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RESEARCH ARTICLE

An infinity of super-Belnap logics

Umberto Rivieccio*

School of Computer Science, The University of Birmingham, Edgbaston, Birmingham, UK;

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We look at extensions (i.e., stronger logics in the same language) of the Belnap-Dunn four-valued logic. We prove the existence of a countable chain of logics that extend the Belnap-Dunn and do not coincide with any of the known extensions (Kleene's logics, Priest's logic of paradox). We characterize the reduced algebraic models of these new logics and prove a completeness result for the first and last element of the chain stating that both logics are determined by a single finite logical matrix. We show that the last logic of the chain is not finitely axiomatizable.

Keywords: extensions of Belnap logic; strong Kleene logic; De Morgan lattices; non-protoalgebraic logics; abstract algebraic logic

1. Introduction

The Belnap-Dunn four-valued logic, also known as the logic of first-degree entailment (FDE), is a many-valued system that is well-known to both logicians and computer scientists. It was introduced by Dunn (1966) and developed by Belnap (1976), Belnap (1977), who proposed a famous interpretation of the four values in terms of what a computer is told about the truth or falsity of a given sentence. In more recent years Belnap's approach was generalized with the introduction of bilattices Ginsberg (1988), Fitting (1990), Arieli and Avron (1996), which opened the way to a variety of new applications of many-valued systems, especially in computer science.

In parallel with the computer science-oriented line of research, the Belnap-Dunn logic has also been investigated from a purely logical point of view, with particular emphasis on its algebraic features Font (1997); Pynko (1995a); A. P. Pynko (1999a). These studies pointed out the existence of an interesting connection between the Belnap-Dunn logic and the algebraic theory of De Morgan lattices, a class of algebras that provide a natural semantics for the logic.

The present work aims at giving a contribution to this last line of investigation, focusing on *extensions* of the Belnap-Dunn logic. Here by "extension" we mean a strengthening, in the same language, of a given logic, as opposed to expansions (such as those considered in A. P. Pynko (1999a)), which are obtained by introducing new connectives. Some extensions of the Belnap-Dunn logic are quite well-known (see Section 4.1) and have applications in computer science and philo-

*Corresponding author. Email: u.rivieccio@cs.bham.co.uk

sophical logic[1]. However, as far as the author is aware of, this topic has never been systematically investigated.

A first question one might ask is the following: how many proper extensions does the Belnap-Dunn logic have? This is the question that gave rise to the present work. One of the main results contained in the paper is that the Belnap-Dunn logic has at least countably many extensions. The next issue would then be to give a precise description of the structure of the lattice of (all) its extensions. This problem in its full generality is still unsolved, and we will give an idea of where the difficulties in tackling it lie. We have, however, obtained some partial results that shed some light on the landscape of extensions of the Belnap-Dunn logic and in the future may hopefully lead to a satisfactory solution to the general problem.

The paper is organized as follows. In Section 2 we introduce definitions and algebraic results on De Morgan lattices that will be needed for our study of extensions of the Belnap-Dunn logic. In Section 3 we formally introduce the logic and recall some known results that will also be used in the study of its extensions. This is developed in Section 4. We first consider the extensions that have already been mentioned in the literature (Section 4.1), then show how to build a countable chain of logics that extend the Belnap-Dunn system and still do not collapse to any of the known ones (Section 4.2). We prove completeness results for the first and last member of such chain and we prove that the latter logic is not finitely axiomatizable. Finally, in Section 5 we mention some further results and discuss future prospects.

2. De Morgan lattices

De Morgan lattices are well-known algebraic structures whose origin traces back to the 1930s. They can be seen as one of the possible generalization of Boolean algebras (in fact, in Rasiowa (1974) bounded De Morgan lattices are called *quasi-Boolean algebras*). It was clear since its very introduction that the Belnap-Dunn logic had some non-trivial relation with De Morgan lattices. In fact, the concrete four-element lattice used by Dunn and Belnap as a semantics for their logic is a particular example of De Morgan lattice (see Section 3). Nowadays we also know from Font (1997) that De Morgan lattices constitute, according to the theory of algebraizability of logics developed in Font and Jansana (2009), the canonical algebraic counterpart of the Belnap-Dunn logic.

In the present work we are going to exploit this relation to obtain some insight on the structure of extensions of the Belnap-Dunn logic. In this section we therefore recall the necessary algebraic results on De Morgan lattices (unless otherwise specified, the reader can consult Balbes and Dwinger (1974) for proofs and further details).

Notation. *The algebraic (as well as the logical) language considered in this paper will always be assumed to be* $\langle \wedge, \vee, \neg \rangle$*, sometimes expanded with the constants* 0 *and* 1*. On a logical level,* $\wedge$ *is interpreted as a* conjunction*,* $\vee$ *as a* disjunction *and* $\neg$ *as a* negation*. The constants 0 and 1 stand for* false *and* true *as in the classical case. We use uppercase boldfaced letters* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *etc. to denote algebras in this language (with universes* A, B, C *etc.). We denote by* $\mathbf{Fm}$ *the formula algebra (with universe* $Fm = \{\varphi, \psi, \ldots\}$*) freely generated by a countable set of variables* $Var = \{p, q, r, \ldots, x, y, z, \ldots\}$*. We use* $\approx, \Rightarrow, \&$ *as formal symbols for, respectively,*

[1] A previously unknown extension has recently been introduced in Pietz and Rivieccio (2011), to which we refer for a discussion of the usefulness and potential applications of this new logic.

the equality relation, implication and conjunction in the (first-order) language that we use to speak about algebras. We will also abbreviate the equation $\varphi \approx \varphi \wedge \psi$ *as* $\varphi \preceq \psi$.

Definition 2.1. *A* De Morgan lattice *is an algebra* $\mathbf{A} = \langle A, \wedge, \vee, \neg \rangle$ *of type* $\langle 2, 2, 1 \rangle$ *such that* $\langle A, \wedge, \vee \rangle$ *is a distributive lattice and the following equations are satisfied:*

$$\neg(x \wedge y) \approx \neg x \vee \neg y \qquad (\mathbf{neg\,1})$$

$$\neg(x \vee y) \approx \neg x \wedge \neg y \qquad (\mathbf{neg\,2})$$

$$x \approx \neg\neg x. \qquad (\mathbf{neg\,3})$$

A De Morgan algebra *is an algebra* $\mathbf{A} = \langle A, \wedge, \vee, \neg, 0, 1 \rangle$ *of type* $\langle 2, 2, 1, 0, 0 \rangle$ *such that* $\langle A, \wedge, \vee, \neg \rangle$ *is a De Morgan lattice and* $\langle A, \wedge, \vee, 0, 1 \rangle$ *is a bounded lattice. A* Kleene lattice (algebra) *is a De Morgan lattice (algebra) that additionally satisfies the following equation:*

$$x \wedge \neg x \preceq y \vee \neg y. \qquad (\mathbf{Klee})$$

($\mathbf{neg\,1}$) *and* ($\mathbf{neg\,2}$) *are usually called* De Morgan laws*, while* ($\mathbf{neg\,3}$) *is referred to as* double negation law *(or* involutivity of negation*).*

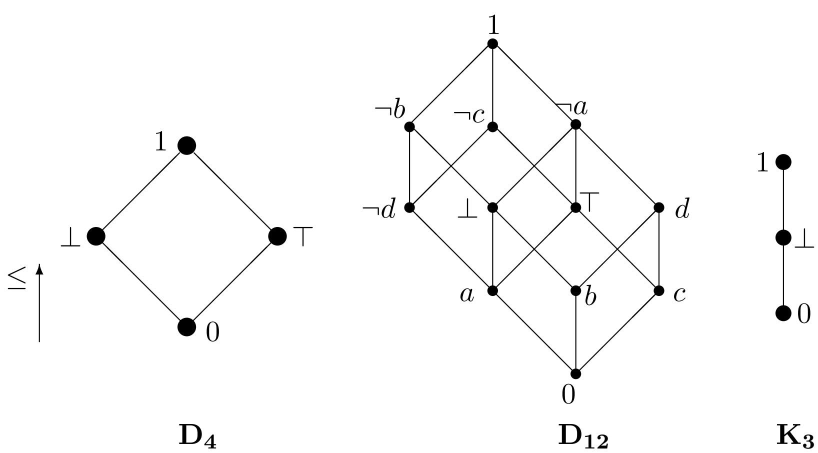

Figure 1. Some De Morgan lattices.

Figure 1 shows the Hasse diagram of some De Morgan lattices. The lattice operations in these algebras are determined by the diagram, while the behaviour of negation is indicated by the names of the elements (in all cases $0 = \neg 1$ and $1 = \neg 0$). Notice also that we denote by $\perp$ and $\top$ the elements that are fixed points of negation.

As mentioned above, Boolean algebras are a particular example of Kleene lattices (or Kleene algebras, if we include the constants in the signature), which can be axiomatized for instance by adding the following equation: $x \wedge \neg x \preceq y$. Both the class of De Morgan lattices (denoted DMLat) and of De Morgan algebras (denoted DMAlg) are varieties. Also, both classes have only two proper non-trivial subvarieties, namely (bounded) Kleene lattices (KLat) and Boolean algebras.

Up to algebraic language, bounded De Morgan lattices coincide thus with De Morgan algebras. However, the choice of the algebraic language makes a substantial difference when we look at the theory of sub-quasi-varieties of these classes.

A. P. Pynko (1999b) showed that De Morgan lattices have only four proper non-trivial sub-quasi-varieties (which are not varieties). On the other hand, the lattice of sub-quasi-varieties of Kleene algebras has the cardinality of the continuum Gaitán and Perea (2004). Thus, the lattice of sub-quasi-varieties of De Morgan algebras has at least the cardinality of the continuum. In particular, (Gaitán & Perea, 2004, Section 3) showed that any quasi-equation β_n of the following form

$$x_1 \preceq \neg x_1 \ \ \& \ \ \ldots \ \ \& \ \ x_n \preceq \neg x_n \ \ \& \ \ x_1 \vee \ldots \vee x_n \approx 1 \quad \Rightarrow \quad x \approx y \qquad (\beta_n)$$

for any natural number $n \geq 1$ defines a proper sub-quasi-variety of De Morgan algebras and these quasi-varieties are all distinct. The quasi-variety defined by the infinite set of quasi-equations (β_n) for all $n \geq 1$ is precisely $\mathbb{Q}(\mathbf{D_{12}})$, the quasi-variety generated by the De Morgan algebra $\mathbf{D_{12}}$ shown in Figure 1. Gaitán and Perea proved that $\mathbb{Q}(\mathbf{D_{12}})$ is not finitely based, i.e., it cannot be axiomatized by a finite set of quasi-equations. For our purposes, the following fact will be important:

Lemma 2.2. *Within the variety of De Morgan algebras, the quasi-equation (β_n) is equivalent, for any $n \geq 1$, to the following one:*

$$(x_1 \wedge \neg x_1) \vee \ldots \vee (x_n \wedge \neg x_n) \approx 1 \quad \Rightarrow \quad x \approx 1.$$

Proof: It is obvious that a De Morgan algebra $\mathbf{A}$ satisfies the equation $x \approx y$ if and only if $\mathbf{A}$ satisfies $x \approx 1$ if and only if $\mathbf{A}$ is trivial. Now suppose a $\mathbf{A}$ satisfies (β_n) for some $n \geq 1$ and there are elements $a_1, \ldots, a_n \in A$ such that $(a_1 \wedge \neg a_1) \vee \ldots \vee (a_n \wedge \neg a_n) = 1$. Then, letting $b_i := a_i \wedge \neg a_i$ for $1 \leq i \leq n$, we have that $b_i \leq \neg b_i$ for each $1 \leq i \leq n$ and $b_1 \vee \ldots \vee b_n = 1$. Thus, the premisses of (β_n) are satisfied. Then $a = b$ for any $a, b \in A$, which means that $\mathbf{A}$ is trivial.
Conversely, assume $\mathbf{A}$ satisfies the quasi-equation stated in the Lemma and there are elements $a_1, \ldots, a_n \in A$ such that $a_i \leq \neg a_i$ for each $1 \leq i \leq n$ and $a_1 \vee \ldots \vee a_n = 1$. Then $a_i = a_i \wedge \neg a_i$ for any $1 \leq i \leq n$. Hence,

$$a_1 \vee \ldots \vee a_n = (a_1 \wedge \neg a_1) \vee \ldots \vee (a_n \wedge \neg a_n) = 1.$$

Then, by hypothesis, we can conclude that $\mathbf{A} \vDash x \approx 1$, which implies that $\mathbf{A}$ is trivial.

Let us notice that the quasi-equation shown in Lemma 2.2 has a clearer logical flavor. In fact, we will see that the equivalence established by the lemma will allow us to associate to each of the above-mentioned quasi-varieties of De Morgan algebras an extension of the Belnap-Dunn logic.

3. The Belnap-Dunn logic

In this section we recall some known results on the Belnap-Dunn logic and highlight its relation with De Morgan lattices.

Let us introduce some logical terminology and general results that we will need (see Font and Jansana (2009) for proofs and further details). By a *(sentential) logic* we mean a pair $\mathcal{L} = \langle \mathbf{Fm}, \vdash_{\mathcal{L}} \rangle$, where $\mathbf{Fm}$ is the formula algebra of our similarity type and $\vdash_{\mathcal{L}}$ is a structural closure operator over the set Fm. Notice that we use the symbol $\vdash$ to denote any operator of this kind, independently of the way it is defined (syntactical, semantical etc.). Instead, we will use $\vDash_{\mathsf{K}}$ to refer to the equational consequence associated with a class of algebras K. We say that a logic $\mathcal{L}'$ is an *extension* of (or: is stronger than) $\mathcal{L}$ when, for all $\Gamma \cup \{\varphi\}$, if $\Gamma \vdash_{\mathcal{L}} \varphi$, then

$\Gamma \vdash_{\mathcal{L}'} \varphi$. We abbreviate this as $\mathcal{L} \leq \mathcal{L}'$ (we write $\mathcal{L} < \mathcal{L}'$ to indicate that moreover the two logics do not coincide).

We consider logical matrices as models of logics, by a *logical matrix* meaning a pair $\langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is an algebra and $D \subseteq A$ is a set of designated elements. A matrix is a *model* of a logic $\mathcal{L}$ when $\Gamma \vdash_{\mathcal{L}} \varphi$ implies that, for any valuation h (i.e., for any homomorphism $h \colon \mathbf{Fm} \to \mathbf{A}$), if $h(\gamma) \in D$ for all $\gamma \in \Gamma$, then $h(\varphi) \in D$. In particular, this implies that D is closed under any rule

$$\frac{\varphi_1, \ldots, \varphi_n}{\varphi}$$

of $\mathcal{L}$, i.e., for all valuations h, if $h(\varphi_i) \in D$ with $1 \leq i \leq n$, then $h(\psi) \in D$.

Conversely, any matrix $\langle \mathbf{A}, D \rangle$ determines a logic $\vdash_{\langle \mathbf{A}, D \rangle}$ by defining $\Gamma \vdash_{\langle \mathbf{A}, D \rangle} \varphi$ if and only if, for all homomorphisms $h \colon \mathbf{Fm} \to \mathbf{A}$, $h(\gamma) \in D$ for all $\gamma \in \Gamma$ implies $h(\varphi) \in D$. In the same way any family of matrices $M = \{\langle \mathbf{A}_i, D_i \rangle : i \in I\}$ defines a logic by letting $\Gamma \vdash_M \varphi$ iff $\Gamma \vdash_{\langle \mathbf{A}_i, D_i \rangle} \varphi$ for any $i \in I$. A remarkable fact, which we will use in the next section, is that *any logic $\mathcal{L}$ defined by a finite family of finite matrices is finitary* (this result can be found, for instance, in Wójcicki (1988)). By *finitary* we mean that, if $\Gamma \vdash_{\mathcal{L}} \varphi$, then there is a finite set $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathcal{L}} \varphi$.

A *matrix congruence* of a matrix $\langle \mathbf{A}, D \rangle$ is a congruence of $\mathbf{A}$ such that whenever two elements $a, b \in A$ are related and $a \in D$, then $b \in D$ as well. Any matrix has a greatest matrix congruence; we say that a matrix is *reduced* when it has just one matrix congruence (which needs to be the identity).

Any logic $\mathcal{L}$ is complete with respect to the class of all its matrix models, in the sense that $\Gamma \vdash_{\mathcal{L}} \varphi$ iff, for any matrix $\langle \mathbf{A}, D \rangle$ of $\mathcal{L}$, $h(\gamma) \in D$ for all $\gamma \in \Gamma$ implies $h(\varphi) \in D$. More interestingly, it is known that *any logic is complete (in the above sense) with respect to the class of its reduced matrix models.* This implies in particular that, when trying to disprove something, it is sufficient to look at reduced models. In fact, if $\Gamma \nvdash_{\mathcal{L}} \varphi$, then there must be some reduced matrix $\langle \mathbf{A}, D \rangle$ for $\mathcal{L}$ and some valuation h such that $h(\gamma) \in D$ for all $\gamma \in \Gamma$ but $h(\varphi) \notin D$.

This last fact will play an important role in our approach to the extensions of the Belnap-Dunn logic. Another straightforward but very useful result is the following: for any logics $\mathcal{L}, \mathcal{L}'$ such that $\mathcal{L} \leq \mathcal{L}'$, it holds that $\mathbf{Mod}^*(\mathcal{L}') \subseteq \mathbf{Mod}^*(\mathcal{L})$, where $\mathbf{Mod}^*(\mathcal{L})$ and $\mathbf{Mod}^*(\mathcal{L}')$ denote the classes of reduced matrix models of, respectively, $\mathcal{L}$ and $\mathcal{L}'$ (the same obviously holds for non-reduced models).

Using the above terminology, we can introduce the Belnap-Dunn four-valued logic (which we will denote by $\mathcal{B}$) as the logic defined by the logical matrix $\langle \mathbf{D_4}, \{1, \top\} \rangle$ or, equivalently (Font, 1997, Proposition 2.3), by the matrix $\langle \mathbf{D_4}, \{1, \bot\} \rangle$, where $\mathbf{D_4}$ is the four-element De Morgan lattice shown in Figure 1. It can also be proved that $\mathcal{B}$ is the logic determined by the class of all matrices of the form $\langle \mathbf{A}, D \rangle$ where $\mathbf{A} \in \mathsf{DMLat}$ and $D \subseteq A$ is a lattice filter of $\mathbf{A}$ or is empty (Font, 1997, Corollary 2.6).

It follows from the result mentioned above that $\mathcal{B}$ is finitary. It is also easy to show that the Belnap-Dunn logic has no theorems, i.e., there is no formula φ such that $\emptyset \vdash_{\mathcal{B}} \varphi$.

Font (1997) proved that the Belnap-Dunn logic is axiomatized by the calculus shown in Table 1. It is obvious that all rules of Table 1 are also also rules of classical logic: thus, the classical propositional calculus is an extension of the Belnap-Dunn logic. It is also easy to show by algebraic means that any extension of the Belnap-Dunn logic must at the same time be a weakening of classical logic.

Notice that, because of the absence of theorems, any calculus for the Belnap-Dunn logic will have only proper rules and no axioms. The fact that $\mathcal{B}$ is a theoremless logic has a crucial consequence for our purposes: it implies that *the Belnap-*

$$\text{(R1)}\ \frac{p \wedge q}{p} \qquad \text{(R2)}\ \frac{p \wedge q}{q} \qquad \text{(R3)}\ \frac{p \qquad q}{p \wedge q}$$

$$\text{(R4)}\ \frac{p}{p \vee q} \qquad \text{(R5)}\ \frac{p \vee q}{q \vee p} \qquad \text{(R6)}\ \frac{p \vee p}{p}$$

$$\text{(R7)}\ \frac{p \vee (q \vee r)}{(p \vee q) \vee r} \qquad \text{(R8)}\ \frac{p \vee (q \wedge r)}{(p \vee q) \wedge (p \vee r)} \qquad \text{(R9)}\ \frac{(p \vee q) \wedge (p \vee r)}{p \vee (q \wedge r)}$$

$$\text{(R10)}\ \frac{p \vee r}{\neg\neg p \vee r} \qquad \text{(R11)}\ \frac{\neg\neg p \vee r}{p \vee r} \qquad \text{(R12)}\ \frac{\neg(p \vee q) \vee r}{(\neg p \wedge \neg q) \vee r}$$

$$\text{(R13)}\ \frac{(\neg p \wedge \neg q) \vee r}{\neg(p \vee q) \vee r} \qquad \text{(R14)}\ \frac{\neg(p \wedge q) \vee r}{(\neg p \vee \neg q) \vee r} \qquad \text{(R15)}\ \frac{(\neg p \vee \neg q) \vee r}{\neg(p \wedge q) \vee r}$$

Table 1. A calculus for the Belnap-Dunn logic.

Dunn logic is not algebraizable in the sense of Blok and Pigozzi (1989). In fact, it is not even protoalgebraic (Font, 1997, Theorem 2.11).

This means that we cannot employ a well-known result of the theory of algebraization of logics, namely that the lattice of extensions of an algebraizable logic is isomorphic to the lattice of sub-quasi-varieties of its algebraic counterpart. This is one of the major difficulties in the study of extensions of the Belnap-Dunn logic.

It follows then that $\mathcal{B}$ cannot have an *equivalent* algebraic semantics in the sense of (Blok & Pigozzi, 1989, Definition 2.8). However, an algebraic semantics (Blok & Pigozzi, 1989, Definition 2.2) for this logic is provided by the class of De Morgan lattices, as shown by the following result (Font, 1997, Proposition 2.5).

Proposition 3.1. *For any $\varphi_1, \ldots, \varphi_n, \varphi \in Fm$, the following are equivalent:*

(i) $\{\varphi_1, \ldots, \varphi_n\} \vDash_{\mathcal{B}} \varphi$
(ii) $\mathbf{D_4} \vDash \varphi_1 \wedge \ldots \wedge \varphi_n \preceq \varphi$
(iii) $\mathsf{DMLat} \vDash \varphi_1 \wedge \ldots \wedge \varphi_n \preceq \varphi$.

The equivalence between (i) and (iii) might be paraphrased by saying that the Belnap-Dunn logic is the logic of the lattice order of De Morgan lattices. In fact, (Font, 1997, Theorem 4.1) showed that the class of De Morgan lattices is the algebraic counterpart of the Belnap-Dunn logic according to the criteria of Font and Jansana (2009). It may be also interesting to note that the De Morgan lattices are not the equivalent algebraic semantics of any algebraizable logic (Font, 1997, Proposition 2.12).

For our study of extensions of the Belnap-Dunn logic, the following characterization of reduced models of $\mathcal{B}$ will be especially useful (Font, 1997, Theorem 3.14).

Theorem 3.2. *For any non-trivial algebra $\mathbf{A}$, the matrix $\langle \mathbf{A}, D \rangle$ is a reduced model of $\mathcal{B}$ if and only if $\mathbf{A} \in \mathsf{DMLat}$ and D is a lattice filter such that, for all $a, b \in A$ with $a < b$, there is $c \in A$ such that either (i) or (ii) holds:*

(i) $a \vee c \notin D$ *and* $b \vee c \in D$ (DP)
(ii) $\neg a \vee c \in D$ *and* $\neg b \vee c \notin D$.

The importance of this result can be easily seen if we take into account the fact that, as mentioned above, any (reduced) model of any extension of the Belnap-

Dunn logic will also be a (reduced) model of the Belnap-Dunn logic.

We will call the property defined in Theorem 3.2 *disjunction property*, abbreviated (DP). The term has already been used in the literature for a similar separation property of lattices (without negation): see, for instance, Wallman (1938), Cignoli (1991). Notice that the assumption that $a < b$ can be replaced by the weaker requirement that $a \neq b$. In fact, if a and b are incomparable and $a \neq b$, then $a \wedge b < b$, so by (DP) we have that there is c such that either $(a \wedge b) \vee c \notin D$ and $b \vee c \in D$ or $\neg(a \wedge b) \vee c \in D$ and $\neg b \vee c \notin D$. In the first case, using distributivity, we have that $(a \vee c) \wedge (b \vee c) \notin D$ and since $b \vee c \in D$ and D is a lattice filter, we conclude that $a \vee c \notin D$. In the second case we use De Morgan laws to obtain $\neg a \vee \neg b \vee c \in D$, so defining $c' = \neg b \vee c$ we have that $\neg a \vee c' \in D$ and $\neg b \vee c' \notin D$.

4. Extensions of the Belnap-Dunn logic

We are now going to look at extensions of $\mathcal{B}$. We begin by briefly reviewing the ones that have already been studied in the literature.

4.1 *The known*

Kleene's strong three-valued logic $\mathcal{K}$ Kleene (1950) is the logic defined by the matrix $\langle \mathbf{K_3}, \{1\} \rangle$, where $\mathbf{K_3}$ is the three-element Kleene lattice shown in Figure 1. It is easy to check that this logic has no theorems as well. However, $\mathcal{K}$ validates the following rule

$$\frac{(p \wedge \neg p) \vee q}{q}$$

which is not sound w.r.t. the semantics of the Belnap-Dunn logic. In fact, it is possible to prove that $\mathcal{K}$ can be axiomatized by adding this rule to those of Table 1.

Kleene's logic of order $\mathcal{K}_{\leq}$ considered in Font (1997) is the logic that corresponds to the lattice order of Kleene lattices. It can be defined by the set of matrices $\{\langle \mathbf{K_3}, \{1\} \rangle, \langle \mathbf{K_3}, \{1, \bot\} \rangle\}$. The name "logic of order" is justified by the following property (Cf. Proposition 3.1):

Proposition 4.1. *For any $\varphi_1, \ldots, \varphi_n, \varphi \in Fm$, the following are equivalent:*

(i) $\{\varphi_1, \ldots, \varphi_n\} \vDash_{\mathcal{K}_{\leq}} \varphi$
(ii) $\mathbf{K_3} \vDash \varphi_1 \wedge \ldots \wedge \varphi_n \preceq \varphi$
(iii) $\mathsf{KLat} \vDash \varphi_1 \wedge \ldots \wedge \varphi_n \preceq \varphi$.

It is claimed in (Font, 1997, Section 5.1) that $\mathcal{K}_{\leq}$ can be axiomatized by adding the following rule to those of Table 1:

$$\frac{p \wedge \neg p}{q \vee \neg q}$$

This is incorrect, as we will see in the next section. It is not difficult to prove that in order to obtain an axiomatization for $\mathcal{K}_{\leq}$ the following stronger rule should be added instead:

$$\frac{(p \wedge \neg p) \vee r}{(q \vee \neg q) \vee r}$$

It is also easy to check that $\mathcal{K}_{\leq} < \mathcal{K}$.

Priest's logic of paradox $\mathcal{LP}$ Priest (1979) is the logic defined by the single matrix $\langle \mathbf{K_3}, \{1, \bot\}\rangle\}$. Unlike the two above-mentioned Kleene's logics, $\mathcal{LP}$ is an axiomatic extension of the Belnap-Dunn logic. Thus, it has theorems. For instance, it holds that

$$\vdash_{\mathcal{LP}} \; p \vee \neg p.$$

In fact, it is proved in Pynko (1995b) that $\mathcal{LP}$ can be axiomatized by adding the axiom $p \vee \neg p$ to the rules of Table 1. It is also obvious from its semantic definition that $\mathcal{K}_{\leq} < \mathcal{LP}$.

The three logics mentioned above are, as far as the author is aware of, the only extensions of the Belnap-Dunn logic that have been considered so far in the literature (besides, of course, classical logic, which we denote by $\mathcal{CL}$). The inclusion relations among these logics are displayed in Figure 2. In the next section we look at the yet unexplored extensions.

$$\begin{tikzcd}
 & \mathcal{CL} & \\
\mathcal{LP} \arrow[ur] & & \mathcal{K} \arrow[ul] \\
 & \mathcal{K}_{\leq} \arrow[ul] \arrow[ur] & \\
 & \mathcal{B} \arrow[u] &
\end{tikzcd}$$

Figure 2. Known extensions of the Belnap-Dunn logic.

4.2 *The unknown*

This section contains the main new results of the paper. Before entering into the details, let us try to give an idea of the strategy we have followed and to sketch an overall picture of the (rather complicated, as it seems) landscape of extensions of the Belnap-Dunn logic.

As mentioned above, any logic $\mathcal{L}$ is determined by the class of its reduced matrix models, denoted $\mathbf{Mod}^*(\mathcal{L})$. We also have that if $\mathcal{L} \geq \mathcal{B}$, then $\mathbf{Mod}^*(\mathcal{L}) \subseteq \mathbf{Mod}^*(\mathcal{B})$. Conversely, any subclass of $\mathbf{Mod}^*(\mathcal{B})$ defines an extension of the Belnap-Dunn logic. For instance, in the previous section we have seen that, using two different matrices based on the same algebra $\mathbf{K_3}$, it is possible to define three new logics. Notice that both matrices $\langle \mathbf{K_3}, \{1\}\rangle$ and $\langle \mathbf{K_3}, \{1, \bot\}\rangle$ are in fact reduced, because $\mathbf{K_3}$ is a simple algebra. So both matrices belong to $\mathbf{Mod}^*(\mathcal{B})$.

It is certainly true that many different subfamilies of $\mathbf{Mod}^*(\mathcal{B})$ will define the same logic: for instance we have seen that the consequence determined by the whole class $\mathbf{Mod}^*(\mathcal{B})$ coincides with that of the single matrix $\langle \mathbf{D_4}, \{1, \top\}\rangle$. However, it is possible to show that in many cases (at least countably many, as we will see) the logics defined by different subclasses do not coincide. Unfortunately, we do not know of any general procedure for establishing whether two classes of matrices define the same logic or not. This partly explains the difficulty (and also the interest) of the problem we are facing.

Let us begin by defining a logic, denoted $\mathcal{B}_1$, that we will later use to construct a countable chain $\mathcal{B} < \mathcal{B}_1 < \mathcal{B}_2 < \mathcal{B}_3 \ldots$ of extensions of $\mathcal{B}$.

Definition 4.2. *Let $\mathcal{B}_1$ be the extension of $\mathcal{B}$ obtained by adding the following rule (that we call* disjunctive syllogism*) to the calculus of Table 1:*

$$\frac{p \wedge (\neg p \vee q)}{q} \tag{DS}$$

It is easy to check that (DS) is not sound w.r.t. the semantics of the Belnap-Dunn logic, therefore $\mathcal{B}_1$ is indeed a proper extension of $\mathcal{B}$. Moreover, we will see that $\mathcal{B}_1$ does not coincide with classical logic nor, as far as the author is aware, with any known logic [1]. It is also easy to check that (DS) is satisfied by the matrix $\langle \mathbf{D_4}, \{1\} \rangle$, therefore $\mathcal{B}_1$ is weaker than the logic determined by $\langle \mathbf{D_4}, \{1\} \rangle$. In fact, we are going to prove that $\mathcal{B}_1$ is exactly the logic of the matrix $\langle \mathbf{D_4}, \{1\} \rangle$. We will need a few lemmas.

Proposition 4.3. *For any $\varphi_1, \ldots, \varphi_n, \varphi \in Fm$, the following are equivalent:*

(i) $\{\varphi_1, \ldots, \varphi_n\} \vDash_{\langle \mathbf{D_4}, \{1\} \rangle} \varphi$
(ii) $\mathbf{D_4} \vDash \varphi_1 \wedge \ldots \wedge \varphi_n \preceq \neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \vee \varphi$
(iii) $\mathsf{DMLat} \vDash \varphi_1 \wedge \ldots \wedge \varphi_n \preceq \neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \vee \varphi$.

Proof: (i)$\Rightarrow$(ii). Let $h : \mathbf{Fm} \to \mathbf{D_4}$ be a valuation and let us abbreviate $\psi := \varphi_1 \wedge \ldots \wedge \varphi_n$. Assume (i). If $h(\psi) = 0$, we are done. If $h(\psi) = \bot$, then $h(\neg\psi) = \bot$ as well and obviously $\bot \leq \bot \vee a$ for any $a \in D_4$. The same reasoning applies to the case where $h(\psi) = \top$. Finally, if $h(\psi) = 1$, then by (i) $h(\varphi) = 1$ as well and since $1 \leq \neg 1 \vee 1 = 1$, we are done.
(ii)$\Rightarrow$(i). Assume $h(\varphi_i) = 1$ for $1 \leq i \leq n$. Then $h(\psi) = 1$, so by (ii) we have $1 \leq \neg 1 \vee h(\varphi) = h(\varphi)$, which obviously implies $h(\varphi) = 1$.
(ii)$\Leftrightarrow$(iii). Follows from the fact that $\mathbf{D_4}$ generates the variety of De Morgan lattices.

As mentioned above, the logic determined by any finite matrix is finitary. This implies that the previous proposition characterizes the consequence relation determined by $\langle \mathbf{D_4}, \{1\} \rangle$ in full generality and not only for finite sets of premisses. Proposition 4.3 also allows us to obtain the following completeness result.

Theorem 4.4. *$\mathcal{B}_1$ is the logic determined by the matrix $\langle \mathbf{D_4}, \{1\} \rangle$.*

Proof: We have mentioned earlier that $\mathcal{B}_1$ is weaker than the logic of the matrix $\langle \mathbf{D_4}, \{1\} \rangle$. Hence, to obtain the desired result it will be sufficient to prove that $\mathcal{B}_1$ is also stronger than the logic determined by $\langle \mathbf{D_4}, \{1\} \rangle$, which implies that the two consequence relations coincide. To see this, suppose $\Gamma \nvdash_{\mathcal{B}_1} \psi$ but $\Gamma \vDash_{\langle \mathbf{D_4}, \{1\} \rangle} \psi$. As mentioned above, we can without loss of generality assume that Γ is finite because both logics are finitary ($\mathcal{B}_1$ is finitary because it is defined by a finite set of finite rules). Let then $\Gamma := \{\varphi_1, \ldots, \varphi_n\}$ and $\varphi := \varphi_1 \wedge \ldots \wedge \varphi_n$. The assumptions imply that there is some reduced model $\langle \mathbf{A}, D \rangle$ of $\mathcal{B}_1$ and a valuation h such that $h(\varphi_i) \in D$ for all $1 \leq i \leq n$ and $h(\psi) \notin D$. Since $\mathcal{B}_1$ is an extension of $\mathcal{B}$, $\langle \mathbf{A}, D \rangle$ is a model of $\mathcal{B}$ as well. As proved in (Font, 1997, Theorem 3.14), this implies that $\mathbf{A}$ is a De Morgan lattice and D is a lattice filter. Moreover, by assumption D is closed under (DS). Thus we have that $h(\varphi_1) \wedge \ldots \wedge h(\varphi_n) = h(\varphi_1 \wedge \ldots \wedge \varphi_n) = h(\varphi) \in D$. By Proposition 4.3, $\{\varphi_1, \ldots, \varphi_n\} \vDash_{\langle \mathbf{D_4}, \{1\} \rangle} \psi$ implies that the equation $\varphi \preceq \neg\varphi \vee \psi$

[1] The logic $\mathcal{B}_1$ was introduced in Pietz and Rivieccio (2011), with a different motivation, under the name *ETL*. The completeness proof presented here (Theorem 4.4) is adapted from (Pietz & Rivieccio, 2011, Theorem 3.4).

is valid in any De Morgan lattice. Then $h(\varphi) \leq \neg h(\varphi) \vee h(\psi)$ and, since D is an up-set w.r.t. to the lattice order, we have that $\neg h(\varphi) \vee h(\psi) \in D$. But D is closed under (DS), so we should have $h(\psi) \in D$, which is against the hypothesis.

It is now easy to check that $\mathcal{B}_1$ does not coincide with classical logic, for $\mathcal{B}_1$ has no valid formulas. To see this, it suffices to note that the constant valuation $h \colon \mathbf{Fm} \to \mathbf{D_4}$ defined by $h(p) = \bot$ for all $p \in Var$ is a homomorphism. Therefore, there is no formula φ such that $h(\varphi) = 1$ for all valuations $h \colon \mathbf{Fm} \to \mathbf{D_4}$, which means that there is no φ such that $\vdash_{\mathcal{B}_1} \varphi$.

The previous argument can be generalized as follows. Let $\langle \mathbf{A}, D \rangle$ be a matrix such that $\mathbf{A} \in \mathsf{DMLat}$ and there is an element $a \in A$ such that $a = \neg a \notin D$. Then $\{a\}$ is the universe of a subalgebra of $\mathbf{A}$, therefore the constant map $h(p) = a$ is a homomorphism from $\mathbf{Fm}$ to $\mathbf{A}$. As a consequence, since $a \notin D$, there cannot be any formula φ such that $h(\varphi) \in D$ for all valuations h. Hence, any logic having $\langle \mathbf{A}, D \rangle$ among its models (and in particular the logic determined by $\langle \mathbf{A}, D \rangle$ itself) has no valid formulas.

The above reasoning proves that both Kleene's logics considered in Section 4.1 are also theoremless. It also implies that $\mathcal{B}_1$ does not coincide with Priest's logic of paradox, because $\mathcal{LP}$ does have theorems.

It is easy to check that the rule

$$\frac{p \wedge \neg p}{q \vee \neg q} \tag{1}$$

mentioned in Section 4.1 is satisfied by the matrix $\langle \mathbf{D_4}, \{1\} \rangle$. Thus, it is a rule of $\mathcal{B}_1$. However, the rule

$$\frac{(p \wedge \neg p) \vee r}{(q \vee \neg q) \vee r} \tag{2}$$

fails in $\langle \mathbf{D_4}, \{1\} \rangle$. Thus, (2) does not follow from (1) together with the rules of Table 1. Since (2) is valid in $\mathcal{K}_{\leq}$ (this can be directly checked in the matrices $\langle \mathbf{K_3}, \{1\} \rangle$ and $\langle \mathbf{K_3}, \{1, \bot\} \rangle$ or using Proposition 4.1), we may conclude that the axiomatization of $\mathcal{K}_{\leq}$ proposed in (Font, 1997, Section 5.1) is not correct.

Next we give a characterization of reduced models of $\mathcal{B}_1$ that will prove to be particularly useful for our purposes.

Proposition 4.5. *Let $\langle \mathbf{A}, D \rangle$ be a reduced model of $\mathcal{B}_1$ with $\mathbf{A}$ non-trivial. Then $\mathbf{A}$ is a bounded De Morgan lattice and $D = \{1\}$.*

Proof: Since $\langle \mathbf{A}, D \rangle$ is a reduced model of the Belnap-Dunn logic, we already know that $\mathbf{A}$ is a De Morgan lattice and D is a lattice filter. It will be then sufficient to prove that D is a singleton. Reasoning by contradiction, suppose there are $a, b \in D$ such that $a \neq b$. We may assume that $a < b$, otherwise we could take $a \wedge b$ and $a \vee b$ (both belong to D since it is a lattice filter). By the disjunction property (DP), there must be $c \in A$ such that either $b \vee c \in D$ and $a \vee c \notin D$ or $\neg a \vee c \in D$ and $\neg b \vee c \notin D$. Since $a \in D$, it is obviously impossible that $a \vee c \notin D$. It follows that $\neg a \vee c \in D$ and $\neg b \vee c \notin D$. Using the fact that D is a lattice filter, we can conclude that $a \wedge (\neg a \vee c) \in D$. Now observe that $a \wedge (\neg a \vee c) \leq \neg(a \wedge (\neg a \vee c)) \vee \neg b \vee c.$. This is so because, by De Morgan and double negation laws, we have that

$$\neg(a \wedge (\neg a \vee c)) \vee \neg b \vee c = \neg a \vee (a \wedge \neg c) \vee \neg b \vee c \geq \neg a \vee c$$

while obviously $a \wedge (\neg a \vee c) \leq \neg a \vee c$. Now, since D is closed under (DS), we should have $\neg b \vee c \in D$, which contradicts the hypothesis. So $a = b$ and, since D is a lattice filter, a must be the top element of $\mathbf{A}$.

From the previous result together with Theorem 3.2 we immediately obtain the following.

Theorem 4.6. *Let $\langle \mathbf{A}, D \rangle$ be a matrix, with $\mathbf{A}$ non-trivial. Then $\langle \mathbf{A}, D \rangle$ is a reduced model of $\mathcal{B}_1$ if and only if:*

(i) *$\mathbf{A}$ is bounded De Morgan lattice*
(ii) *$D = \{1\}$*
(iii) *$\langle \mathbf{A}, \{1\} \rangle$ satisfies the disjunction property (DP).*

It is easy to check that, for any bounded De Morgan lattice $\mathbf{A}$, the matrix $\langle \mathbf{A}, \{1\} \rangle$ is a model of $\mathcal{B}_1$. While these models are not necessarily reduced, Theorem 4.6 tells us that all the reduced models of $\mathcal{B}_1$ have this form.

Taken together, these facts imply that $\mathcal{B}_1$ *is complete with respect to the class of all matrices of the form $\langle \mathbf{A}, \{1\} \rangle$ where $\mathbf{A}$ is a bounded De Morgan lattice and 1 is the top element of* $\mathbf{A}$.

The main relevance of Theorem 4.6, as we will see, comes from the fact that a reduced matrix $\langle \mathbf{A}, \{1\} \rangle$ for $\mathcal{B}_1$ satisfies a rule of the form

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

if and only if the De Morgan algebra $\mathbf{A}$ satisfies the quasi-equation

$$\varphi_1 \approx 1 \quad \& \quad \ldots \quad \& \quad \varphi_n \approx 1 \quad \Rightarrow \quad \psi \approx 1.$$

Thus, we have a way of translating any logical rule into a quasi-equation in the language of De Morgan algebras (notice that we had to include the constants 0 and 1 in the algebraic language).

Using the completeness result of Theorem 4.4, it is easy to check that $\mathcal{B}_1$ satisfies the following rule, sometimes called (ECQ) for *ex contradictione quodlibet*:

$$\frac{p \wedge \neg p}{q} \tag{ECQ}$$

This implies that $\mathcal{B}_1$, unlike the Belnap-Dunn logic, is not paraconsistent in the usual sense. Neither does $\mathcal{B}_1$ belong to the family of relevant logics, as in (ECQ) there is no relation whatsoever between the premiss and the conclusion.

It is not difficult to see that (ECQ) does not imply (DS). Consider, for instance, the matrix $\langle \mathbf{D_{12}}, \{1, \neg a\} \rangle$, where $\mathbf{D_{12}}$ is the twelve-element De Morgan algebra shown in Figure 1. It is easy to check that $\langle \mathbf{D_{12}}, \{1, \neg a\} \rangle$ satisfies (ECQ). On the other hand, (DS) can fail, as shown by the following example. Let $h \colon \mathbf{Fm} \to \mathbf{D_{12}}$ be a valuation such that $h(p) = \neg a$ and $h(q) = d$. Then $h(p \wedge (\neg p \vee q)) = \neg a \wedge (\neg\neg a \vee d) = \neg a$ but $h(q) \notin \{1, \neg a\}$. Thus, $\mathcal{B}$ + (ECQ) is a new logic that is strictly weaker than $\mathcal{B}_1$. The problem of characterizing the (reduced) models of this logic has yet to be addressed, and the completeness results for $\mathcal{B}_1$ stated above seem to indicate that the strategy adopted in this paper will not applicable to $\mathcal{B}$ + (ECQ).

While we have seen that $\mathcal{B}_1$ satisfies (ECQ), it is easy to check that the following stronger explosion rule is not sound w.r.t. the semantics of $\mathcal{B}_1$:

$$\frac{(p_1 \wedge \neg p_1) \vee (p_2 \wedge \neg p_2)}{q} \tag{ECQ$_2$}$$

To see this, just consider a valuation $h\colon \mathbf{Fm} \to \mathbf{D_4}$ such that $h(p_1) = h(q) = \bot$ and $h(p_2) = \top$. On a logical level, the failure of (ECQ$_2$) implies that reasoning by cases is not possible in $\mathcal{B}_1$, because a disjunction of two formulas can take a designated value even if none of the values of the disjuncts is designated (further considerations on this unusual feature of $\mathcal{B}_1$ and also on alternative versions of (ECQ) of can be found in Pietz and Rivieccio (2011)).

So, if we add (ECQ$_2$) to our syntactical presentation of $\mathcal{B}_1$, we obtain a new logic, which we will denote by $\mathcal{B}_2$, such that $\mathcal{B}_1 < \mathcal{B}_2$. This procedure can be generalized in order to construct a denumerable chain of extensions of the Belnap-Dunn logic as follows.

Let $\mathcal{B}_n$ denote the logic obtained by adding the rule (ECQ$_n$) to $\mathcal{B}$, defined, for any natural number $n \geq 1$, as

$$\frac{(p_1 \wedge \neg p_1) \vee \ldots \vee (p_n \wedge \neg p_n)}{q} \tag{ECQ$_n$}$$

Then we are able to prove the following result.

Theorem 4.7. *There exists a denumerable chain of extensions of the Belnap-Dunn logic*

$$\mathcal{B} < \mathcal{B} + (\mathrm{ECQ}) < \mathcal{B}_1 < \mathcal{B}_2 < \mathcal{B}_3 < \ldots < \mathcal{B}_n < \ldots < \mathcal{B}_\infty < \mathcal{K}$$

such that $\mathcal{B}_n < \mathcal{B}_{n+1}$ for any $n \geq 1$. The chain has an upper bound (strictly weaker than $\mathcal{K}$) given by the logic $\mathcal{B}_\infty$ axiomatized by adding to $\mathcal{B}_1$ the infinite family of rules (ECQ$_n$) *for all $n \geq 1$.*

Proof: Let us consider the logic $\mathcal{B}_n$ for an arbitrary $n \geq 2$. The fact that $\mathcal{B}_1 \leq \mathcal{B}_n$ implies that $\mathbf{Mod}^*(\mathcal{B}_n) \subseteq \mathbf{Mod}^*(\mathcal{B}_1)$, i.e., any reduced matrix for $\mathcal{B}_n$ must have the form $\langle \mathbf{A}, \{1\} \rangle$ and satisfy the three properties stated in Theorem 4.6. Moreover, $\langle \mathbf{A}, \{1\} \rangle$ will satisfy (ECQ$_n$). This means that the De Morgan lattice $\mathbf{A}$ satisfies the quasi-equation

$$(x_1 \wedge \neg x_1) \vee (x_2 \wedge \neg x_2) \approx 1 \quad \Rightarrow \quad x \approx 1.$$

As mentioned in Section 2, the latter is equivalent to the quasi-equation (β_n) introduced in Gaitán and Perea (2004). Since $\mathbf{A}$ is a bounded De Morgan lattice, we can view it as a De Morgan algebra, and we have that $\mathbf{A} \in \mathbb{Q}_n$, where $\mathbb{Q}_n$ denotes the sub-quasi-variety of De Morgan algebras axiomatized by adding (β_n) to the equations that define De Morgan algebras.
Within the context of extensions of $\mathcal{B}$, it is easily proved that (ECQ$_n$) implies (ECQ$_{n-1}$) for all $n \geq 2$. We have thus a countable chain of logics:

$$\mathcal{B} < \mathcal{B} + (\mathrm{ECQ}) < \mathcal{B}_1 < \mathcal{B}_2 \leq \mathcal{B}_3 \leq \ldots \leq \mathcal{B}_n \leq \ldots$$

We have already seen that the first three inequalities are strict. Let us check that $\mathcal{B}_n < \mathcal{B}_{n+1}$ holds in general. (Gaitán & Perea, 2004, Section 3) proved that, for any quasi-equation β_n, there is a De Morgan algebra $\mathbf{A}_n$ such that $\mathbf{A}_n \vDash \beta_n$ but $\mathbf{A}_n \nvDash \beta_{n+1}$. We have then that the matrix $\langle \mathbf{A}_n, \{1\} \rangle$ satisfies ECQ$_n$ but does not satisfy ECQ$_{n+1}$. This means that $\langle \mathbf{A}_n, \{1\} \rangle$ is a model of $\mathcal{B}_n$ but not a model of $\mathcal{B}_{n+1}$. Hence, $\mathcal{B}_n \neq \mathcal{B}_{n+1}$. Thus, we have that

$$\mathcal{B} < \mathcal{B} + (\mathrm{ECQ}) < \mathcal{B}_1 < \mathcal{B}_2 < \mathcal{B}_3 < \ldots < \mathcal{B}_n < \ldots < \mathcal{B}_\infty$$

The above reasoning also implies that $\mathcal{B}_\infty$ does not coincide with $\mathcal{B}_n$ for any $n \geq 1$. In order to prove that $\mathcal{B}_\infty \leq \mathcal{K}$, it is sufficient to check that the matrix $\langle \mathbf{K_3}, \{1\} \rangle$, which defines $\mathcal{K}$, satisfies (ECQ$_n$) for all $n \geq 1$. To prove that $\mathcal{B}_\infty \neq \mathcal{K}$, just notice that the matrix $\langle \mathbf{D_{12}}, \{1\} \rangle$, where $\mathbf{D_{12}}$ is the twelve-element De Morgan algebra shown in Figure 1, is a (reduced) model of $\mathcal{B}_\infty$ but not of $\mathcal{K}$.

Theorem 4.7 allows us to draw an improved diagram of the inclusion relations among extensions of the Belnap-Dunn logic (Figure 3). Notice that, as shown by the diagram, all the logics between $\mathcal{B}$ + (ECQ) and $\mathcal{B}_\infty$ are incomparable with $\mathcal{K}_\leq$ and with $\mathcal{LP}$. This follows, on the one hand, from the fact that any logic weaker than $\mathcal{B}_\infty$ has some reduced model $\langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is a De Morgan lattice (hence, $\langle \mathbf{A}, D \rangle$ is not a reduced the model of $\mathcal{K}_\leq$ or $\mathcal{LP}$). On the other hand, neither $\mathcal{K}_\leq$ and $\mathcal{LP}$ satisfy (ECQ).

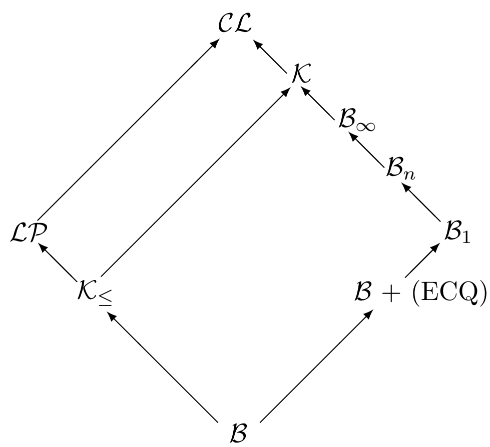

Figure 3. Some more extensions of the Belnap-Dunn logic.

We are now going to prove that $\mathcal{B}_\infty$ is precisely the logic determined by the matrix $\langle \mathbf{D_{12}}, \{1\} \rangle$, where $\mathbf{D_{12}}$ is the twelve-element De Morgan algebra shown in Figure 1. It is easy to check that $\langle \mathbf{D_{12}}, \{1\} \rangle$ is a model of $\mathcal{B}_\infty$. Thus, the logic defined by $\langle \mathbf{D_{12}}, \{1\} \rangle$ is stronger than $\mathcal{B}_\infty$. To prove the converse, we will use the following result (Gaitán & Perea, 2004, Theorem 3.5).

Theorem 4.8. *Let $\mathbf{A}$ be a De Morgan algebra and let $\mathbb{Q}(\mathbf{D_{12}})$ be the quasi-variety generated by $\mathbf{D_{12}}$. Then $\mathbf{A} \in \mathbb{Q}(\mathbf{D_{12}})$ if and only if $\mathbf{A}$ satisfies (β_n) for all $n \geq 1$.*

On a logical level, Theorem 4.8 has the following important consequence.

Lemma 4.9. *Let $\langle \mathbf{A}, \{1\} \rangle$ be a matrix such that $\mathbf{A} \in \mathsf{DMAlg}$. The following are equivalent:*

(i) $\langle \mathbf{A}, \{1\} \rangle$ *is a model of* $\mathcal{B}_\infty$
(ii) $\mathbf{A} \in \mathbb{Q}(\mathbf{D_{12}})$.

Proof: (i)⇒(ii). By (i), the matrix $\langle \mathbf{A}, \{1\} \rangle$ satisfies all the rules of $\mathcal{B}_\infty$, in particular, for all $n \geq 1$,

$$(\varphi_1 \wedge \neg\varphi_1) \vee \ldots \vee (\varphi_n \wedge \neg\varphi_n) \vdash \varphi.$$

Therefore $\mathbf{A}$ satisfies any quasi-equation of the form

$$\mathbf{A} \ \vDash \ (x_1 \wedge \neg x_1) \vee \ldots \vee (x_n \wedge \neg x_n) \approx 1 \quad \Rightarrow \quad x \approx 1$$

which, by Theorem 4.8, means that $\mathbf{A} \in \mathbb{Q}(\mathbf{D_{12}})$.
(ii)⇒(i). We already know that, for any $\mathbf{A} \in \mathsf{DMAlg}$, the matrix $\langle \mathbf{A}, \{1\} \rangle$ is a model of all the rules of $\mathcal{B}_1$. Moreover, the assumption that $\mathbf{A} \in \mathbb{Q}(\mathbf{D_{12}})$ implies that $\mathbf{A}$ satisfies any quasi-equation that is valid in $\mathbf{D_{12}}$, in particular those of the form

$$\varphi_1 \approx 1 \;\;\&\;\; \ldots \;\;\&\;\; \varphi_n \approx 1 \;\;\Rightarrow\;\; \varphi \approx 1.$$

This means that the matrix $\langle \mathbf{A}, \{1\} \rangle$ satisfies the rule (ECQ$_n$) for all $n \geq 1$. Thus, $\langle \mathbf{A}, \{1\} \rangle$ is a model of $\mathcal{B}_\infty$.

Another important consequence of Theorem 4.8 is the announced completeness result.

Theorem 4.10. *$\mathcal{B}_\infty$ is the logic determined by the matrix $\langle \mathbf{D_{12}}, \{1\} \rangle$.*

Proof: Let $\mathcal{L}$ be the logic determined by the matrix $\langle \mathbf{D_{12}}, \{1\} \rangle$. As mentioned above, we only need to prove that $\mathcal{B}_\infty \geq \mathcal{L}$. Reasoning by contradiction, assume there are formulas $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash_{\mathcal{L}} \varphi$ but $\Gamma \nvdash_{\mathcal{B}_\infty} \varphi$. Since $\mathcal{L}$ is determined by a single finite matrix, it is a finitary logic. So we may assume that there is a finite set of formulas $\{\gamma_1, \ldots, \gamma_m\} \subseteq \Gamma$ such that $\{\gamma_1, \ldots, \gamma_m\} \vdash_{\mathcal{L}} \varphi$ and obviously $\{\gamma_1, \ldots, \gamma_m\} \nvdash_{\mathcal{B}_\infty} \varphi$. This last assumption implies that there are a reduced model $\langle \mathbf{A}, D \rangle$ of $\mathcal{B}_\infty$ and a valuation $h \colon \mathbf{Fm} \to \mathbf{A}$ such that $h(\gamma_i) \in D$ for $1 \leq i \leq m$ but $h(\varphi) \notin D$. By definition, $\mathcal{B}_\infty \geq \mathcal{B}_1$, so we know by Theorem 4.6 that $\mathbf{A}$ is a De Morgan algebra with top element 1 and $D = \{1\}$. We have then that $h(\gamma_i) = 1$ for $1 \leq i \leq m$ but $h(\varphi) \neq 1$. This means that $\mathbf{A}$ does not satisfy the quasi-equation

$$\gamma_1 \approx 1 \;\;\&\;\; \ldots \;\;\&\;\; \gamma_m \approx 1 \;\;\Rightarrow\;\; \varphi \approx 1$$

which is instead satisfied by $\mathbf{D_{12}}$. It follows that $\mathbf{A} \notin \mathbb{Q}(\mathbf{D_{12}})$. So, by Theorem 4.8, $\mathbf{A}$ should not satisfy the quasi-equation

$$(x_1 \wedge \neg x_1) \vee \ldots \vee (x_n \wedge \neg x_n) \approx 1 \;\;\Rightarrow\;\; x \approx 1$$

for some $n \geq 1$. But from the definition of $\mathcal{B}_\infty$ it follows that $\langle \mathbf{A}, \{1\} \rangle$ satisfies any rule of the form

$$(\varphi_1 \wedge \neg\varphi_1) \vee \ldots \vee (\varphi_n \wedge \neg\varphi_n) \vdash \varphi$$

for all $n \geq 1$, and this is equivalent, as we have seen, to the fact that $\mathbf{A}$ satisfies the quasi-equation

$$(x_1 \wedge \neg x_1) \vee \ldots \vee (x_n \wedge \neg x_n) \approx 1 \;\;\Rightarrow\;\; x \approx 1$$

for any n. Thus we have a contradiction. Therefore, any reduced model of $\mathcal{B}_\infty$ is a model of $\mathcal{L}$, which implies that $\mathcal{B}_\infty \geq \mathcal{L}$.

At this point one might wonder whether the logic that we have defined by means of an infinite set of rules can be finitely axiomatized. The answer is negative, as shown by the following result.

Theorem 4.11. *The logic $\mathcal{B}_\infty$ cannot be axiomatized by a finite set of finite rules.*

Proof: Assume there is a finite set of finite rules of the form

$$\frac{\varphi_i^1, \ \ldots, \ \varphi_i^{n_i}}{\psi_i} \qquad (\alpha_i)$$

with $1 \leq i \leq m$ for some natural number $m \geq 1$, which axiomatizes $\mathcal{B}_\infty$. Then a matrix $\langle \mathbf{A}, D \rangle$ is a model of $\mathcal{B}_\infty$ if and only if it satisfies (α_i) for $1 \leq i \leq m$. In particular, this would hold for any matrix of the form $\langle \mathbf{A}, \{1\} \rangle$ with $\mathbf{A} \in \mathsf{DMAlg}$. By Lemma 4.9, this means that $\mathbf{A} \in \mathbb{Q}(\mathbf{D_{12}})$ if and only if the matrix $\langle \mathbf{A}, \{1\} \rangle$ satisfies (α_i) for $1 \leq i \leq m$. But, as mentioned above, $\langle \mathbf{A}, \{1\} \rangle$ satisfies a rule (α_i) if and only if $\mathbf{A}$ satisfies the quasi-equation

$$\varphi_i^1 \approx 1 \;\;\&\;\; \ldots \;\;\&\;\; \varphi_i^{n_i} \approx 1 \quad \Rightarrow \quad \psi_i \approx 1. \qquad (\alpha_i')$$

Thus, for any De Morgan algebra $\mathbf{A}$, we have that $\mathbf{A} \in \mathbb{Q}(\mathbf{D_{12}})$ if and only if $\mathbf{A} \vDash \alpha_i'$ with $1 \leq i \leq m$. This would imply that the quasi-variety $\mathbb{Q}(\mathbf{D_{12}})$ can be axiomatized by a finite set of quasi-equations, which is against what was proved by Gaitán and Perea (2004).

Using the above results it is easy to obtain a characterization of the class $\mathbf{Mod}^*(\mathcal{B}_n)$ for any $n \geq 2$.

Theorem 4.12. *Let $\langle \mathbf{A}, D \rangle$ be a matrix, with $\mathbf{A}$ non-trivial. Then $\langle \mathbf{A}, D \rangle$ is a reduced model of $\mathcal{B}_n$ with $n \geq 2$ if and only if:*

(i) $\mathbf{A} \in \mathbb{Q}_n$
(ii) $D = \{1\}$
(iii) $\langle \mathbf{A}, \{1\} \rangle$ *satisfies the disjunction property (DP).*

Proof: Assume $\langle \mathbf{A}, D \rangle$ is a reduced model of $\mathcal{B}_n$. Since $\mathcal{B}_1 \leq \mathcal{B}_n$, we have that $\langle \mathbf{A}, D \rangle$ is also a reduced model of $\mathcal{B}_1$. Then, by Theorem 4.6, we have that $\mathbf{A}$ is bounded (so we can view it as a De Morgan algebra) and $\langle \mathbf{A}, \{1\} \rangle$ satisfies the (DP). Moreover, $\langle \mathbf{A}, \{1\} \rangle$ satisfies (ECQ$_n$), so $\mathbf{A}$ will satisfy the quasi-equation (β_n). That is, $\mathbf{A} \in \mathbb{Q}_n$.
Conversely, assume (i), (ii) and (iii) hold. By (i), $\mathbf{A}$ is a De Morgan algebra. Then, by (iii) and Theorem 3.2, we have that the matrix $\langle \mathbf{A}, \{1\} \rangle$ is a reduced model of the Belnap-Dunn logic. Moreover, as mentioned above, $\langle \mathbf{A}, \{1\} \rangle$ satisfies (DS), so it is a model of $\mathcal{B}_1$. Finally, since $\mathbf{A} \in \mathbb{Q}_n$, we have that $\mathbf{A}$ satisfies the quasi-equation (β_n), so the rule (ECQ$_n$) holds in $\langle \mathbf{A}, \{1\} \rangle$. This means that $\langle \mathbf{A}, \{1\} \rangle$ is a model of $\mathcal{B}_n$ and this concludes our proof.

5. Further work

The results proved in the previous section can be used to obtain further information on the known extensions of $\mathcal{B}$ mentioned in Section 4.1. For instance, since we have seen that $\mathcal{B}_\infty < \mathcal{K}$, we can apply Theorem 4.12 to obtain a characterization of reduced models of $\mathcal{K}$. Similar results can be used to prove interesting facts on $\mathcal{K}_\leq$ and $\mathcal{LP}$ as well. For instance, it is possible to show that there is no logic $\mathcal{L}$ such that $\mathcal{K} < \mathcal{L} < \mathcal{CL}$ or such that $\mathcal{K}_\leq < \mathcal{L} < \mathcal{LP}$. On the other hand, if add (ECQ) to $\mathcal{LP}$, we obtain a new logic that is strictly weaker than $\mathcal{CL}$. It is also easy to obtain a characterization of the algebraic reducts of generalized models of the above-mentioned logics, i.e., the classes of algebras that, according to the criteria of Font and Jansana (2009), constitute the algebraic counterparts of $\mathcal{K}$, $\mathcal{K}_\leq$ and $\mathcal{LP}$.

All the above-mentioned results will perhaps be published in a future work. However, the landscape of extensions of the Belnap-Dunn logic is still largely unexplored and the most interesting questions have yet to be addressed. For instance, we do not know whether there is any logic $\mathcal{L}$ such that $\mathcal{B}_n < \mathcal{L} < \mathcal{B}_{n+1}$ for some $\mathcal{B}_n$ belonging to our infinite chain of logics. In general, the structure of the lattice of all extensions of the Belnap-Dunn logic (or even of its upper part) is still not quite understood. This issue is clearly beyond the scope of the present paper and will perhaps require the development of a more general method for tackling the problem. It is our hope that our work will serve as an invitation for other researchers to gain an interest in this topic.

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