A bilattice for contextual reasoning

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ABSTRACT. Bilattices, introduced by Matthew Ginsberg as a uniform framework for inference in Artificial Intelligence, are algebraic structures that proved useful in many fields, but have never been applied to contextual reasoning so far. My aim is to sketch one such possible application. The basic idea is to treat contexts as "truth values" that form a bilattice of the kind introduced by Ginsberg for the "justifications" of a *Truth Maintenance System*. My proposal is that the same machinery may be applied to sets of formulas representing not premises in the usual logical sense but contexts.

1 Introduction

Bilattices are algebraic structures introduced by Ginsberg [4] as a uniform framework for inference in Artificial Intelligence. In the last two decades the bilattice formalism proved useful in many fields: however it has never been applied to contextual reasoning so far. My aim here is to sketch one such possible application.

The basic idea is to treat contexts as "truth values" that form a bilattice, that is a lattice equipped with two partial orders. We shall employ a bilattice construction introduced in [4] for the "justifications" of a *Truth Maintenance System*, i.e. sets of premises used for derivations, but we will apply it to sets of formulas representing not premises in the usual logical sense but cognitive contexts.

The usual logical connectives will then be defined as lattice operators on the set of truth values, for instance conjunction and disjunction correspond respectively to the *meet* and *join* with respect to the so called "truth ordering". Non classical connectives may also be defined, such as those corresponding to the *meet* and *join* w.r.t. the second partial order, usually called the "knowledge ordering".

The next step would be to construct a suitable inference mechanism for contextual bilattices. This has been done for the general case (see for instance [4] and [1]), but it remains to show that such mechanisms may be successfully applied to contextual reasoning. In section 4 we shall see an example of a possible application.

2 Bilattices

Given a set B, a bilattice may be defined as a structure $\langle B, \leq_t, \leq_k \rangle$ where $\langle B, \leq_t \rangle$ and $\langle B, \leq_k \rangle$ are both complete lattices.

The elements of B are intended to represent truth values ordered according to the degree of truth and the degree of knowledge (or information): the relation $\leq_t$ corresponds to the truth ordering and $\leq_k$ to the knowledge ordering. Intuitively, given two sentences p and q, $v(p) \leq_t v(q)$ means that the agent has stronger evidence for the truth of q than for the truth of p and weaker evidence for the falsity of q than for that of p, while $v(p) \leq_k v(q)$ means that the agent has stronger evidence for *both* the truth and falsity of q than for the truth and falsity of p (thus allowing for inconsistency).

The *meet* and *join* operations on the two lattices correspond to propositional connectives in the bilattice-based logics. Conjunction and disjunction are defined respectively as the greatest lower bound and least upper bound with respect to the truth ordering. Given two sentences p, q and a valuation v, we have:

$$v(p \wedge q) = \mathrm{glb}_t \{v(p), v(q)\}$$

$$v(p \vee q) = \mathrm{lub}_t \{v(p), v(q)\}$$

The corresponding connectives relative to the knowledge ordering have been called *consensus* ($\otimes$) and *gullability* ($\oplus$) operator (see [3]). They are defined as follows:

$$v(p \otimes q) = \mathrm{glb}_k \{v(p), v(q)\}$$

$$v(p \oplus q) = \mathrm{lub}_k \{v(p), v(q)\}$$

Intuitively, we may interpret $v(p) \otimes v(q)$ as the most information that $v(p)$ and $v(q)$ agree on, while the gullability operator $\oplus$ simply accepts any information from both $v(p)$ and $v(q)$.

We have a bilattice negation if there is a function $\neg : B \rightarrow B$ such that:

1. if $v(p) \leq_t v(q)$ then $\neg v(q) \leq_t \neg v(p)$,
2. if $v(p) \leq_k v(q)$ then $\neg v(p) \leq_k \neg v(q)$,
3. $v(p) = \neg\neg v(p)$.

In other words, we require that the negation be an involutive operator that reverses the truth ordering while leaving the knowledge ordering unchanged: this corresponds to the intuition that the amount of information one has concerning some sentence p should not be altered when considering its negation $\neg p$. The existence of such a negation operator is a minimal requirement for logical bilattices (indeed some authors consider it part of the basic definition of bilattice); for the kind of bilattices we will construct there is a straightforward way to define it (see below, section 3).

The smallest non-trivial bilattice is the one corresponding to Belnap's logic (see [2]), which has four elements, that is exactly the least and greatest elements with respect to the two orderings. This bilattice can be constructed from the cartesian product of the classical two-point truth set with itself: $\{0,1\} \times \{0,1\} = \{(0,1),(1,0),(0,0),(1,1)\}$. We may interpret the first element of each ordered pair as representing the evidence for the truth of a sentence p, while the second element represents the evidence for the falsity of p. In this way we can understand Belnap's values in terms of the classical ones: $(1,0)$ corresponds to "at least true", $(0,1)$ to "at least false", $(0,0)$ to "unknown" (i. e. not known to be either true or false) and $(1,1)$ to "contradictory" (i. e. known to be both true and false).

Several more complex bilattices have been introduced in the literature for a variety of applications (for instance to deal with default reasoning, with modal operators etc.), many of them built like Belnap's using two copies of some lattice, the first for the positive evidence and the second for the negative. In the next section we shall employ this procedure to construct a bilattice for contextual reasoning.

3 Contexts as truth values

In cognitive processes, the notion of context may be defined as a part of the epistemic state of an agent, i.e. as a set of implicit assumptions. These assumptions enable us to assign a reference to indexical expressions such as "this", "here", "now" etc., and so to determine the truth value of the sentences involving them. The simplest way to formalize this is to identify contexts with subsets of the knowledge base, i.e. sets of formulas.

Let F be the set of all formulas in the knowledge base and let $C_1, \ldots, C_n \subseteq$

F be sets of formulas intended to represent contexts. To each sentence p we may associate the set $C^+ = \{C_1, \ldots, C_n\}$ of all contexts in which p holds. We assume each C_i to be a set of sentences, possibly cointaining contextual "axioms" such as "Speaker = ...", " Time = ... " etc., that logically imply p. We shall denote this writing $v(p) = [C^+]$. This is the basic idea that provides a link with the multi-valued setting of bilattices, that is the idea to treat contexts as truth values.

If we want to handle inconsistent beliefs, we may also consider the set C^- of all contexts in which $\neg p$ holds, without requiring that C^+ and C^- be disjoint, so that we may have some context in which both p and $\neg p$ hold. Therefore, instead of writing only $v(p) = [C^+]$, meaning that the value of p is given by the contexts in which p holds, we shall write $v(p) = [C^+, C^-]$, meaning that the value of p is given by the contexts in which p holds together with the contexts in which $\neg p$ holds.

Now we proceed to define an order relation on these "truth values". We may order contexts in a natural way by set inclusion. For instance, $C_2 \subseteq C_1$ intuitively means that C_2 is more general than C_1, since C_2 requires fewer assumptions than C_1. (The most general context is thus the empty context, corresponding to sentences that are completely context-independent, while the least general context is simply the set of all formulas.) This intuition may be extended to sets of contexts as follows.

Given two sets of contexts $C = \{C_1, \ldots, C_m\}$ and $D = \{D_1, \ldots, D_n\}$, we set $C \leq D$ iff for all $C_i \in C$ there is some $D_j \in D$ such that $D_j \subseteq C_i$. This means that, for every context in C, there is some context in D which is more general: so if we know that p holds in D and q holds in C, we can conclude that p is less context-dependent than q.

So far the relation we have defined is in fact only a preorder, since we may have $C \leq D$ and $D \leq C$ but $C \neq D$. This happens for instance if $C = \{\{p\}\}$ and $D = \{\{p\}, \{p, q\}\}$. However, to obtain an order we just need to consider the equivalence classes under this preorder; or, equivalently, we might require that each set of contexts be minimal, in the sense that for every $C = \{C_1, \ldots, C_m\}$ there should be no $C_i, C_j \in C$ such that $C_i \subseteq C_j$. In other words, what we are assuming is that all sets are free of redundant contexts (such as C_j in our example).

Of course this is not the only possible way to define an order relation on (sets of) contexts. For instance one might consider the logical (instead of just the set inclusion) relationship between the propositions representing contexts.

For suppose we have two contexts C and D such that $C \nsubseteq D$ and $D \nsubseteq C$ but $D \subseteq \mathrm{th}(C)$, that is $C \vDash p$ for each sentence $p \in D$. Since D is contained in the logical consequences of C, from a deductive point of view we might expect to have $C \leq D$. According to this intuition, we should replace the previous definition with the more general one: $C \leq D$ iff $D \subseteq \mathrm{th}\,(C)$.

However, we shall not employ this definition here because it would not allow to construct a lattice of contexts in an effective way. In fact, in order to determine if $C \leq D$ we would have to check if $D \subseteq \mathrm{th}\,(C)$, which is notoriously a complex task from a computational point of view.

Instead, we prefer to adopt the simpler set inclusion definition, delaying the hard part of the job to a later stage of the inference process. So if p holds in C and q holds in D, with $D \subseteq \mathrm{th}\,(C)$, this relation will not be reflected in the values assigned to p and q by our initial valuation until we have applied some suitable closure operator (such as the one introduced in [4]).

Adopting the set inclusion order relation we are now able to define a lattice of sets of contexts. Let F be the set of all formulas in the knowledge base and $P\,(F)$ its power set, that is the set of all possible contexts. If we denote the set of all sets of contexts by $L = P\,(P\,(F))$, then the structure $\langle L, \leq \rangle$ is the lattice of sets of contexts.

As we have said, in order to consider inconsistent beliefs we employ two copies of L, one for the contexts in which a sentence p holds and the other for those in which $\neg p$ holds. In this way we obtain a structure that may be called a "contextual bilattice" $\langle L \times L, \leq_t, \leq_k \rangle$, the underlying set being formed by the ordered pairs $[C^+, C^-]$ of elements of the lattice of sets of contexts. In this structure the truth and knowledge order relations may be defined as follows. For any two elements $[C^+, C^-]$, $[D^+, D^-] \in L \times L$:

$$\left[C^+, C^-\right] \leq_t \left[D^+, D^-\right] \quad \text{iff} \quad C^+ \leq D^+ \quad \text{and} \quad C^- \geq D^-$$

$$\left[C^+, C^-\right] \leq_k \left[D^+, D^-\right] \quad \text{iff} \quad C^+ \leq D^+ \quad \text{and} \quad C^- \leq D^-$$

It can be verified that our definition reflects the previous considerations on the two orderings. The logical connectives on the contextual bilattice may then be easily defined as lattice operators. Let $\langle L, \leq, \cdot, + \rangle$ be the lattice of sets of contexts, with $\cdot$ and $+$ denoting respectively the meet and join operations. It is easy to see that the propositional connectives on $\langle L \times L, \leq_t, \leq_k \rangle$ result as follows. For any two elements $[C^+, C^-]\,, [D^+, D^-] \in L \times L$:

$$[C^+, C^-] \wedge [D^+, D^-] = [(C^+ \cdot D^+), (C^- + D^-)]$$

$$[C^+, C^-] \vee [D^+, D^-] = [(C^+ + D^+), (C^- \cdot D^-)]$$

$$[C^+, C^-] \otimes [D^+, D^-] = [(C^+ \cdot D^+), (C^- \cdot D^-)]$$

$$[C^+, C^-] \oplus [D^+, D^-] = [(C^+ + D^+), (C^- + D^-)]$$

Negation is simply defined as a function swapping the "truth" and "falsity degree", that is we have $\neg [C^+, C^-] = [C^-, C^+]$ for all $[C^+, C^-] \in L \times L$. It can be easily verified that this operation meets all the requirements stated in the previous section.

We may also note that our contextual bilattice has nice structural properties, for instance that it is *interlaced*, that is for any $x, y, z \in L \times L$ we have:

(i) $x \leq_t y$ implies $x \otimes z \leq_t y \otimes z$ and $x \oplus z \leq_t y \oplus z$

(ii) $x \leq_k y$ implies $x \wedge z \leq_t y \wedge z$ and $x \vee z \leq_t y \vee z$.

This means that each operation associated with one of the lattice orderings is monotonic with respect to the other ordering. As noted by Fitting [3], this is another kind of connection between the two orderings besides that provided by the negation operator.

4 Reasoning with contexts

The next step would be to construct a suitable inference mechanism for "contextual bilattices". As we have anticipated, there exist general inference procedures for bilattices (see [4] and [1]), but it remains to show that they may be successfully applied to contextual reasoning. As a preliminary result, we will see an application to our setting of the closure operator defined by Ginsberg [4].

As we have said in the previous section, we begin with an arbitrary valuation and then apply to it a closure operator to derive the logical consequences of the set of initial beliefs we are interested in.

First of all we need to define what it means for a valuation to be closed in the bilattice framework. Let $v : F \to B$ be a valuation from the set of all formulas into a bilattice B. We say that v is *closed* iff, for any formulas $p, q \in F$, the following conditions hold:

(i) if $p \vDash q$ then $v(p) \leq_t v(q)$

(ii) $v(p) \wedge v(q) \leq_k v(p \wedge q)$

(iii) $v(\neg p) = \neg v(p)$.

Item (i) says that if p implies (in classical logic) q, then the value of q should be at least as true as that of p.

Item (ii) says that one should know at least as much about a conjunction as about each one of the conjuncts, and in some cases one may know more. For instance, consider the four-point bilattice corresponding to Belnap's logic, and recall that $(0, 0)$ and $(0, 1)$ may be intuitively interpreted as (respectively) "unknown" and "at least false". We may have $v(p) = v(\neg p) = (0{,}0)$ for some sentence p, but still we would expect to have $v(p \wedge \neg p) = (0, 1)$ instead of $v(p \wedge \neg p) = v(p) \wedge v(\neg p) = (0, 0)$. So we see that in some cases it is reasonable to have $v(p \wedge \neg p) >_k v(p) \wedge v(q)$.

Item (iii) just states the intuitive requirement that the negation should map the truth value assigned to p to that assigned to $\neg p$.

Given two valuations v and w, we will say that w is an *extension* of v in case w is "more informed" than v, that is in case for each formula $p \in F$ we have $v(p) \leq_k w(p)$. We shall write $v \leq_k w$ to indicate that w is an extension of v.

Now we define $\mathrm{cl}(v)$, the *closure* of a valuation $v : F \to B$, as follows:

$$\mathrm{cl}(v) = \bigotimes \{w : v \leq_k w \text{ and } w \text{ is closed}\}$$

where the symbol $\bigotimes$ denotes the infinitary version of the consensus operator, i.e. the meet with respect to the knowledge lattice. As Ginsberg [4] has shown, it is possible to construct an effective procedure that computes $\mathrm{cl}(v)$ for a given valuation v.

Now we shall see how this closure operator works with the contextual bilattice. Consider a set A of assumptions that may represent the agent's initial beliefs. We define an initial valuation $v : F \to L \times L$ as follows. For each sentence $p \in F$:

$$v(p) = \begin{cases} [(\{p\}), (F)] & \text{if } p \in A \\ [(F), (F)] & \text{otherwise.} \end{cases}$$

In this way we are labeling each sentence in A as self-justified, i.e associated with a context consisting only of itself (recall that F, the set of all formulas, is the least element in the lattice of contexts L, since it amounts to having no information at all about in which contexts a sentence holds). Then we apply the closure operator to v. We now have the following result:

THEOREM 1.1. *Let v be a valuation defined as above. Then for each $\{p_1, \ldots, p_n, q\} \subseteq A$:*

$$cl\left(v\left(q\right)\right) \geq_k \left[\left(\{p_1, \ldots, p_n\}\right), \left(F\right)\right] \quad \textit{iff} \quad p_1, \ldots, p_n \vDash q$$

$$cl\left(v\left(q\right)\right) \geq_k \left[\left(F\right), \left(\{p_1, \ldots, p_n\}\right)\right] \quad \textit{iff} \quad p_1, \ldots, p_n \vDash \neg q$$

Proof. Let $\bigoplus$ denote the infinitary version of the gullability operator $\oplus$ and let $\bar{v}$ be a valuation defined as follows. For any $q \in F$:

$$\bar{v}\left(q\right) = \bigoplus_{X \subseteq A} \left\{\left[\left(X\right), \left(F\right)\right] : X \vDash q\right\} \oplus \bigoplus_{Y \subseteq A} \left\{\left[\left(F\right), \left(Y\right)\right] : Y \vDash \neg q\right\}.$$

Clearly, to prove the theorem it is sufficient to show that $\bar{v}\left(q\right) = \mathrm{cl}\left(v\left(q\right)\right)$ for any $q \in F$. First we show that for every closed valuation w, if $v\left(q\right) \leq_k w\left(q\right)$ for any $q \in F$, then

$$\bigoplus_{X \subseteq A} \left\{\left[\left(X\right), \left(F\right)\right] : X \vDash q\right\} \leq_k w\left(q\right).$$

Since w is closed, for any set $\{p_1, \ldots, p_n\} \subset F$ such that $\{p_1, \ldots, p_n\} \vDash q$ we have $\wedge_i w\left(p_i\right) \leq_k w\left(\wedge_i p_i\right) \leq_t w\left(q\right) \quad (1 \leq i \leq n)$. By hypothesis $v\left(p_i\right) \leq_k w\left(p_i\right)$: since we are in an interlaced bilattice, this implies $\wedge_i v\left(p_i\right) \leq_k \wedge_i w\left(p_i\right)$ for all $p_i \in F$. By the definition of v we have $\wedge_i v\left(p_i\right) = \wedge_i \left[\left(\{p_i\}\right), \left(F\right)\right] = \left[\left(\{p_1, \ldots, p_n\}\right), \left(F\right)\right]$. So we also have $\left[\left(\{p_1, \ldots, p_n\}\right), \left(F\right)\right] = \wedge_i v\left(p_i\right) \leq_k \wedge_i w\left(p_i\right) \leq_k w\left(\wedge_i p_i\right) \leq_t w\left(q\right)$. Since F is the minimal element in the lattice of contexts L, this implies that $\left[\left(\{p_1, \ldots, p_n\}\right), \left(F\right)\right] \leq_k w\left(q\right)$. Hence

$$\bigoplus_{X \subseteq A} \left\{\left[\left(X\right), \left(F\right)\right] : X \vDash q\right\} \leq_k w\left(q\right).$$

Recalling that $w\left(\neg q\right) = \neg w\left(q\right)$ for all $q \in F$, we may show in the same way that

$$\bigoplus_{Y \subseteq A} \left\{\left[\left(F\right), \left(Y\right)\right] : Y \vDash \neg q\right\} \leq_k w\left(q\right).$$

So we have that $\bar{v}\left(q\right) \leq_k w\left(q\right)$ for any closed extension w of v.

Now it remains only to show that $\bar{v}$ is closed, for then it will clearly coincide with the greatest lower bound of all the closed extensions of v. We consider each item of the definition of closed valuation.

(i) Let $p, q \in F$ such that $p \vDash q$. We have to show that $\bar{v}(p) \leq_t \bar{v}(q)$. Note that $Z \vDash p$ implies $Z \vDash q$ for any $Z \subset F$, so any set of formulas appearing in

$$\bigoplus_{S \subseteq A} \{[(S), (F)] : S \vDash p\}$$

will also be in

$$\bigoplus_{X \subseteq A} \{[(X), (F)] : X \vDash q\}.$$

Hence

$$\bigoplus_{S \subseteq A} \{[(S), (F)] : S \vDash p\} \leq_k \bigoplus_{X \subseteq A} \{[(X), (F)] : X \vDash q\}$$

and since the falsity component is the same in both elements (F), this implies that

$$\bigoplus_{S \subseteq A} \{[(S), (F)] : S \vDash p\} \leq_t \bigoplus_{X \subseteq A} \{[(X), (F)] : X \vDash q\}.$$

Similarly, for all $Z \subset F$, if $Z \vDash \neg q$ then by contraposition $Z \vDash \neg p$. Hence we have

$$\bigoplus_{T \subseteq A} \{[(F), (T)] : T \vDash \neg p\} \leq_t \bigoplus_{Y \subseteq A} \{[(F), (Y)] : Y \vDash \neg q\}.$$

Recalling again that the bilattice is interlaced, we may conclude that $\bar{v}(p) \leq_t \bar{v}(q)$.

(ii) We have to show that $\bar{v}(p) \wedge \bar{v}(q) \leq_k \bar{v}(p \wedge q)$ for all $p, q \in F$. For this it is sufficient so show that in the lattice of contexts we have

$$\sum_{X \subseteq A} \{[(X)] : X \vDash p\} \cdot \sum_{Y \subseteq A} \{[(Y)] : Y \vDash q\} \leq \sum_{Z \subseteq A} \{[(Z)] : Z \vDash p \wedge q\}$$

where $\cdot$ and $\sum$ denote respectively the meet and the infinitary join in this lattice.

Note that $X \vDash p$ and $Y \vDash q$ imply $X \cup Y \vDash p \wedge q$, so for any X and Y appearing in

$$\sum_{X \subseteq A} \{[(X)] : X \vDash p\} \cdot \sum_{Y \subseteq A} \{[(Y)] : Y \vDash q\}$$

there will be some Z in

$$\sum_{Z\subseteq A}\{[(Z)]:Z\vDash p\wedge q\}$$

such that $Z=X\cup Y$. Therefore we have

$$\sum_{X,Y\subseteq A}\{[(X\cup Y)]:X\vDash p \text{ and } Y\vDash q\}\leq\sum_{Z\subseteq A}\{[(Z)]:Z\vDash p\wedge q\}.$$

So we just need to observe that

$$\sum_{X,Y\subseteq A}\{[(X)\cdot(Y)]:X\vDash p \text{ and } Y\vDash q\}=\sum_{X,Y\subseteq A}\{[(X\cup Y)]:X\vDash p \text{ and } Y\vDash q\}$$

and we are done.

(iii) We have to show that for all $q\in F$ we have $\neg\bar{v}(q)=\bar{v}(\neg q)$. This is straightforward, since in any bilattice B we have $\neg(a\oplus b)=\neg a\oplus\neg b$ for all $a,b\in B$. Hence:

$$\neg\left(\bigoplus_{Y\subseteq A}\{[(Y),(F)]:Y\vDash\neg q\}\oplus\bigoplus_{X\subseteq A}\{[(F),(X)]:X\vDash q\}\right)=$$

$$=\neg\bigoplus_{Y\subseteq A}\{[(Y),(F)]:Y\vDash\neg q\}\oplus\neg\bigoplus_{X\subseteq A}\{[(F),(X)]:X\vDash q\}=$$

$$=\bigoplus_{Y\subseteq A}\{[(F),(Y)]:Y\vDash\neg q\}\oplus\bigoplus_{X\subseteq A}\{[(X),(F)]:X\vDash q\}.$$

■

We see therefore that, whenever a sentence q (or its negation) holds in some context $C=\{p_1,\ldots,p_n\}$, this information is punctually reflected in the value assigned to q once we have applied the closure operator.

5 Future work

What we have sketched in the present paper is of course just a proposal: it is clear that most of the work is still to be done.

The closure operator we have seen in the previous paragraph is just one of the possible definition for a consequence relation on the contextual bilattice. Arieli and Avron [1] introduced several other consequence relations for bilattices, with corresponding Gentzen-style deduction systems, that could be employed as well. All these inference mechanism, however, have been introduced with other applications in mind: it could be desirable to investigate the possibility of developing inference systems specifically designed for contextual bilattices.

Another related issue would be to incorporate inference rules that are *local* in the sense of [5], that is relative to a given context. I believe these may be interesting topics for future investigation.

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