\neg$ whose semantics is the one proposed by Bou et. al (2011):

$$v(\Diamond\varphi, w) := \bigvee\{R(w, w') * v(\varphi, w') : w' \in W\}.$$

- Technically, these advantages are related to the fact that $\langle \mathbf{Four}, \wedge, \vee, *, \rightarrow, \neg, \top \rangle$ is an (involutive) residuated lattice but, e.g., $\langle \mathbf{Four}, \otimes, \oplus, *, \rightarrow, \neg, \top \rangle$ is not.

Kripke semantics

Modal consequence relations

$M = \langle W, R, v \rangle$ is a **Four**-valued model. As in classical modal logic, we define:

- Satisfaction at $w \in W$: $M, w \models \varphi$ iff $v(\varphi, w) \in \{\mathrm{t}, \top\}$
- Local consequence: $\Gamma \models_l \varphi$ iff $\forall M \ \forall w \in W$

  $M, w \models \Gamma \Rightarrow M, w \models \varphi$
- Global consequence: $\Gamma \models_g \varphi$ iff $\forall M$

  $(\forall w \in W \ M, w \models \Gamma) \Rightarrow (\forall w \in W \ M, w \models \varphi)$

Kripke semantics

Remarks

As in classical modal logic:

- $\models_l \leqslant \models_g$
- $\varnothing \models_l \varphi$ iff $\varnothing \models_g \varphi$
- $\varphi \to \psi \models_g \Box\varphi \to \Box\psi$ but $\varphi \to \psi \not\models_l \Box\varphi \to \Box\psi$

Unlike classical modal logic:

- $\not\models \Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$ (normality fails)
- $\varphi \not\models_g \Box\varphi$ (necessitation fails)

Syntax

Following the recipe of Bou et. al (2011), we conjectured the following Hilbert-style axiomatization.

Axioms

The set of axioms is the least $\Sigma \subseteq Fm$ closed under substitutions such that:

- $\varphi \in \Sigma$ for any theorem φ of the non-modal logic of **Four**
- $\Box \mathrm{t} \leftrightarrow \mathrm{t} \in \Sigma$
- $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q) \in \Sigma$
- $\Box(\bot \to p) \leftrightarrow (\bot \to \Box p) \in \Sigma$
- if $p \in \Sigma$ and $p \supset q \in \Sigma$, then $q \in \Sigma$
- if $p \to q \in \Sigma$, then $\Box p \to \Box q \in \Sigma$.

Syntax

Following the recipe of Bou et. al (2011), we conjectured the following Hilbert-style axiomatization.

Rules

- Modus ponens is a rule of both the local ($\vdash_l$) and the global calculus ($\vdash_g$):

$$\frac{p \quad p \supset q}{q}$$

- Monotonicity is a rule of $\vdash_g$ only:

$$\frac{p \rightarrow q}{\Box p \rightarrow \Box q}$$

Obs.: Monotonicity does not imply necessitation.

Completeness?

Bad news

- Standard modal logic techniques are not applicable because of the lack of normality.
- The recipe of Bou et. al (2011) only works for the local consequence, and proofs involve complicated syntactical lemmas.

Good news

We have algebraic logic and duality.

Algebraic completeness

Algebraic models

- As in classical modal logic, our calculus $\vdash_g$ is algebraizable.
- $\mathbf{Alg}(\vdash_g)$ is the variety of modal bilattices, i.e., algebras $\langle \mathbf{A}, \Box \rangle$ such that $\mathbf{A}$ is a bilattice with implication and

$$\begin{aligned} \Box \mathsf{t} &= \mathsf{t} \\ \Box(x \wedge y) &= \Box x \wedge \Box y \\ \Box(\bot \to x) &= \bot \to \Box x \end{aligned}$$

- Reduced models of $\vdash_g$ are matrices $\langle \mathbf{A}, F_0 \rangle$ where $\mathbf{A}$ is a modal bilattice and F_0 is the least bifilter of $\mathbf{A}$.

Algebraic completeness

Algebraic models

- $\mathbf{Alg}(\vdash_g) = \mathbf{Alg}(\vdash_l) = \mathbf{Alg}^*(\vdash_l)$.
- Reduced models of $\vdash_l$ are matrices $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is a modal bilattice and F is a a bifilter of $\mathbf{A}$.
- Hence, $\vdash_l$ is complete w.r.t. to the above-mentioned class of matrices.

Completeness?

A strategy

- We have defined algebra-based semantics for our calculus, namely modal bilattices with additional structure.
- We have a topological duality theory for bilattices (Jung & R., 2012).
- Modal bilattices are just bilattices with a finite meet-preserving operator that satisfies one additional axiom:
  $\Box(\bot \to x) \; = \; \bot \to \Box x.$
- We can thus extend the above duality to a duality for modal bilattices. This would tell us that the algebra-based and the topological semantics are equivalent.

Completeness?

A strategy

- Algebraic, topological and Kripke semantics are related (Jónsson-Tarski duality).
- Thus, for a completeness proof, we assume $\Gamma \nvdash \varphi$ and we can:
  1. find an algebraic counter-model
  2. turn this model into a topological one
  3. turn the topological model into a Kripke model to conclude that $\Gamma \nvDash \varphi$.
- This works, but the topological counter-model that we obtain is essentially two-valued: a Stone space X together with a two-valued relation $R_{\Box} \subseteq X \times X$.

  A close topological analysis is required to transform this model into a **Four**-valued Kripke model as required by our semantics.

Completeness?

An alternative strategy

- We still use algebraic completeness and Jónsson-Tarski duality, but we also exploit the twist-structure representation.
- For this we need to extend the twist-structure construction to modal bilattices.
- We can thus view the algebraic counter-model mentioned above as a modal twist-structure $\mathbf{B}^{\bowtie}$.
- This gives us a topological counter-model on the dual space of $\mathbf{B}$, which is a Stone space $X(\mathbf{B})$ endowed with two (two-valued) relations $R_{\boxplus}$ and $R_{\boxminus}$.

Completeness?

An alternative strategy

- $R_{\boxplus}$ and $R_{\boxminus}$ can then be combined into one **Four**-valued relation R_4, so that $\langle X(\mathbf{B}), R_4 \rangle$ is a **Four**-valued Kripke frame.
- The twist-structure construction allows us to define a valuation $v \colon Fm \times X(\mathbf{B}) \to \mathbf{Four}$ such that $\langle X(\mathbf{B}), R_4, v \rangle$ is a **Four**-valued Kripke model, as required by our semantics.
- This is the counter-model we were looking for, and the same proof works for both $\vdash_l$ and $\vdash_g$.

Completeness!

Proof (sketch)

- Suppose $\Gamma \nvdash_l \varphi$.
- By algebraic completeness, there is a model $\langle \mathbf{A}, F \rangle$ and a homomorphism $h \colon Fm \to A$ such that $h[\Gamma] \subseteq F$ and $h(\varphi) \notin F$, with $\mathbf{A}$ a modal bilattice and F a filter such that $\top \in F$.
- Since the lattice reduct of $\mathbf{A}$ is distributive, we can extend F to a prime filter $F' \supseteq F$ with $h(\varphi) \notin F'$.
- By the twist-structure representation, we can assume $\mathbf{A} = \mathbf{B}^{\bowtie}$ for some bimodal Boolean algebra $\mathbf{B}$.
- In this case $F' = U \times B$, where U is an ultrafilter of $\mathbf{B}$. Moreover, $\pi_1[h[\Gamma]] \subseteq U$ and $\pi_1(h(\varphi)) \notin U$.

Completeness!

Proof (sketch, continued)

- On the Stone space $X(\mathbf{B})$ we have relations $R_{\boxplus}$ and $R_{\boxminus}$ defined by

$$\langle P, Q\rangle \in R_{\boxplus} \quad \text{iff} \quad \boxplus^{-1}[P] \subseteq Q$$
$$\langle P, Q\rangle \in R_{\boxminus} \quad \text{iff} \quad \boxminus^{-1}[P] \subseteq Q.$$

- We combine them into one **Four**-valued relation R_4 as follows:

$$R_4(P,Q) = \begin{cases} \mathsf{t} & \text{iff } \langle P, Q\rangle \in R_{\boxplus} \text{ and } \langle P, Q\rangle \in R_{\boxminus} \\ \top & \text{iff } \langle P, Q\rangle \in R_{\boxplus} \text{ and } \langle P, Q\rangle \notin R_{\boxminus} \\ \bot & \text{iff } \langle P, Q\rangle \notin R_{\boxplus} \text{ and } \langle P, Q\rangle \in R_{\boxminus} \\ \mathsf{f} & \text{iff } \langle P, Q\rangle \notin R_{\boxplus} \text{ and } \langle P, Q\rangle \notin R_{\boxminus} \end{cases}$$

Completeness!

Proof (sketch, continued)

- Thus, $\langle X(\mathbf{B}), R_4 \rangle$ is a **Four**-valued Kripke frame.
- Define valuations $v_+, v_- \colon \mathit{Var} \to \mathit{Clop}(X(\mathbf{B}))$ as follows:

$$v_+(p) := \{Q \in X(\mathbf{B}) : \pi_1(h(p)) \in Q\}$$
$$v_-(p) := \{Q \in X(\mathbf{B}) : \pi_2(h(p)) \in Q\}.$$

- Combine v_+ and v_- into one **Four**-valued valuation $v_4 \colon \mathit{Var} \times X(\mathbf{B}) \to \mathbf{Four}$ as follows:

$$v_4(p, Q) = \begin{cases} \mathsf{t} & \text{iff } Q \in v_+(p) \text{ and } Q \notin v_-(p) \\ \top & \text{iff } Q \in v_+(p) \text{ and } Q \in v_-(p) \\ \bot & \text{iff } Q \notin v_+(p) \text{ and } Q \notin v_-(p) \\ \mathsf{f} & \text{iff } Q \notin v_+(p) \text{ and } Q \in v_-(p) \end{cases}$$

Completeness!

Proof (sketch, continued)

- Thanks to the twist-structure construction, v_4 can be homomorphically extended to arbitrary formulas. In particular,

$$\Box \langle v_+(\psi), v_-(\psi)\rangle = \langle \boxplus\, v_+(\psi) \cap \boxminus\, (v_-(\psi))^c, (\boxplus\, (v_-(\psi))^c)^c \rangle$$

  where, for $S \in Clop(X(\mathbf{B}))$,

$$\boxplus\, S := \{Q \in X(\mathbf{B}) : R_{\boxplus}[Q] \subseteq S\}$$
$$\boxminus\, S := \{Q \in X(\mathbf{B}) : R_{\boxminus}[Q] \subseteq S\}$$

- So, $M = \langle X(\mathbf{B}), R_4, v_4\rangle$ is a **Four**-valued Kripke model.

Completeness!

Proof (sketch, continued)

- Recall that $\pi_1[h[\Gamma]] \subseteq U$ and $\pi_1(h(\varphi)) \notin U$. This means that $U \in v_+(\gamma)$ for all $\gamma \in \Gamma$, but $U \notin v_+(\varphi)$.
- That is, $v_4[\Gamma] \subseteq \{\top, \mathrm{t}\}$ but $v_4(\varphi) \in \{\bot, \mathrm{f}\}$.
- That is, $M, U \models \Gamma$ but $M, U \not\models \varphi$.
- Hence, $\Gamma \not\models_l \varphi$.

The completeness proof for $\vdash_g$ is essentially the same, replacing $\langle \mathbf{A}, F \rangle$ with $\langle \mathbf{A}, F_0 \rangle$.

. . . and beyond

Concluding remarks

- We have axiomatized the least modal logic over the Belnap lattice: what about extensions? (e.g., more restricted classes of frames).
- In fact, our result still holds if we replace **Four** by any complete bilattice with implication.

. . . and beyond

Concluding remarks

- The splitting of $\Box$ into the pair $\langle \boxplus, \boxminus \rangle$ provides further insights. For instance, the extension obtained by adding the normality axiom $\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$ is exactly the logic of idempotent frames, i.e., those where $R(w, w') \neq \bot$. The algebraic models of this logic are twist-structures $\mathbf{B}^{\bowtie}$ s.t. $\mathbf{B} \models \boxplus x \leqslant \boxminus x$.
- Similar axioms characterize consistent frames $(R(w, w') \neq \top)$ and crisp frames $(R(w, w') \in \{\mathrm{f}, \mathrm{t}\})$.
- Applying the same kind of reasoning, we might be able to say something interesting about Sahlqvist formulas.

References

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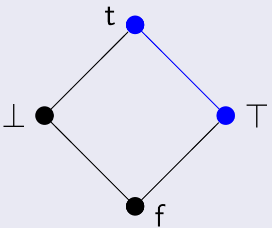
Four as a residuated lattice

$\langle \mathbf{Four}, \wedge, \vee, *, \rightarrow, \neg \rangle$ is a (commutative) residuated lattice with underlying monoid $\langle \mathbf{Four}, *, \top \rangle$, where $x * y = \neg(x \rightarrow \neg y)$.



$*$	f	$\bot$	$\top$	t
f	f	f	f	f
$\bot$	f	f	$\bot$	$\bot$
$\top$	f	$\bot$	$\top$	t
t	f	$\bot$	t	t

$\rightarrow$	f	$\bot$	$\top$	t
f	t	t	t	t
$\bot$	$\bot$	t	$\bot$	t
$\top$	f	$\bot$	$\top$	t
t	f	$\bot$	f	t

Back

The logic of **Four** and its semantics

- The logic of the matrix $\langle \mathbf{Four}, \{\top, \mathsf{t}\}\rangle$ in the language $\langle \wedge, \vee, *, \to, \neg, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$ is algebraizable.
- Its equivalent algebraic semantics is the variety generated by **Four** in this language (bounded classical implicative bilattices).
- Bilattice connectives are definable:

$$a \otimes b = (a \wedge \bot) \vee (b \wedge \bot) \vee (a \wedge b) \qquad \leqslant_k \text{ lattice}$$
$$a \oplus b = (a \wedge \top) \vee (b \wedge \top) \vee (a \wedge b)$$
$$a \supset b = (a \to (a \to b)) \vee b \qquad \text{`weak' implication}$$

- Every algebra in this variety is representable as a twist-structure over a Boolean algebra.

The logic of **Four** and its semantics

Twist-structures

Let $\mathbf{B} = \langle B, \wedge, \vee, \to, 0, 1 \rangle$ be Boolean algebra. The twist-structure over $\mathbf{B}$ is the algebra $\mathbf{B}^{\bowtie} = \langle B \times B, \wedge, \vee, *, \to, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ where

$$\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle := \langle a_1 \wedge b_1,\ a_2 \vee b_2 \rangle$$
$$\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle := \langle a_1 \vee b_1,\ a_2 \wedge b_2 \rangle$$
$$\langle a_1, a_2 \rangle * \langle b_1, b_2 \rangle := \langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2) \rangle$$
$$\langle a_1, a_2 \rangle \to \langle b_1, b_2 \rangle := \langle (a_1 \to b_1) \wedge (b_2 \to a_2), a_1 \wedge b_2 \rangle$$
$$\neg \langle a_1, a_2 \rangle := \langle a_2,\ a_1 \rangle$$
$$\mathsf{f} := \langle 0, 1 \rangle \qquad \mathsf{t} := \langle 1, 0 \rangle \qquad \bot := \langle 0, 0 \rangle \qquad \top := \langle 1, 1 \rangle$$

Back

Completeness?

Modal twist-structures

A bimodal Boolean algebra $\mathbf{B} = \langle B, \wedge, \vee, \sim, \boxplus, \boxminus, 0, 1 \rangle$ is a Boolean algebra with two operators that preserve finite meets:

$$\boxplus 1 = \boxminus 1 = 1$$
$$\boxplus (x \wedge y) = \boxplus x \wedge \boxplus y$$
$$\boxminus (x \wedge y) = \boxminus x \wedge \boxminus y.$$

The modal twist-structure over $\mathbf{B}$ is the algebra $\mathbf{B}^{\bowtie}$ defined as in the non-modal case, with

$$\Box \langle a_1, a_2 \rangle := \langle \boxplus a_1 \wedge \boxminus (\sim a_2), \sim \boxplus \sim a_2 \rangle.$$

Obs.: our construction generalizes Odintsov & Wansing (2010), who had $\Box \langle a_1, a_2 \rangle := \langle \boxplus a_1, \sim \boxplus \sim a_2 \rangle$.

Completeness?

Modal twist-structures

Theorem: Every modal bilattice is representable as a twist-structure over a bimodal Boolean algebra.

Obs.: this representation uses crucially the constants, for we need to construct terms such as $\Box(\neg(a \supset \mathsf{f}) \vee \top)$.

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