Umberto Rivieccio

Algebraic Semantics for Bilattice Public Announcement Logic

Abstract. Building on recent work on bilattice modal logic and extensions of public announcement logic to a non-classical base, we introduce a simple dynamic epistemic logic having the logic of bilattices as its propositional support. Bilattice logic is both paraconsistent and paracomplete, thus suited for applications in contexts with multiple sources of information, where lacking as well as potentially contradictory evidence must be accounted for. We introduce an algebra-based semantics for bilattice public announcement logic and axiomatize the resulting consequence relation by means of a Hilbert-style calculus. Our results and methodology relate to recent work on non-classical dynamic epistemic logics such as intuitionistic public announcement logic.

Keywords: Bilattices, public announcement, paraconsistent logic, epistemic updates, dynamic logic, modal logic, many-valued logic.

Introduction

Dynamic logics are language expansions of classical (modal) logic designed to reason about changes induced by actions of different kinds, e.g., updates on the memory state of a computer, displacements of a moving robot, belief-revisions changing the common ground among different cognitive agents, knowledge update. Semantically, an action is represented as a transformation of a model describing a given state of affairs into a new one that encodes the state of affairs after the action has been performed.

The logic of public announcements (PAL) [14, 2, 6, 4] is a simple and well-known dynamic logic that models the epistemic change brought about on the cognitive state of a group of agents once a given proposition has become publicly known. To each proposition α one associates a *dynamic* modal operator $\langle\alpha\rangle$ whose semantic interpretation is given by the transformation of models corresponding to its action-parameter α.

The present paper builds on the logic of public announcements developed in [13, 12, 2] on the one hand and on the bilattice-valued modal logic [11] on the other. [13, 12] introduce a semantically justified definition of dynamic epistemic logic on a base that is weaker than classical logic. The main methodological feature of these papers is the dual characterization of epistemic updates via Stone-type dualities.

It is well known that epistemic updates induced by public announcements are formalized in relational models by means of the relativization construction, which creates a submodel of the original model. In [13] the corresponding submodel injection map is dually represented as a quotient construction between the complex algebras of the original model and of the updated one. This construction allows one to study epistemic updates within mathematical environments having a support that is weaker than classical logic. In the present paper we develop a similar study in a context that is yet more general than that of [13]. As propositional base we take the bilattice logic introduced by Arieli and Avron [1], which is both an inconsistency-tolerant and a paracomplete logic. Epistemic modalities are modeled using the framework of the bilattice modal logic introduced in [11].

The algebraic framework of bilattices [9] and their associated logic builds on seminal ideas of Belnap [3] motivated by the problem of dealing with incomplete and potentially inconsistent information. This setting has been further developed in [1] and generalized to weaker logics in, e.g., [10], [5]. In particular, [11] expands the language of bilattice logic with modal operators that are interpreted in many-valued analogues of Kripke frames.

In the present paper generalize the quotient construction of [13] to the algebraic semantics of bilattice modal logic, which allows us to define a natural interpretation of the language of PAL on modal bilattices. In this way we establish which interaction axioms among dynamic modalities are sound with respect to our intended semantics. The resulting calculus defines a bilattice-based version of public announcement logic (called *bilattice public announcement logic*, BPAL), which we prove to be complete with respect to our algebra-based semantics analogously to classical PAL. This paper aims at paving the way to a semantically-grounded analysis of epistemic updates in the presence of incomplete and/or inconsistent information. It is also a contribution to the research line initiated in [13, 12] which aims at introducing methods of algebraic logic, duality and proof theory in the study of mathematical foundations of dynamic logic (see also [7, 8]).

1. Bilattice modal logic

In this section we introduce the setting of bilattice modal logic and recall facts and definitions that will be needed to develop our bilattice public announcement logic. The reader is referred to [11] for proofs and further details. The non-modal base of bilattice modal logic is the logic introduced by Arieli and Avron [1], which can be defined through Belnap's (bi)lattice FOUR (Figure 1). We view FOUR as an algebra having op-

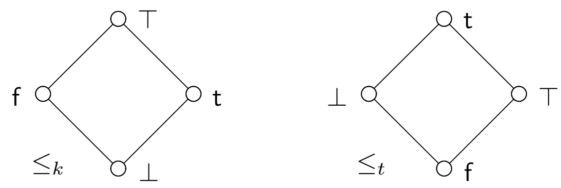

Figure 1. The four-element Belnap bilattice FOUR in its two orders

erations $\langle \wedge, \vee, \otimes, \oplus, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ of type $\langle 2, 2, 2, 2, 2, 1, 0, 0, 0, 0 \rangle$. Both $\langle \mathsf{FOUR}, \wedge, \vee, \mathsf{f}, \mathsf{t} \rangle$ and $\langle \mathsf{FOUR}, \otimes, \oplus, \bot, \top \rangle$ are bounded distributive lattices, as shown in Figure 1, whose lattice orders are denoted, respectively, by $\leq_t$ (*truth order*) and $\leq_k$ (*knowledge order*). We have, moreover, a binary *weak implication* operation $\supset$ defined by $x \supset y := y$ if $x \in \{\mathsf{t}, \top\}$ and $x \supset y := \mathsf{t}$ otherwise. *Negation* is a unary operation $\neg$ having $\bot$ and $\top$ as fixed points and s.t. $\neg\mathsf{f} = \mathsf{t}$ and $\neg\mathsf{t} = \mathsf{f}$.

The operations $\otimes$ and $\oplus$ need not be included in the primitive signature as they can be retrieved as terms in the language $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$. Thus, we will consider them as abbreviations of the terms shown below, together with the following defined operations: $x \otimes y := (x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y)$; $x \oplus y := (x \wedge \top) \vee (y \wedge \top) \vee (x \wedge y)$; $x \to y := (x \supset y) \wedge (\neg y \supset \neg x)$; ${\sim} x := x \supset \mathsf{f}$; $x * y := \neg(y \to \neg x)$; $x \equiv y := (x \supset y) \wedge (y \supset x)$; $x \leftrightarrow y := (x \to y) \wedge (y \to x)$. The operation $\sim$ provides an alternative negation, while $\to$ is an alternative implication called *strong implication.*

The bilattice logic of [1] can then be introduced as the propositional logic defined by the matrix $\langle \mathsf{FOUR}, \{\mathsf{t}, \top\} \rangle$ as follows. Starting from a countable set of propositional variables Var, one constructs the formula algebra $\mathbf{Fm} = \langle Fm, \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ in the usual way. Given formulas $\Gamma, \{\phi\} \subseteq Fm$, one then sets $\Gamma \vDash_{\mathsf{FOUR}} \phi$ iff, for all homomorphisms $v \colon \mathbf{Fm} \to \mathsf{FOUR}$, if $v(\gamma) \in \{\mathsf{t}, \top\}$ for all $\gamma \in \Gamma$, then also $v(\phi) \in \{\mathsf{t}, \top\}$. This logic can be axiomatized through the Hilbert-style calculus of [11, Section III B]. It is sufficient to take all axioms of classical logic in the language $\langle \wedge, \vee, \supset, \mathsf{f}, \mathsf{t} \rangle$ plus the following:

$$
\begin{array}{lll}
\top \wedge \neg\top & \neg(p \supset q) \equiv (p \wedge \neg q) & \neg(p \wedge q) \equiv (\neg p \vee \neg q) \\
(\bot \vee \neg\bot) \supset \mathsf{f} & & \neg(p \vee q) \equiv (\neg p \wedge \neg q)
\end{array}
$$

The only rule is *modus ponens* (mp): $p, p \supset q \vdash q$. This logic can be semantically expanded with modal operators by considering *four-valued Kripke models.* These are structures $\langle W, R, v \rangle$ such that both R and v are four-valued. That is, one defines $R\colon W \times W \to \mathsf{FOUR}$ and $v\colon \mathbf{Fm} \times W \to \mathsf{FOUR}$. We then call $\langle W, R\rangle$ a *four-valued Kripke frame*. Valuations are required to be homomorphisms in their first argument, so they preserve all non-modal connectives (including all four constants) of the logic of FOUR. The modal operators $\Box$ is defined as follows: for every $w \in W$ and every $\phi \in Fm$, $v(\Box\phi, w) := \bigwedge\{R(w, w') \to v(\phi, w') : w' \in W\}$, where $\bigwedge$ denotes the infinitary version of $\wedge$ in FOUR and $\to$ is the strong implication introduced above. The dual operator $\Diamond$ is defined as $v(\Diamond\phi, w) := \bigvee\{R(w, w') * v(\phi, w') : w' \in W\}$, where $\bigvee$ denotes the infinitary version of $\vee$ in FOUR and $*$ is the *fusion* operation introduced above. It is straightforward to check that $v(\Box\phi, w) = v(\neg\Diamond\neg\phi, w)$ for all $w \in W$ and all valuations v. Thus, as happens in the classical case (and unlike the intuitionistic), the two modal operators are inter-definable. In the present paper we take $\Diamond$ as primitive.

A modal consequence relation can now be defined in the usual way. We say that a point $w \in W$ of a four-valued model $M = \langle W, R, v\rangle$ satisfies a formula $\phi \in Fm$ iff $v(\phi, w) \in \{\mathsf{t}, \top\}$, and we write $M, w \vDash \phi$. For a set of formulas $\Gamma \subseteq Fm$, we write $M, w \vDash \Gamma$ to mean that $M, w \vDash \gamma$ for each $\gamma \in \Gamma$. The (local) consequence $\Gamma \vDash \phi$ holds if, for every model $M = \langle W, R, v\rangle$ and every $w \in W$, it is the case that $M, w \vDash \Gamma$ implies $M, w \vDash \phi$.

The above-defined consequence is axiomatized in [11]. The set of axioms is the least set $\Sigma \subseteq Fm$ containing all substitution instances of the schemata axiomatizing non-modal bilattice logic plus the following ones: (i) $\Box\mathsf{t} \leftrightarrow \mathsf{t}$, (ii) $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$, (iii) $\Box(\bot \to p) \leftrightarrow (\bot \to \Box p)$. Moreover, Σ must satisfy: (val-mp) if ϕ and $\phi \supset \psi$ are in Σ, then so is ψ; (val-mono) if $\phi \to \psi$ is in Σ, then so is $\Box\phi \to \Box\psi$. The only inference rule is (mp).

This calculus is complete not only with respect to the semantics of four-valued Kripke models, but also with respect to an algebra-based semantics given by the class of *modal bilattices*. We give a brief account of these results in the remaining part of this section, as we will build on them later on. We begin with completeness with respect to Kripke models [11, Theorem 19].

THEOREM 1.1 (Relational completeness). *For all $\Gamma, \{\phi\} \subseteq Fm$, $\Gamma \vdash \phi$ iff $M, w \vDash \Gamma$ implies $M, w \vDash \phi$ for every four-valued Kripke model $M = \langle W, R, v\rangle$ and every $w \in W$.*

In order to state the algebraic completeness theorem we need to introduce a class of algebras providing an alternative semantics for the local and global calculi. A *modal bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \supset, \sim, \Diamond, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$ such that the $\Diamond$-free reduct of $\mathbf{B}$ is an *implicative bilattice*, that is, the algebra $\langle B, \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$ belongs to the variety generated by FOUR, and moreover the following identities are satisfied: (i) $\Diamond\mathsf{f} = \mathsf{f}$, (ii) $\Diamond(x \vee y) =$

$(\Diamond x \vee \Diamond y)$, (iii) $\Box(x \supset \bot) = \Diamond x \supset \bot$. Thus, in particular, $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t}\rangle$ is a bounded distributive lattice. It is easy to show that identities (i)-(iii) correspond, respectively, to axioms (i)-(iii) of our calculus, and that the presentation of modal bilattices given here is equivalent to that of [11].

Given a modal bilattice $\mathbf{B}$ and a subset $F \subseteq B$, we say that F is a *bifilter* if F is a lattice filter of $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t}\rangle$ and moreover $\top \in F$. Given a pair $\langle \mathbf{B}, F\rangle$ and formulas $\Gamma, \{\phi\} \subseteq Fm$, we write $\Gamma \models_{\langle B,F\rangle} \phi$ to mean that, for every modal bilattice homomorphism $v \colon \mathbf{Fm} \to \mathbf{B}$, if $v(\gamma) \in F$ for all $\gamma \in \Gamma$, then also $v(\phi) \in F$. We can then state the announced algebraic completeness result as follows [11, Theorem 10].

THEOREM 1.2 (Algebraic completeness). *For all $\Gamma, \{\phi\} \subseteq Fm$, $\Gamma \vdash \phi$ iff $\Gamma \models_{\langle B,F\rangle} \phi$ for any modal bilattice $\mathbf{B}$ and any bifilter $F \subseteq B$.*

Just as in the case of classical modal logic, the relational and the algebraic semantics for bilattice modal logic are related to one another via a Stone-type duality [11, Theorem 18]. In the case of bilattices, another essential ingredient is the so-called *twist-structure* representation. Let $\mathbf{A} = \langle A, \wedge, \vee, \sim, \Diamond_+, \Diamond_-, 0, 1\rangle$ be a *bimodal Boolean algebra* [11, Definition 11], i.e., a structure such that $\langle A, \wedge, \vee, \sim, 0, 1\rangle$ is a Boolean algebra and $\Diamond_+$ and $\Diamond_-$ are unary operators that preserve finite joins (no relation between the two is required). The dual operators $\Box_+$ and $\Box_-$ are defined in the usual way as $\Box_+ x := \sim \Diamond_+ \sim x$ and $\Box_- x := \sim \Diamond_- \sim x$. The *twist-structure over* $\mathbf{A}$ is the algebra $\mathbf{A}^{\bowtie} = \langle A \times A, \wedge, \vee, \supset, \neg, \Diamond, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$ with operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A \times A$, by: $\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle := \langle a_1 \wedge b_1,\ a_2 \vee b_2\rangle$; $\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle := \langle a_1 \vee b_1,\ a_2 \wedge b_2\rangle$; $\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle := \langle \sim a_1 \vee b_1, a_1 \wedge b_2\rangle$; $\neg\langle a_1, a_2\rangle := \langle a_2,\ a_1\rangle$; $\Diamond\langle a_1, a_2\rangle := \langle \Diamond_+ a_1,\ \Box_+ a_2 \wedge \sim \Diamond_- a_1\rangle$; $\mathsf{f} := \langle 0, 1\rangle$; $\mathsf{t} := \langle 1, 0\rangle$; $\bot := \langle 0, 0\rangle$; $\top := \langle 1, 1\rangle$. It is straightforward to check that any twist-structure is a modal bilattice. Conversely, *any modal bilattice is isomorphic to a twist-structure* [11, Theorem 12]. This means that instead of working directly with modal bilattices, one can (as we will in the following sections) without loss of generality focus only on twist-structures.

2. Pseudo-quotients on modal bilattices

When considering epistemic updates in the context of bilattice logic, we have to take into account that validity of a formula in our logic only depends on its "positive part". By this we mean that any two formulas ϕ, ψ are logically equivalent if and only if, for every valuation $v \colon Fm \to \mathsf{FOUR}$, it holds that $\pi_1(v(\phi)) = \pi_1(v(\psi))$, where π_1 denotes first component projection defined by the twist-structure representation of FOUR. For instance, t and $\top$

(seen as propositional constants) are both valid formulas (hence, logically equivalent) because $\pi_1(\mathtt{t}) = \pi_1(\top) = 1$. Thus, in particular, both the public announcement of $\mathtt{t}$ and of $\top$ should be vacuous. An alternative characterization of logical equivalence is the following. Any two formulas ϕ, ψ are logically equivalent if and only if $v({\sim}{\sim}\phi) = v({\sim}{\sim}\psi)$ for any valuation v. This remark motivates our definition of pseudo-quotients. Let $\mathbf{B}$ be a modal bilattice and $a \in B$. We define a relation $\equiv_a$ as follows: for all $b, c \in B$, we let $b \equiv_a c$ iff $b \wedge {\sim}{\sim}a = c \wedge {\sim}{\sim}a$. This definition is adapted from (and can indeed be seen as a special of) that of [13]. The only difference is that, as noted above, here we need to consider only the "positive part" of $a \in B$, hence the term ${\sim}{\sim}a$. We are now going to prove that the above-defined relation is indeed a congruence of the non-modal reduct of any modal bilattice.

LEMMA 2.1. *Let $\mathbf{A}^{\bowtie}$ be a twist-structure over a Boolean algebra $\mathbf{A}$. Then, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle, \langle c_1, c_2\rangle \in A \times A$, we have $\langle b_1, b_2\rangle \equiv_{\langle a_1, a_2\rangle} \langle c_1, c_2\rangle$ iff $b_1 \equiv_{a_1} c_1$ and $b_2 \equiv_{a_1} c_2$, where $\equiv_{a_1}$ is defined as in [13, Section 3.2], i.e., $x \equiv_{a_1} y$ iff $x \wedge a_1 = y \wedge a_1$.*

PROOF. Assume $\langle b_1, b_2\rangle \equiv_{\langle a_1, a_2\rangle} \langle c_1, c_2\rangle$, which by definition means $\langle b_1, b_2\rangle \wedge {\sim}{\sim}\langle a_1, a_2\rangle = \langle c_1, c_2\rangle \wedge {\sim}{\sim}\langle a_1, a_2\rangle$. Applying the definitions of operations in a twist-structure, we obtain $\langle b_1, b_2\rangle \wedge {\sim}{\sim}\langle a_1, a_2\rangle = \langle b_1, b_2\rangle \wedge \langle a_1, {\sim} a_1\rangle = \langle a_1 \wedge b_1, {\sim} a_1 \vee b_2\rangle$. and similarly $\langle c_1, c_2\rangle \wedge {\sim}{\sim}\langle a_1, a_2\rangle = \langle a_1 \wedge c_1, {\sim} a_1 \vee c_2\rangle$. Thus, $a_1 \wedge b_1 = a_1 \wedge c_1$ and ${\sim} a_1 \vee b_2 = {\sim} a_1 \vee c_2$. The first equality means that $b_1 \equiv_{a_1} c_1$ holds in $\mathbf{A}$, while the second (applying Boolean negation to both sides) implies ${\sim} b_2 \equiv_{a_1} {\sim} c_2$. Since $\equiv_{a_1}$ is a congruence of $\mathbf{A}$ [13, Fact 8], we conclude ${\sim}{\sim} b_2 = b_2 \equiv_{a_1} c_2 = {\sim}{\sim} c_2$. Conversely, if $b_1 \equiv_{a_1} c_1$ and $b_2 \equiv_{a_1} c_2$, then clearly $b_1 \wedge a_1 = c_1 \wedge a_1$ and ${\sim} a_1 \vee b_2 = {\sim} a_1 \vee c_2$, which means $\langle b_1, b_2\rangle \equiv_{\langle a_1, a_2\rangle} \langle c_1, c_2\rangle$ as required. ■

FACT 2.2. *For any modal bilattice $\mathbf{B}$ and any $a \in B$, the relation $\equiv_a$ is a congruence of the non-modal reduct of $\mathbf{B}$.*

PROOF. It is sufficient to check that the statement holds in a twist-structure $\mathbf{B} = \mathbf{A}^{\bowtie}$. Assume $\langle b_1, b_2\rangle \equiv_{\langle a_1, a_2\rangle} \langle c_1, c_2\rangle$ and $\langle d_1, d_2\rangle \equiv_{\langle a_1, a_2\rangle} \langle e_1, e_2\rangle$. By Lemma 2.1, this is equivalent to $b_1 \equiv_{a_1} c_1$, $b_2 \equiv_{a_1} c_2$, $d_1 \equiv_{a_1} e_1$, $d_2 \equiv_{a_1} e_2$. Since $\equiv_{a_1}$ is a congruence of the Boolean algebra $\mathbf{A}$, we have, for instance, ${\sim} b_1 \vee d_1 \equiv_{a_1} {\sim} c_1 \vee e_1$ and $b_1 \wedge d_2 \equiv_{a_1} c_1 \wedge e_2$. By Lemma 2.1 again, this means that $\langle b_1, b_2\rangle \supset \langle d_1, d_2\rangle \equiv_{\langle a_1, a_2\rangle} \langle c_1, c_2\rangle \supset \langle e_1, e_2\rangle$. Compatibility with all the other bilattice operations can be shown in a similar way. ■

As happened in [13], our relation $\equiv_a$ is in general not compatible with the modal operators. The next step is thus to find a suitable definition for modal operators on the pseudo-quotient. We begin with the following observation (the proof is essentially the same as as [13, Fact 6], replacing a by $\sim\sim a$).

FACT 2.3. *Let* $\mathbf{B}$ *be a modal bilattice and* $a \in B$. *Then, (i)* $[b \wedge \sim\sim a] = [b]$ *for every* $b \in B$. *Hence, for every* $b \in B$, *there exists a unique* $c \in B$ *such that* $c \in [b]_a$ *and* $c \leq_t \sim\sim a$. *(ii)* $[b] \leq_t [c]$ *iff* $b \wedge \sim\sim a \leq_t c \wedge \sim\sim a$ *for all* $b, c \in B$.

Item (i) of Fact 2.3 implies that for each equivalence class modulo $\equiv_a$ we can choose a canonical representative, namely the unique element in the given class that is below $\sim\sim a$ in the truth order. Hence we can define an (injective) map $i' = i'_a \colon \mathbf{B}^a \to \mathbf{B}$ given, for every $[b] \in B^a$, by $i'[b] := b \wedge \sim\sim a$. Notice also that $\pi \cdot i'$ is the identity on $\mathbf{B}^a$. At this point we are ready to introduce modal operator(s) on the pseudo-quotient. We define $\Diamond^a[b] := [\Diamond(b \wedge \sim\sim a)] = [\Diamond(b \wedge \sim\sim a) \wedge \sim\sim a]$ for all $a, b \in \mathbf{B}$. The dual operator is defined as $\Box^a[b] := \neg\Diamond^a\neg[b]$. Using Fact 2.2 and the identities of modal bilattices, it is easy to check that, in keeping with [13, Section 3.3.2], $\Box^a[b] = [\Box(a \supset b)] = [a \supset \Box(a \supset b)]$. This could thus be taken as an alternative but equivalent definition. The following result shows that our definition indeed suits our purpose (cf. [13, Fact 10]).

FACT 2.4. *For every modal bilattice* $\mathbf{B}$ *and all* $a, b, c \in B$: *(i)* $\Diamond^a[\mathsf{f}] = [\mathsf{f}]$. *(ii)* $\Diamond^a([b] \vee [c]) = \Diamond^a[b] \vee \Diamond^a[c]$. *(iii)* $\Box^a([b] \supset [\bot]) = \Diamond^a[b] \supset [\bot]$. *(iv) Hence,* $(\mathbf{B}^a, \Diamond^a)$ *is a modal bilattice.*

PROOF. (i) Immediate, since $[\Diamond(\mathsf{f} \wedge \sim\sim a)] = [\Diamond\mathsf{f}]$ and $\Diamond\mathsf{f} = \mathsf{f}$ in any modal bilattice.

(ii)
$$\begin{aligned}
\Diamond^a([b] \vee [c]) &= \Diamond^a([b \vee c]) = [\Diamond((b \vee c) \wedge \sim\sim a)] && \text{Fact 2.2}\\
&= [\Diamond((b \wedge \sim\sim a) \vee (c \wedge \sim\sim a)] && \text{distributivity}\\
&= [\Diamond(b \wedge \sim\sim a) \vee \Diamond(c \wedge \sim\sim a)] && \Diamond \text{ preserves } \vee\\
&= [\Diamond(b \wedge \sim\sim a)] \vee [\Diamond(c \wedge \sim\sim a)] && \text{Fact 2.2}\\
&= \Diamond^a[b] \vee \Diamond^a[c].
\end{aligned}$$

(iii) We preliminary observe that (1) $\neg(x \supset \bot) \wedge \sim\sim y = \neg((y \wedge x) \supset \bot)$ is valid in any modal bilattice. Then:

$$
\begin{aligned}
\Box^a([b] \supset [\bot]) &= \Box^a[b \supset \bot] && \text{Fact 2.2}\\
&= \neg\Diamond^a\neg[b \supset \bot] && \Box^a[x] := \neg\Diamond^a\neg[x]\\
&= \neg\Diamond^a[\neg(b \supset \bot)] && \text{Fact 2.2}\\
&= \neg[\Diamond((\neg(b \supset \bot)) \wedge {\sim}{\sim} a)] && \\
&= \neg[\Diamond\neg((b \wedge a) \supset \bot)] && (1)\\
&= [\neg\Diamond\neg((b \wedge a) \supset \bot)] && \text{Fact 2.2}\\
&= [\neg\neg\Box\neg\neg((b \wedge a) \supset \bot)] && \Diamond x = \neg\Box\neg x\\
&= [\Box((b \wedge a) \supset \bot)] && \neg\neg x = x\\
&= [\Box((b \wedge {\sim}{\sim} a) \supset \bot)] && (x \wedge y) \supset z = (x \wedge {\sim}{\sim} y) \supset z\\
&= [\Diamond(b \wedge {\sim}{\sim} a) \supset \bot] && \Box(x \supset \bot) = \Diamond x \supset \bot\\
&= [\Diamond(b \wedge {\sim}{\sim} a)] \supset [\bot] && \text{Fact 2.2}\\
&= \Diamond^a[b] \supset [\bot]. && \text{Fact 2.2}
\end{aligned}
$$

(iv) It follows from (i) and (ii) above that $\Diamond^a$ is a $\vee$-preserving operator, which implies that $\Box^a$ is a $\wedge$-preserving operator. Finally, by (iii), the algebra $(\mathbf{B}^a, \Diamond^a)$ is a modal bilattice. ■

3. Axiomatization of BPAL

Our calculus for bilattice public announcement logic is defined over the language $\langle \wedge, \vee, \supset, \neg, \Diamond, \langle\alpha\rangle, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$, where $\alpha \in Fm$. Derived connectives $\langle {\sim}, \Box, \otimes, \oplus, \to, *, \leftrightarrow \rangle$ are introduced as before. Moreover, we let $[\alpha]\phi := \neg\langle\alpha\rangle\neg\phi$. BPAL is axiomatically defined by the axioms and rules of the (local) calculus for bilattice modal logic augmented with the following axioms:

Interaction with constants	$\langle\alpha\rangle\mathsf{f} \leftrightarrow \mathsf{f}$	$\langle\alpha\rangle\mathsf{t} \leftrightarrow {\sim}{\sim}\alpha$
	$\langle\alpha\rangle\top \leftrightarrow (\alpha \wedge \top)$	$\langle\alpha\rangle\bot \leftrightarrow \neg(\alpha \supset \bot)$
Interaction with $\wedge$	$\langle\alpha\rangle(\phi \wedge \psi) \leftrightarrow (\langle\alpha\rangle\phi \wedge \langle\alpha\rangle\phi)$	
Interaction with $\vee$	$\langle\alpha\rangle(\phi \vee \psi) \leftrightarrow (\langle\alpha\rangle\phi \vee \langle\alpha\rangle\phi)$	
Interaction with $\supset$	$\langle\alpha\rangle(\phi \supset \psi) \leftrightarrow ({\sim}{\sim}\alpha \wedge (\langle\alpha\rangle\phi \supset \langle\alpha\rangle\phi))$	
Interaction with $\neg$	$\langle\alpha\rangle\neg\phi \leftrightarrow ({\sim}{\sim}\alpha \wedge \neg\langle\alpha\rangle\phi)$	
Interaction with $\Diamond$	$\langle\alpha\rangle\Diamond\phi \leftrightarrow ({\sim}{\sim}\alpha \wedge \Diamond\langle\alpha\rangle\phi)$	
Preservation of facts	$\langle\alpha\rangle p \leftrightarrow ({\sim}{\sim}\alpha \wedge p)$	

where ϕ, ψ, α are arbitrary formulas, while p is a propositional variable.

4. Algebraic models and completeness

In this section we introduce an algebra-based semantics and we prove completeness with respect to the calculus introduced in Section 3. We define an *algebraic model* as a tuple $M = (\mathbf{B}, F, v)$ where $\mathbf{B}$ is a modal bilattice, $F \subseteq B$ a bifilter, and $v \colon Var \to B$. The *extension* map $[\![\cdot]\!]_M \colon Fm \to \mathbf{B}$ is defined as follows:

$$
\begin{aligned}
[\![p]\!]_M &:= v(p) \\
[\![c]\!]_M &:= c^{\mathbf{B}} && \text{for } c \in \{\mathsf{f}, \mathsf{t}, \bot, \top\} \\
[\![\circ\phi]\!]_M &:= \circ^{\mathbf{B}} [\![\phi]\!]_M && \text{for } \circ \in \{\neg, \Diamond\} \\
[\![\phi \bullet \psi]\!]_M &:= [\![\phi]\!]_M \bullet^{\mathbf{B}} [\![\psi]\!]_M && \text{for } \bullet \in \{\wedge, \vee, \supset\} \\
[\![\langle\alpha\rangle\phi]\!]_M &:= {\sim}{\sim}[\![\alpha]\!]_M \wedge^{\mathbf{B}} i'([\![\phi]\!]_{M^\alpha}) \\
[\![[\alpha]\phi]\!]_M &:= [\![\alpha]\!]_M \supset^{\mathbf{B}} i'([\![\phi]\!]_{M^\alpha})
\end{aligned}
$$

where $M^\alpha = (\mathbf{B}^\alpha, v^\alpha)$ is given by $\mathbf{B}^\alpha = \mathbf{B}^{[\![\alpha]\!]_M}$ and $v^\alpha = \pi \circ v \colon Var \to \mathbf{B}^\alpha$. That is, $[\![p]\!]_{M^\alpha} = V^\alpha(p) = \pi(V(p)) = \pi([\![p]\!]_M)$ for every $p \in Var$. We define $\Gamma \vDash_{BPAL} \phi$ iff, for every algebraic model $M = (\mathbf{B}, F, v)$, we have that $[\![\gamma]\!] \in F$ for all $\gamma \in \Gamma$ implies $[\![\phi]\!] \in F$. We are now going to see that the calculus introduced in Section 3 is sound and complete with respect to the semantics provided by the above-defined algebraic models. The following lemmas are needed to establish this result (cf. [13, Lemmas 29-34]).

LEMMA 4.1. *Let $M = (\mathbf{B}, v)$ be an algebraic model and ϕ a formula such that $[\![\phi]\!]_{M^\alpha} = \pi([\![\phi]\!]_M)$ for any $\alpha \in Fm$. Then $[\![\langle\alpha\rangle\phi]\!]_M = {\sim}{\sim}[\![\alpha]\!]_M \wedge [\![\phi]\!]_M$ and $[\![[\alpha]\phi]\!]_M = [\![\alpha]\!]_M \supset [\![\phi]\!]_M$.*

PROOF. Concerning the first statement, we have $[\![\langle\alpha\rangle\phi]\!]_M = {\sim}{\sim}[\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha}) = {\sim}{\sim}[\![\alpha]\!]_M \wedge i'(\pi([\![\phi]\!]_M)) = {\sim}{\sim}[\![\alpha]\!]_M \wedge ([\![\phi]\!]_M \wedge [\![{\sim}{\sim}\alpha]\!]_M) = {\sim}{\sim}[\![\alpha]\!]_M \wedge [\![\phi]\!]_M$. Concerning the second:

$$
\begin{aligned}
[\![[\alpha]\phi]\!]_M &= [\![\alpha]\!]_M \supset i'([\![\phi]\!]_{M^\alpha}) = [\![\alpha]\!]_M \supset i'(\pi([\![\phi]\!]_M)) \\
&= [\![\alpha]\!]_M \supset ([\![\phi]\!]_M \wedge [\![{\sim}{\sim}\alpha]\!]_M) \\
&= ([\![\alpha]\!]_M \supset [\![\phi]\!]_M) \wedge ([\![\alpha]\!]_M \supset [\![{\sim}{\sim}\alpha]\!]_M) && (1) \\
&= ([\![\alpha]\!]_M \supset [\![\phi]\!]_M) \wedge ([\![\alpha]\!]_M \supset {\sim}{\sim}[\![\alpha]\!]_M) \\
&= ([\![\alpha]\!]_M \supset [\![\phi]\!]_M) \wedge \mathsf{t} && \mathsf{t} = x \supset {\sim}{\sim}x \\
&= [\![\alpha]\!]_M \supset [\![\phi]\!]_M. && x \leq_t \mathsf{t}
\end{aligned}
$$

Here (1) holds because the equation $x \supset (y \wedge z) = (x \supset y) \wedge (x \supset z)$ is satisfied by every modal bilattice. ■

Fact 4.2. *Let* $\mathbf{B}$ *be modal bilattice,* $a \in B$*, and let* $i' =: \mathbf{B}^a \to \mathbf{B}$ *be given, for every* $[b] \in B^a$*, by* $i'[b] := b \wedge \sim\sim a$*. Then, for every* $[b], [c] \in B^a$*,*

(i) $i'([b] \wedge [c]) = i'[b] \wedge i'[c]$

(ii) $i'([b] \vee [c]) = i'[b] \vee i'[c]$

(iii) $i'([b] \supset [c]) = \sim\sim a \wedge (i'[b] \supset i'[c])$

(iv) $i'(\neg[b]) = \sim\sim a \wedge \neg i'[b]$

(v) $i'(\Diamond^a[b]) = \sim\sim a \wedge \Diamond(i'[b]) = \sim\sim a \wedge \Diamond(\sim\sim a \wedge i'[b])$

(vi) $i'(\Box^a[b]) = \sim\sim a \wedge \Box(a \supset i'[b])$.

Proof. (i) Using Fact 2.2, we have $i'([b] \wedge [c]) = i'([b \wedge c]) = (b \wedge c) \wedge \sim\sim a = (b \wedge \sim\sim a) \wedge (c \wedge \sim\sim a) = i'[b] \wedge i'[c]$.

(ii)
$$\begin{aligned} i'([b] \vee [c]) = i'([b \vee c]) &= (b \vee c) \wedge \sim\sim a && \text{Fact 2.2} \\ &= (b \wedge \sim\sim a) \vee (c \wedge \sim\sim a) && \text{distributivity} \\ &= i'[b] \vee i'[c]. \end{aligned}$$

(iii) We are going to use Fact 2.2 together with the following identities: $\sim\sim x \wedge (y \supset z) = \sim\sim x \wedge ((y \wedge \sim\sim x) \supset z)$, $\mathsf{t} = (x \wedge y) \supset \sim\sim y$, and $(x \supset y) \wedge (x \supset z) = x \supset (y \wedge z)$, which are valid in any modal bilattice. We have:

$$\begin{aligned} i'([b] \supset [c]) = i'[b \supset c] &= \sim\sim a \wedge (b \supset c) \\ &= \sim\sim a \wedge ((b \wedge \sim\sim a) \supset c) \\ &= \sim\sim a \wedge (((b \wedge \sim\sim a) \supset c) \wedge \mathsf{t}) \\ &= \sim\sim a \wedge (((b \wedge \sim\sim a) \supset c) \wedge ((b \wedge \sim\sim a) \supset \sim\sim a)) \\ &= \sim\sim a \wedge ((b \wedge \sim\sim a) \supset (c \wedge \sim\sim a)) \\ &= \sim\sim a \wedge (i'[b] \supset i'[c]). \end{aligned}$$

(iv)
$$\begin{aligned} i'(\neg[b]) = i'([\neg b]) &= \sim\sim a \wedge \neg b && \text{Fact 2.2} \\ &= (\sim\sim a \wedge \neg b) \vee \mathsf{f} && \mathsf{f} \leq_t x \\ &= (\sim\sim a \wedge \neg b) \vee (\sim\sim a \wedge \neg \sim\sim a) && \mathsf{f} = \sim\sim x \wedge \neg\sim\sim x \\ &= \sim\sim a \wedge (\neg b \vee \neg \sim\sim a) && \text{distributivity} \\ &= \sim\sim a \wedge \neg(b \wedge \sim\sim a) && \text{De Morgan law} \\ &= \sim\sim a \wedge \neg i'[b]. \end{aligned}$$

(v) Straightforward, because we have on the one hand $i'(\Diamond^a[b]) = i'[\Diamond(b \wedge \sim\sim a)] = \sim\sim a \wedge \Diamond(b \wedge \sim\sim a) = \sim\sim a \wedge \Diamond(i'[b])$, and on the other $i'(\Diamond^a[b]) = i'[\Diamond(b \wedge \sim\sim a)] = i'[\Diamond(b \wedge \sim\sim a) \wedge \sim\sim a] = \sim\sim a \wedge \sim\sim a \wedge \Diamond(b \wedge \sim\sim a) = \sim\sim a \wedge \Diamond(i'[b])$. ■

LEMMA 4.3. *For any algebraic model $M = (\mathbf{B}, v)$ with underlying modal bilattice $\mathbf{B} = \langle B, \wedge, \vee, \supset, \neg, \Diamond, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ and for and all formulas $\alpha, \phi, \psi \in Fm$,*

(i) $[\![\langle\alpha\rangle(\phi \wedge \psi)]\!]_M = [\![\langle\alpha\rangle\phi]\!]_M \wedge [\![\langle\alpha\rangle\psi]\!]_M$

(ii) $[\![\langle\alpha\rangle(\phi \vee \psi)]\!]_M = [\![\langle\alpha\rangle\phi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M$

(iii) $[\![\langle\alpha\rangle(\phi \supset \psi)]\!]_M = \sim\sim[\![\alpha]\!]_M \wedge ([\![\langle\alpha\rangle\phi]\!]_M \supset [\![\langle\alpha\rangle\psi]\!]_M)$

(iv) $[\![\langle\alpha\rangle\neg\phi]\!]_M = \sim\sim[\![\alpha]\!]_M \wedge \neg[\![\langle\alpha\rangle\phi]\!]_M$

(v) $[\![[\alpha]\phi]\!]_M = [\![\neg\langle\alpha\rangle\neg\phi]\!]_M$

(vi) $[\![\langle\alpha\rangle\Diamond\phi]\!]_M = \sim\sim[\![\alpha]\!]_M \wedge \Diamond[\![\langle\alpha\rangle\phi]\!]_M$

(vii) $[\![\langle\alpha\rangle\Box\phi]\!]_M = \sim\sim[\![\alpha]\!]_M \wedge \Box[\![[\alpha]\phi]\!]_M$.

PROOF. (i)

$$
\begin{aligned}
&[\![\langle\alpha\rangle(\phi \wedge \psi)]\!]_M = \\
&= \sim\sim[\![\alpha]\!]_M \wedge i'([\![\phi \wedge \psi]\!]_{M^\alpha}) \\
&= \sim\sim[\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha} \wedge [\![\psi]\!]_{M^\alpha}) \\
&= \sim\sim[\![\alpha]\!]_M \wedge i'[\![\phi]\!]_{M^\alpha} \wedge i'[\![\psi]\!]_{M^\alpha} && \text{Fact 4.2 (i)} \\
&= (\sim\sim[\![\alpha]\!]_M \wedge i'[\![\phi]\!]_{M^\alpha}) \wedge (\sim\sim[\![\alpha]\!]_M \wedge i'[\![\psi]\!]_{M^\alpha}) \\
&= [\![\langle\alpha\rangle\phi]\!]_M \wedge [\![\langle\alpha\rangle\psi]\!]_M.
\end{aligned}
$$

(ii) Using Fact 4.2 (ii) and distributivity, we have:

$$
\begin{aligned}
[\![\langle\alpha\rangle(\phi \vee \psi)]\!]_M &= \sim\sim[\![\alpha]\!]_M \wedge i'([\![\phi \vee \psi]\!]_{M^\alpha}) \\
&= \sim\sim[\![\alpha]\!]_M \wedge (i'[\![\phi]\!]_{M^\alpha} \vee i'[\![\psi]\!]_{M^\alpha}) \\
&= (\sim\sim[\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha})) \vee (\sim\sim[\![\alpha]\!]_M \wedge i'([\![\psi]\!]_{M^\alpha})) \\
&= [\![\langle\alpha\rangle\phi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M.
\end{aligned}
$$

(iii) We notice that $\sim\sim x \wedge (y \supset z) = \sim\sim x \wedge ((\sim\sim x \wedge y) \supset z)$ and $(\sim\sim x \wedge y) \supset (\sim\sim x \wedge z) = (\sim\sim x \wedge y) \supset z$ are both valid in every modal bilattice (this can be easily checked using the twist-structure representation). Using this together with Fact 4.2 (iii), we have

$$
\begin{aligned}
&[\![\langle\alpha\rangle(\phi \supset \psi)]\!]_M = \sim\sim[\![\alpha]\!]_M \wedge i'[\![\phi \supset \psi]\!]_{M^\alpha} = \\
&= \sim\sim[\![\alpha]\!]_M \wedge \sim\sim[\![\alpha]\!]_M \wedge (i'[\![\phi]\!]_{M^\alpha} \supset i'[\![\psi]\!]_{M^\alpha}) \\
&= \sim\sim[\![\alpha]\!]_M \wedge (i'[\![\phi]\!]_{M^\alpha} \supset i'[\![\psi]\!]_{M^\alpha}) \\
&= \sim\sim[\![\alpha]\!]_M \wedge ((\sim\sim[\![\alpha]\!]_M \wedge i'[\![\phi]\!]_{M^\alpha}) \supset i'[\![\psi]\!]_{M^\alpha}) \\
&= \sim\sim[\![\alpha]\!]_M \wedge ((\sim\sim[\![\alpha]\!]_M \wedge i'[\![\phi]\!]_{M^\alpha}) \supset (\sim\sim[\![\alpha]\!]_M \wedge i'[\![\psi]\!]_{M^\alpha})) \\
&= \sim\sim[\![\alpha]\!]_M \wedge ([\![\langle\alpha\rangle\phi]\!]_M \supset [\![\langle\alpha\rangle\psi]\!]_M).
\end{aligned}
$$

(iv) We notice that identities (1) $\mathsf{f} = \sim\sim x \wedge \sim\sim\sim x$, (2) $\sim\sim\sim x = \neg\sim\sim x$ and (3) $\neg x \vee \neg y = \neg(x \wedge y)$ hold in every modal bilattice. Then, using Fact 4.2 (iv), we have:

$$\begin{aligned}
[\![\langle\alpha\rangle\neg\phi]\!]_M &= \sim\sim[\![\alpha]\!]_M \wedge i'[\![\neg\phi]\!]_{M^\alpha} \\
&= \sim\sim[\![\alpha]\!]_M \wedge \sim\sim[\![\alpha]\!]_M \wedge \neg i'[\![\phi]\!]_{M^\alpha} \\
&= \sim\sim[\![\alpha]\!]_M \wedge \neg i'[\![\phi]\!]_{M^\alpha} = \mathsf{f} \vee (\sim\sim[\![\alpha]\!]_M \wedge \neg i'[\![\phi]\!]_{M^\alpha}) \\
&= (\sim\sim[\![\alpha]\!]_M \wedge \sim\sim\sim[\![\alpha]\!]_M) \vee (\sim\sim[\![\alpha]\!]_M \wedge \neg i'[\![\phi]\!]_{M^\alpha}) && (1) \\
&= \sim\sim[\![\alpha]\!]_M \wedge (\sim\sim\sim[\![\alpha]\!]_M \vee \neg i'[\![\phi]\!]_{M^\alpha}) && \text{distrib.} \\
&= \sim\sim[\![\alpha]\!]_M \wedge (\neg\sim\sim[\![\alpha]\!]_M \vee \neg i'[\![\phi]\!]_{M^\alpha}) && (2) \\
&= \sim\sim[\![\alpha]\!]_M \wedge (\neg\sim\sim[\![\alpha]\!]_M \vee \neg i'[\![\phi]\!]_{M^\alpha}) \\
&= \sim\sim[\![\alpha]\!]_M \wedge \neg(\sim\sim[\![\alpha]\!]_M \wedge i'[\![\phi]\!]_{M^\alpha}) && (3) \\
&= \sim\sim[\![\alpha]\!]_M \wedge \neg[\![\langle\alpha\rangle\phi]\!]_M.
\end{aligned}$$

(v)
$$\begin{aligned}
[\![\neg\langle\alpha\rangle\neg\phi]\!]_M &= \neg(\sim\sim[\![\alpha]\!]_M \wedge \neg i'[\![\phi]\!]_{M^\alpha}) && \text{Item (iv)} \\
&= \neg\sim\sim[\![\alpha]\!]_M \vee \neg\neg i'[\![\phi]\!]_{M^\alpha} && \text{De Morgan law} \\
&= \neg\sim\sim[\![\alpha]\!]_M \vee i'[\![\phi]\!]_{M^\alpha} && \neg\neg x = x \\
&= \sim[\![\alpha]\!]_M \vee i'[\![\phi]\!]_{M^\alpha} && \neg\sim\sim x = \sim x \\
&= [\![\alpha]\!]_M \supset i'[\![\phi]\!]_{M^\alpha} && \sim x \vee y = x \supset y \\
&= [\![[\alpha]\phi]\!]_M.
\end{aligned}$$

(vi) Using Fact 4.2 (v), we have $[\![\langle\alpha\rangle\Diamond\phi]\!]_M = \sim\sim[\![\alpha]\!]_M \wedge i'[\![\Diamond\phi]\!]_{M^\alpha} = \sim\sim[\![\alpha]\!]_M \wedge \sim\sim[\![\alpha]\!]_M \wedge \Diamond(\sim\sim[\![\alpha]\!]_M \wedge i'[\![\phi]\!]_{M^\alpha}) = \sim\sim[\![\alpha]\!]_M \wedge \Diamond(\sim\sim[\![\alpha]\!]_M \wedge i'[\![\phi]\!]_{M^\alpha}) = \sim\sim[\![\alpha]\!]_M \wedge \Diamond[\![\langle\alpha\rangle\phi]\!]_{M^\alpha}$. (vii) Follows easily from (v) and (vii) above. ■

Item (v) of the preceding lemma shows that the choice of considering the formula $[\alpha]\phi$ as an abbreviation for $\neg\langle\alpha\rangle\neg\phi$ is sound. The following result easily follows from Lemma 4.3.

FACT 4.4. *For any algebraic model $M = (\mathbf{B}, v)$ with underlying modal bilattice $\mathbf{B} = \langle B, \wedge, \vee, \supset, \neg, \Diamond, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$ and for and all formulas $\alpha, \phi, \psi \in Fm$,*

(i) $[\![[\alpha](\phi \wedge \psi)]\!]_M = [\![[\alpha]\phi]\!]_M \wedge [\![[\alpha]\psi]\!]_M$

(ii) $[\![[\alpha](\phi \vee \psi)]\!]_M = [\![\alpha]\!]_M \supset ([\![\langle\alpha\rangle\phi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M)$

(iii) $[\![[\alpha](\phi \supset \psi)]\!]_M = [\![\langle\alpha\rangle\phi]\!]_M \supset [\![\langle\alpha\rangle\psi]\!]_M$

(iv) $[\![[\alpha]\neg\phi]\!]_M = \neg[\![\langle\alpha\rangle\phi]\!]_M$

(v) $[\![[\alpha]\Diamond\phi]\!]_M = [\![\alpha]\!]_M \supset \Diamond[\![\langle\alpha\rangle\phi]\!]_M$

(vi) $[\![[\alpha]\Box\phi]\!]_M = [\![\alpha]\!]_M \supset \Box[\![[\alpha]\phi]\!]_M$.

The above facts ensure that the axioms introduced in Section 3 are indeed sound w.r.t. our algebraic semantics for BPAL. Completeness can be proved following the same strategy used for classical and intuitionistic PAL [13, Theorem 22], i.e., reducing BPAL to the bilattice modal logic of [11] via the interaction axioms.

THEOREM 4.5. *The calculus for BPAL is sound and complete with respect to algebraic models.*

PROOF. Soundness of the preservation of facts and logical constants axioms follow from Lemma 4.1. For the remaining axioms we only need to invoke Lemma 4.3. The proof of completeness is similar to those for classical and intuitionistic PAL [13, Theorem 22] and follows from the reducibility of BPAL to the bilattice modal logic of [11] via reduction axioms. Let ϕ be a valid BPAL formula. Consider some innermost occurrence of a dynamic modality in ϕ. Hence, the subformula ψ having that occurrence labeling the root of its generation tree has the form $\langle\alpha\rangle\psi'$ for some formula ψ' in the static language. The distribution axioms make it possible to equivalently transform ψ by pushing the dynamic modality down the generation tree, through the static connectives, until it attaches to a proposition letter or to a constant symbol. Here the dynamic modality disappears by applying the appropriate 'preservation of facts' or 'interaction with constant' axiom. The process is repeated for all dynamic modalities of ϕ, so as to obtain a formula ϕ' which is provably equivalent to ϕ. Since ϕ is valid by assumption, and since the process preserves provable equivalence, by soundness we can conclude that ϕ' is valid. By Theorem 1.2, we can conclude that ϕ' is provable in bilattice modal logic and thus in BPAL. This, together with the provable equivalence of ϕ and ϕ', concludes the proof. ■

As mentioned earlier, Kripke-style and algebraic semantics for bilattice modal logic are related, and can indeed be proved to be equivalent via duality. This can be generalized to the setting of BPAL, thus introducing a relational semantics and a suitable notion of epistemic update on four-valued Kripke models. Also, along the line of [13, Section 5], one may think of applying the logic introduced in the present paper to a concrete example of multi-agent reasoning in order to better appreciate the potentiality and limits of our new formalism. We leave these as suggestions for future work.

Acknowledgements. This research has been supported by Vidi grant 016.138.314 of the Netherlands Organization for Scientific Research (NWO).

References

[1] Arieli, O., and A. Avron, 'Reasoning with logical bilattices', *Journal of Logic, Language and Information*, 5 (1996), 1, 25–63.

[2] Baltag, A., Moss, L., and S. Solecki, Arieli, O., and A. Avron, 'The logic of public announcements, common knowledge, and private suspicions', *CWI technical report SEN-R9922*, 1999.

[3] Belnap, N. D., 'How a computer should think', in G. Ryle, (ed.), *Contemporary Aspects of Philosophy*, Oriel Press, Boston, 1976, pp. 30–56.

[4] van Benthem, J., *Logical Dynamics of Information and Interaction*, Cambridge University Press, 2011.

[5] Bou, F., and U. Rivieccio, 'Bilattices with implications', *Studia Logica*, 101 (2013), 4, 651–675.

[6] van Ditmarsch, H., van der Hoek, W., and B. Kooi, *Dynamic Epistemic Logic*, Springer, 2007.

[7] Frittella, S., Greco, G., Kurz, A., Palmigiano, A., and V. Sikimic, 'A Proof-Theoretic Semantic Analysis of Dynamic Epistemic Logic', *Journal of Logic and Computation, Special issue on Sub-structural logic and information dynamics*, forthcoming.

[8] Frittella, S., Greco, G., Kurz, A., Palmigiano, A., and V. Sikimic, 'Multi-type Display Calculus for Propositional Dynamic Logic', *Journal of Logic and Computation, Special issue on Sub-structural logic and information dynamics*, forthcoming.

[9] Ginsberg, M. L., 'Multivalued logics: A uniform approach to inference in artificial intelligence', *Computational Intelligence*, 4 (1988), 265–316.

[10] Jansana, R., and U. Rivieccio, 'Residuated bilattices', *Soft Computing*, 16 (2012), 3, 493–504.

[11] Jung, A., and U. Rivieccio, 'Kripke semantics for modal bilattice logic', *Proceedings of the 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, IEEE Computer Society Press, 2013, 438–447.

[12] Kurz, A., and A. Palmigiano, 'Epistemic Updates on Algebras', *Logical Methods in Computer Science*, 9 (2013), 4. DOI: 10.2168/LMCS-9(4:17)2013.

[13] Ma, M., Palmigiano, A., and M. Sadrzadeh, 'Algebraic semantics and model completeness for Intuitionistic Public Announcement Logic', *Annals of Pure and Applied Logic*, 165 (2014), 963–995.

[14] Plaza, J., 'Logics of Public Communications', *Proceedings 4th International Symposium on Methodologies for Intelligent Systems*, 1989, 201–216.

Umberto Rivieccio
Faculty of Technology, Policy and Management
Delft University of Technology
Jaffalaan 5 – 2628 BX Delft, The Netherlands
u.rivieccio@tudelft.nl