Epistemic updates on algebras: Bilattice Epistemic Action and Knowledge

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Outline

Main goal

- Technical Aim: Obtaining a semantics and a complete axiomatization for a Bilattice-based Logic of Epistemic Action and Knowledge (BEAK)

- **algebraic** and **duality-theoretic** methods.

Histoty and Motivation

- ”Dynamic phenomena” are best analyzed using an appropriate non-classical logic, in many contexts:
  - which are ***inconsistency-tolerant***, ***paracomplete***: multiple sources of information, inconsistent/contradictory evidence
  - where ***truth is procedural***;
- Recent works:

Histoty and Motivation

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  - which are ***inconsistency-tolerant***, ***paracomplete***: multiple sources of information, inconsistent/contradictory evidence
  - where ***truth is procedural***;
- Recent works:
  - Recent work of Alessandra Palmigiano and collaborators provides methods which allow one to: Define a logic of Epistemic Actions and Knowledge on a propositional basis that is weaker than classical logic, for example an intuitionistic basis.
  - Provide a way to apply these methods to a variety of contexts where classical reasoning is not suitable.

Dynamic Epistemic Logic (DEL)

- Family of logics for multiagent interaction;
- describing and reasoning about **information flow**, how it affects epistemic setup of agents.
- Merging of *two* issues:
  - Epistemic: what do agents know, or believe (partial knowledge, incorrect beliefs...)
  - Dynamic: knowledge acquisition, belief updates...

  giving rise to **epistemic actions**.
- Examples: Public announcements, private announcements, ...

Epistemic Action and Knowledge(EAK)

- The logic EAK was introduced by A. Baltag, L.S. Moss and S. Solecki (1999) to deal with “Public Announcements, Common Knowledge and Private Suspicions”.
- The language of EAK is that of modal logic (S5) expanded with dynamic operators $\langle \alpha \rangle$ and $[\alpha]$, where α is an action structure.
- Intended meaning of $\langle \alpha \rangle \phi$: the action α can be executed, and after execution ϕ is the case.
- Dually, $[\alpha]\phi$ means: if the action α can be executed, then after execution ϕ holds.

Language

Language of (classical, single-agent) EAK

$$\phi ::= p \in \mathsf{Var} \mid \neg\phi \mid \phi \vee \phi \mid \Diamond\phi \mid \Box\phi \mid \langle\alpha\rangle\phi \mid [\alpha]\phi,$$

Where α is an action structure:

$$\alpha = (K, k, R_\alpha, Pre_\alpha : K \to Fm).$$

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Kripke semantics

For $M = (W, R, v)$, define

$$M, w \Vdash \langle\alpha\rangle\phi \quad \text{iff} \quad M, w \Vdash Pre_\alpha(k) \text{ and } M^\alpha, (w, k) \Vdash \phi$$

$$M, w \Vdash [\alpha]\phi \quad \text{iff} \quad \text{if } M, w \Vdash Pre_\alpha(k), \text{ then } M^\alpha, (w, k) \Vdash \phi$$

where M^α is the updated model, after execution of α.

Updated model

Intermediate model (pseudo coproduct)

Given $\alpha := (K, k, R_\alpha, Pre_\alpha : K \to Fm)$ and $M = (W, R, v)$, let

$$\coprod_\alpha M := (\coprod_K W, R \times R_\alpha, \coprod_K v)$$

- $\coprod_K W \cong W \times K$
- $(w, j)(R \times R_\alpha)(u, i)$ iff wRu and $jR_\alpha i$
- $(\coprod_K v)(p) := \coprod_K v(p)$.

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The second step, M^α

M^α is the submodel of $\coprod_\alpha M$ with domain

$$W^\alpha := \{(w, j) \mid M, w \Vdash Pre_\alpha(j)\}.$$

Epistemic updates

- Epistemic change is represented in DEL as a transformation from a (relational, algebraic) model representing the current situation to a new model that represents the situation after some epistemic action has occurred.

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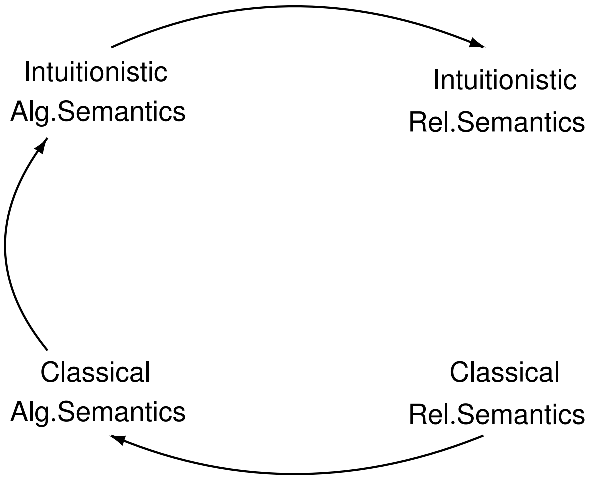
- Epistemic change is represented in DEL as a transformation from a (relational, algebraic) model representing the current situation to a new model that represents the situation after some epistemic action has occurred.
- The update on the epistemic state of agents caused by an action is known as epistemic update.
- Epistemic updates are formalized
  - on Kripke-style models via (pseudo-) co-products and sub-models,
  - on algebras via (pseudo-) products and quotients.

Methodology: dual characterizations



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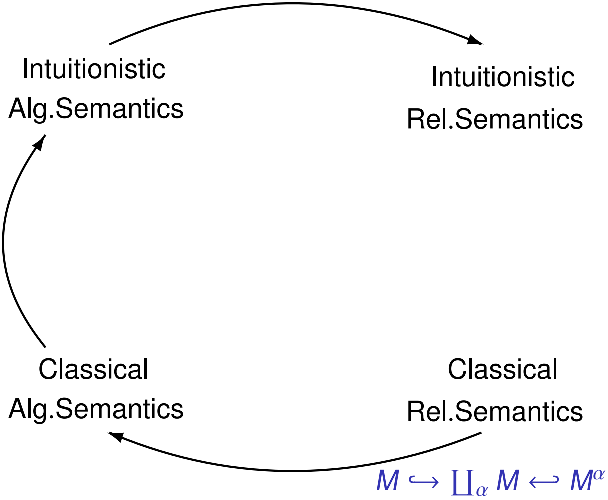

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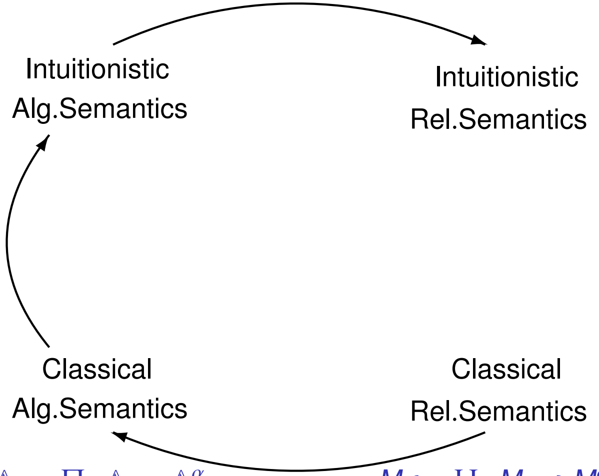

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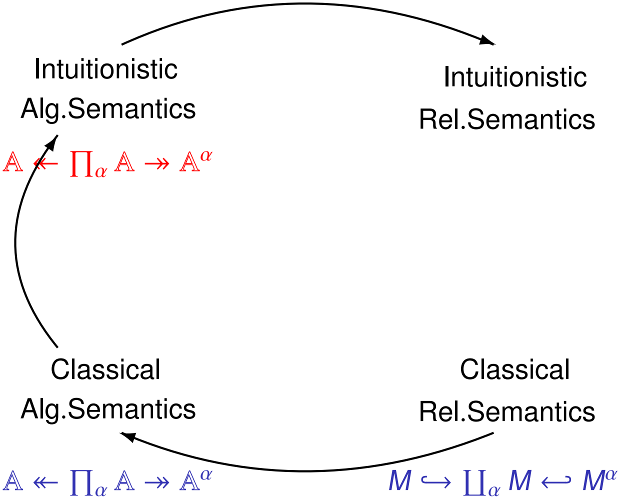

Methodology: dual characterizations

Intuitionistic
Alg.Semantics

$\mathbb{A} \twoheadleftarrow \prod_\alpha \mathbb{A} \twoheadrightarrow \mathbb{A}^\alpha$

Intuitionistic
Rel.Semantics

$M \hookrightarrow \coprod_\alpha M \hookleftarrow M^\alpha$

Classical
Alg.Semantics

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Classical
Rel.Semantics

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Methodology: dual characterizations

Bilattice

Alg.Semantics

$\mathbb{A} \twoheadleftarrow \prod_\alpha \mathbb{A} \twoheadrightarrow \mathbb{A}^\alpha$

Primary definition

Rel.Semantics

$M \hookrightarrow \coprod_\alpha M \hookleftarrow M^\alpha$

Methodology: dual characterizations

Bilattice
Alg.Semantics

$\mathbb{A} \twoheadleftarrow \prod_\alpha \mathbb{A} \twoheadrightarrow \mathbb{A}^\alpha$

Primary definition

Four-valued
Rel.Semantics

$M \hookrightarrow \coprod_\alpha M \hookleftarrow M^\alpha$

Obtained by dual characterization

Modal bilattices

Bimodal Boolean algebra: $(A, \wedge, \vee, \sim, \Diamond_+, \Diamond_-, 0, 1)$ s.t.:

- $(A, \wedge, \vee, \sim, 0, 1)$ is a Boolean algebra;
- $\Diamond_+$ and $\Diamond_-$ preserve finite joins (possibly empty).

Modal twist structures: $\mathbb{A}^{\bowtie} = (A \times A, \wedge, \vee, \supset, \neg, \mathbf{t}, \mathbf{f}, \top, \bot)$ s.t. A is a bimodal Boolean algebra and

$$
\begin{array}{rcl}
(a_1, a_2) \wedge (b_1, b_2) & = & (a_1 \wedge b_1, a_2 \vee b_2) \\
(a_1, a_2) \vee (b_1, b_2) & = & (a_1 \vee b_1, a_2 \wedge b_2) \\
(a_1, a_2) \supset (b_1, b_2) & = & (\sim a_1 \vee b_1, a_1 \wedge b_2) \\
\Diamond(a, b) & = & (\Diamond_+ a, \Box_+ b \wedge \sim\Diamond_- a) \\
\neg(a, b) & = & (b, a) \\
\mathrm{f} & = & (0, 1) \\
\mathrm{t} & = & (1, 0) \\
\top & = & (1, 1) \\
\bot & = & (0, 0)
\end{array}
$$

Intermediate structures

Let $\mathsf{A} \equiv \mathbb{A}^{\bowtie}$ be a **modal bilattice**; $\alpha = (K, k, R_\alpha, Pre_\alpha : K \to \mathsf{A})$ four-valued action structure over A; It means:

$$R_\alpha : K \to \mathsf{FOUR}$$

is a four-valued relation.

$$\prod_\alpha \mathsf{A} := (\mathsf{A}^K, \Diamond^{\prod_\alpha \mathsf{A}}, \Box^{\prod_\alpha \mathsf{A}})$$

For each $f : K \to \mathsf{A}$ and each $j \in K$,

$$(\Diamond^{\prod_\alpha \mathsf{A}} f)(j) = \bigvee \{\Diamond^{\mathsf{A}} f(i) \mid R_\alpha(j, i) \in \{\mathbf{t}, \top\}\}$$

$$(\Box^{\prod_\alpha \mathsf{A}} f)(j) = \bigwedge \{\Box^{\mathsf{A}} f(i) \mid R_\alpha(j, i) \in \{\mathbf{t}, \top\}\}.$$

The modal bilattice $\mathbb{A}^\alpha$

Problem

Defining

$$b \equiv_{Pre_\alpha} c \quad \text{iff} \quad b \wedge Pre_\alpha = c \wedge Pre_\alpha$$

NOT a congruence.

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Fact

$\phi \dashv\vdash \psi \quad \text{iff} \quad v(((\phi \supset \mathbf{f}) \supset \mathbf{f})) = v(((\psi \supset \mathbf{f}) \supset \mathbf{f}))$

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Solution

Define

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$\prod_\alpha A/{\equiv_{Pre_\alpha}}$: Modal bilattice; $[b] \in \prod_\alpha A/{\equiv_{Pre_\alpha}}$,

$$\Diamond^{\alpha}[b] := [\Diamond^{\prod_\alpha \mathbb{A}}({\sim\sim}\, Pre_\alpha \wedge b)]$$

$$\Box^{\alpha}[b] := [\Box^{\prod_\alpha \mathbb{A}}(Pre_\alpha \supset b)].$$

Axiomatization of BEAK

Our calculus for BEAK is defined over the language
$\langle \vee, \supset \neg, \Diamond, \langle \alpha \rangle, \mathsf{f}, \mathsf{t}, \top, \bot \rangle$
BEAK is axiomatically defined by axioms and rules of the calculus for bilattice modal logic of [3] + the following axioms and rules:

Axiomatization of BEAK

Constant axioms	$\langle\alpha\rangle \mathsf{f} \leftrightarrow \mathsf{f}$	$\langle\alpha\rangle \mathsf{t} \leftrightarrow {\sim\sim}\, Pre(\alpha)$
	$\langle\alpha\rangle\top \leftrightarrow (Pre(\alpha) \wedge \top)$	$\langle\alpha\rangle\bot \leftrightarrow \neg(Pre(\alpha) \supset \bot)$
$\vee$ **axiom**	$\langle\alpha\rangle(\phi \vee \psi) \leftrightarrow (\langle\alpha\rangle\phi \vee \langle\alpha\rangle\psi)$	
$\supset$ **axiom**	$\langle\alpha\rangle(\phi \supset \psi) \leftrightarrow ({\sim\sim}\, Pre(\alpha) \wedge (\langle\alpha\rangle\phi \supset \langle\alpha\rangle\psi))$	
$\neg$ **axiom**	$\langle\alpha\rangle\neg\phi \leftrightarrow ({\sim\sim}\, Pre(\alpha) \wedge \neg\langle\alpha\rangle\phi))$	
$\Diamond$ **axiom**	$\langle\alpha\rangle\Diamond\phi \leftrightarrow ({\sim\sim}\, Pre(\alpha) \wedge \bigvee\{\Diamond\langle\alpha_j\rangle\phi \mid \mathsf{R}_\alpha(k,j) \in \{\mathsf{t}, \top\}\})$	
Fact preservation	$\langle\alpha\rangle p \leftrightarrow ({\sim\sim}\, Pre(\alpha) \wedge p)$	

The rule:

$$\text{from } \emptyset \vdash \phi \to \psi \quad \text{infer} \quad \emptyset \vdash \langle\alpha\rangle\phi \to \langle\alpha\rangle\psi.$$

Algebraic semantics

For every algebraic model $M = (\mathbb{A}, v)$, where $\mathbb{A}$ is a modal bilattice and $v\colon \mathrm{Var} \to \mathbb{A}$, the extension map $[\![\cdot]\!]_M : Fm \to \mathbb{A}$ is defined as:

$$
\begin{array}{rcll}
[\![p]\!]_M & = & v(p) & \\
[\![\phi \spadesuit \psi]\!]_M & = & [\![\phi]\!]_M \spadesuit^{\mathbb{A}} [\![\psi]\!]_M & \text{for } \spadesuit \in \{\wedge, \vee, \to, \ldots\} \\
[\![\heartsuit \phi]\!]_M & = & \heartsuit^{\mathbb{A}} [\![\phi]\!]_M & \text{for } \heartsuit \in \{\Diamond, \Box, \neg, \ldots\} \\
[\![\langle\alpha\rangle\phi]\!]_M & = & [\![{\sim\sim}\, Pre(\alpha_k)]\!]_M \wedge^{\mathbb{A}} \pi_k \circ \iota([\![\phi]\!]_{M^\alpha}) & \\
[\![[\alpha]\phi]\!]_M & = & [\![Pre(\alpha_k)]\!]_M \supset^{\mathbb{A}} \pi_k \circ \iota([\![\phi]\!]_{M^\alpha}). &
\end{array}
$$

$\iota\colon [b] \longmapsto b \wedge {\sim\sim}\, Pre_\alpha$ is an injective map that embeds $\prod_\alpha \mathbb{A}/{\equiv_{Pre_\alpha}}$ into $\prod_\alpha \mathbb{A}$.

Results

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- Soundness of the axioms is checked w.r.t. to algebraic models.
- The proof of completeness is analogous to that of classical and intuitionistic EAK, and follows from reducibility of BEAK to bilattice modal logic and the $\langle\alpha\rangle$-monotonicity axiom.
- Soundness and completeness w.r.t. relational models follow by duality.

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Thanks for your attention...