# Bilattice public announcement logic

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**Abstract**

Building on recent work on bilattice modal logic and extensions of public announcement logic to a non-classical setting, we introduce a dynamic epistemic logic having the logic of modal bilattices as propositional support. Bilattice logic is both inconsistency-tolerant and paracomplete, thus suited for applications in contexts with multiple sources of information, where one may have to deal with lacking as well as potentially contradictory evidence. We introduce an algebra-based semantics for bilattice public announcement logic as well as a relational semantics based on many-valued Kripke models. We show via duality that the two semantics are equivalent and axiomatize the resulting logic by means of a Hilbert-style calculus. Our results and methodology extend recent work on non-classical dynamic epistemic logics such as intuitionistic public announcement logic.

*Keywords:* Bilattices, public announcement, epistemic updates, dynamic logic, modal logic, inconsistency-tolerant logic, many-valued logic.

## 1 Introduction

Dynamic logics are language expansions of classical (modal) logic designed to reason about changes induced by actions of different kinds, e.g. updates on the memory state of a computer, displacements of a moving robot, belief-revisions changing the common ground among different cognitive agents, knowledge update. Semantically, an action is represented as a transformation of a model describing a given state of affairs into a new one that represents the state of affairs after the action has been performed.

The logic of public announcements [15], [2], [7], [3] is a simple and well-known dynamic logic that models the epistemic change on the cognitive state of a group of agents resulting from a given proposition becoming publicly known. To each proposition $\alpha$ one associates a *dynamic* modal operator $\langle\alpha\rangle$ whose

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semantic interpretation is given by the transformation of models corresponding to its action-parameter $\alpha$.

The present paper builds on the logic of public announcements (PAL) developed in [14],[13], [2] on the one hand, and on the bilattice-valued modal logic [12] on the other. [14], [13] introduce a semantically justified definition of dynamic epistemic logic on a base that is weaker than classical logic: the main methodological novelty of these papers is the dual characterization of epistemic updates via Stone-type dualities.

It is well known that epistemic updates induced by public announcements are formalized in relational models by means of the relativization construction, which creates a submodel of the original model. In [14] the corresponding submodel injection map is dually represented as a quotient construction between the complex algebras of the original model and of the updated one. This construction allows one to study epistemic updates within mathematical environments having a support that is weaker than classical logic.

Here we develop a similar study in a context that is yet more general. As propositional base we take the bilattice logic introduced by Arieli and Avron [1], which is both an inconsistency-tolerant and a paracomplete logic. Epistemic (i.e. *static*) modalities are modeled using the framework of the bilattice modal logic introduced in [12].

The algebraic framework of bilattices [10] and their associated logic builds on seminal ideas of Belnap [4], [5] motivated by the issue of dealing with incomplete and potentially inconsistent information. This setting has been further developed in [1] and generalized to weaker logics in, e.g., [11], [6]. In particular, [12] expands the language of bilattice logic with modal operators that are interpreted in many-valued analogues of Kripke frames.

In the present paper we generalize the quotient construction of [14] to the algebraic semantics of bilattice modal logic, which allows us to define a natural interpretation of the language of PAL on modal bilattices. In this way we establish which interaction axioms among dynamic modalities are sound with respect to our intended semantics. The resulting calculus defines a bilattice-based version of public announcement logic (called *bilattice public announcement logic*, BPAL), which we prove to be complete with respect to our algebra-based semantics analogously to classical PAL. We also introduce an equivalent relational semantics for BPAL based on many-valued Kripke frames, which is obtained from the algebraic semantics via a Stone-type duality. Preliminary results on BPAL are contained in [16], to which we will sometimes refer in order to shorten our proofs.

The main aim of our work is to pave the way to a semantically-grounded analysis of epistemic updates in the presence of incomplete and/or inconsistent information. It is also a contribution to the research line initiated in [14], [13], which aims at introducing methods of algebraic logic, duality and proof theory in the study of mathematical foundations of dynamic logic (see also [8], [9]).

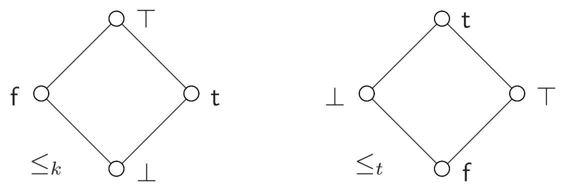


Fig. 1. The four-element Belnap bilattice $\mathcal{FOUR}$ in its two orders

## 2 Bilattice modal logic

In this section we introduce the setting of bilattice modal logic and recall facts and definitions that will be needed to develop a bilattice public announcement logic. We refer the reader to [12] for proofs and further details. The non-modal basis of bilattice modal logic is the logic introduced by Arieli and Avron [1], which can be defined through Belnap's (bi)lattice $\mathsf{FOUR}$ (Figure 1). We view $\mathsf{FOUR}$ as an algebra having operations $\langle \wedge, \vee, \otimes, \oplus, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ of type $\langle 2, 2, 2, 2, 2, 1, 0, 0, 0, 0 \rangle$. Both $\langle \mathsf{FOUR}, \wedge, \vee, \mathsf{f}, \mathsf{t} \rangle$ and $\langle \mathsf{FOUR}, \otimes, \oplus, \bot, \top \rangle$ are bounded distributive lattices, as shown in Figure 1, whose lattice orders are denoted, respectively, by $\leq_t$ (*truth order*) and $\leq_k$ (*knowledge order*). We have, moreover, a binary *weak implication* operation $\supset$ defined by $x \supset y := y$ if $x \in \{\mathsf{t}, \top\}$ and $x \supset y := \mathsf{t}$ otherwise. *Negation* is a unary operation $\neg$ having $\bot$ and $\top$ as fixed points and such that $\neg \mathsf{f} = \mathsf{t}$ and $\neg \mathsf{t} = \mathsf{f}$.

We have included the operations $\otimes$ and $\oplus$ in the primitive signature as they are essential ingredients of bilattices as they were originally introduced, and of the motivation behind them. In the present context, however, they can be retrieved as terms in the language $\langle \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$. We will thus consider them as abbreviations of the terms shown below, together with the following defined operations:

$$
\begin{aligned}
x \otimes y &:= (x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y) \\
x \oplus y &:= (x \wedge \top) \vee (y \wedge \top) \vee (x \wedge y) \\
\sim x &:= x \supset \mathsf{f} \\
x \to y &:= (x \supset y) \wedge (\neg y \supset \neg x) \\
x * y &:= \neg(y \to \neg x) \\
x \equiv y &:= (x \supset y) \wedge (y \supset x) \\
x \leftrightarrow y &:= (x \to y) \wedge (y \to x).
\end{aligned}
$$

The operation $\sim$ provides an alternative negation, while $\to$ is an alternative implication called *strong implication*, which is adjoint to the operation $*$, called strong conjunction or *fusion*. The operations $\langle *, \to \rangle$ form a *residuated pair* (in the residuated lattice sense [11]) w.r.t. the truth order of $\mathsf{FOUR}$, and so $\to$ can be seen as a truth-implication. It might be possible to consider, dually, a knowledge-implication, but we will not pursue this here; as mentioned in [12], this option seems to be technically less viable in a modal logic setting.

The bilattice logic of [1] can be introduced as the propositional logic defined by the matrix $\langle \mathsf{FOUR}, \{\mathsf{t}, \top\}\rangle$ as follows. Starting from a countable set of propositional variables $Var$, one constructs the formula algebra $\mathbf{Fm} = \langle Fm, \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$ in the usual way. Given formulas $\Gamma, \{\varphi\} \subseteq Fm$, one sets $\Gamma \vDash_{\mathsf{FOUR}} \varphi$ iff, for all homomorphisms $v \colon \mathbf{Fm} \to \mathsf{FOUR}$, if $v(\gamma) \in \{\mathsf{t}, \top\}$ for all $\gamma \in \Gamma$, then also $v(\varphi) \in \{\mathsf{t}, \top\}$. This logic can be axiomatized through the Hilbert-style calculus introduced in [12, Section III B]. It is sufficient to take all axioms of classical logic in the language $\langle \wedge, \vee, \supset, \mathsf{f}, \mathsf{t}\rangle$ plus the following:

$$\begin{array}{lll} \top \wedge \neg\top & \neg(p \supset q) \equiv (p \wedge \neg q) & \neg(p \wedge q) \equiv (\neg p \vee \neg q) \\ (\bot \vee \neg\bot) \supset \mathsf{f} & & \neg(p \vee q) \equiv (\neg p \wedge \neg q) \end{array}$$

The only rule is *modus ponens* (mp): $p, p \supset q \vdash q$. Notice that the above axioms involving the bilattice negation $\neg$ are not derivable from those of classical logic, because $\neg$ is not defined from the falsum constant and implication in the usual way.

This logic can be semantically expanded with modal operators by considering *four-valued Kripke models*. These are structures $\langle W, R, v\rangle$ such that both $R$ and $v$ are four-valued. That is, one defines $R \colon W \times W \to \mathsf{FOUR}$ and $v \colon \mathbf{Fm} \times W \to \mathsf{FOUR}$. We then call $\langle W, R\rangle$ a *four-valued Kripke frame*. Valuations are required to be homomorphisms in their first argument, so they preserve all non-modal connectives (including the four constants) of the logic of $\mathsf{FOUR}$. The modal operator $\Box$ is defined as follows: for every $w \in W$ and every $\varphi \in Fm$,

$$v(\Box\varphi, w) := \bigwedge\{R(w, w') \to v(\varphi, w') : w' \in W\}$$

where $\bigwedge$ denotes the infinitary version of $\wedge$ in $\mathsf{FOUR}$ and $\to$ is the strong implication introduced above. If we replace $\mathsf{FOUR}$ by the two-element Boolean algebra and $\wedge, \to$ with classical conjunction and implication, this can be readily seen to be a generalization of the classical case. Notice that *all* worlds $w' \in W$ are taken into account to evaluate $v(\Box\varphi, w)$.

The dual operator $\Diamond$ is defined as

$$v(\Diamond\varphi, w) := \bigvee\{R(w, w') * v(\varphi, w') : w' \in W\}$$

where $\bigvee$ denotes the infinitary version of $\vee$ in $\mathsf{FOUR}$ and $*$ is the *fusion* operation introduced above. It is straightforward to check that $v(\Box\varphi, w) = v(\neg\Diamond\neg\varphi, w)$ for all $w \in W$ and all valuations $v$. Thus, as happens in the classical case (and unlike the intuitionistic one), the two modal operators are inter-definable. In the present paper we will take $\Diamond$ as primitive.

A modal consequence relation can now be defined in the usual way. We say that a point $w \in W$ of a four-valued model $M = \langle W, R, v\rangle$ satisfies a formula $\varphi \in Fm$ iff $v(\varphi, w) \in \{\mathsf{t}, \top\}$, and we write $M, w \vDash \varphi$. For a set of formulas $\Gamma \subseteq Fm$, we write $M, w \vDash \Gamma$ to mean that $M, w \vDash \gamma$ for each $\gamma \in \Gamma$. The (local) consequence $\Gamma \vDash \varphi$ holds if, for every model $M = \langle W, R, v\rangle$ and every $w \in W$, it is the case that $M, w \vDash \Gamma$ implies $M, w \vDash \varphi$.

Notice that this consequence relation inherits from the non-modal fragment the deduction-detachment theorem in the following form: $\Gamma \vDash \varphi$ if and only if $\emptyset \vDash \bigwedge \Gamma \supset \varphi$, where $\bigwedge \Gamma := \bigwedge \{\gamma \in \Gamma\}$. This, which will remain true about its dynamic expansion BPAL, implies that we can without loss of generality restrict our attention to valid formulas.

The above-defined consequence is axiomatized in [12]. The set of axioms is the least set $\Sigma \subseteq Fm$ containing all substitution instances of the schemata axiomatizing non-modal bilattice logic plus the following ones:

| | |
|---|---|
| ($\Box$ $\mathsf{t}$) | $\Box \mathsf{t} \leftrightarrow \mathsf{t}$ |
| ($\Box$ $\wedge$) | $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$ |
| ($\Box$ $\bot$) | $\Box(\bot \rightarrow p) \leftrightarrow (\bot \rightarrow \Box p)$ |

Moreover, $\Sigma$ must satisfy: (val-mp) if $\varphi$ and $\varphi \supset \psi$ are in $\Sigma$, then so is $\psi$; (val-mono) if $\varphi \rightarrow \psi$ is in $\Sigma$, then so is $\Box\varphi \rightarrow \Box\psi$. The only inference rule is (mp). We notice that (val-mono) replaces the more common *necessitation* rule (if $\varphi \in \Sigma$, then $\Box\varphi \in \Sigma$) because the latter would not be sound in our setting [12, Section III.A].

This calculus is complete not only with respect to the semantics of four-valued Kripke models, but also with respect to an algebra-based semantics given by the class of *modal bilattices*. We briefly recall these results in the remaining part of this section, as we will build on them later on. We begin with completeness with respect to Kripke models [12, Theorem 19].

**Theorem 2.1 (Relational completeness)** *For all* $\Gamma, \{\varphi\} \subseteq Fm$*,* $\Gamma \vdash \varphi$ *iff* $M, w \vDash \Gamma$ *implies* $M, w \vDash \varphi$ *for every four-valued Kripke model* $M = \langle W, R, v \rangle$ *and every* $w \in W$*.*

In order to state the algebraic completeness theorem we need to introduce a class of algebras providing an alternative semantics for our calculus. A *modal bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \supset, \neg, \Diamond, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ such that the $\Diamond$-free reduct of $\mathbf{B}$ is an *implicative bilattice* [2], that is, the algebra $\langle B, \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ belongs to the variety generated by $\mathsf{FOUR}$, and moreover the following identities are satisfied:

| | |
|---|---|
| (i) | $\Diamond \mathsf{f} = \mathsf{f}$ |
| (ii) | $\Diamond(p \vee q) = (\Diamond p \vee \Diamond q)$ |
| (iii) | $\Box(x \supset \bot) = \Diamond x \supset \bot$ |

Thus, in particular, $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t} \rangle$ is a bounded distributive lattice. It is easy to show that identities (i)-(iii) correspond, respectively, to axioms (i)-(iii) of our calculus, and that the presentation of modal bilattices given here is equivalent to that of [12].

Given a modal bilattice $\mathbf{B}$ and a subset $F \subseteq B$, we say that $F$ is a *bifilter* if $F$ is a lattice filter of $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t} \rangle$ such that $\top \in F$. Given a pair $\langle \mathbf{B}, F \rangle$

[2] An abstract equational of implicative bilattices can be found in [6].

and formulas $\Gamma, \{\varphi\} \subseteq Fm$, we write $\Gamma \vDash_{\langle B,F \rangle} \varphi$ to mean that, for every modal bilattice homomorphism $v \colon \mathbf{Fm} \to \mathbf{B}$, if $v(\gamma) \in F$ for all $\gamma \in \Gamma$, then also $v(\varphi) \in F$. A valid formula $\varphi$ is one such that $v(\varphi) \geq_t \top^{\mathbf{B}}$ for every $\mathbf{B}$ and $v$. We can then state the algebraic completeness result [12, Theorem 10] as follows.

**Theorem 2.2 (Algebraic completeness)** *For all $\Gamma, \{\varphi\} \subseteq Fm$, $\Gamma \vdash \varphi$ iff $\Gamma \vDash_{\langle B,F \rangle} \varphi$ for any modal bilattice $\mathbf{B}$ and any bifilter $F \subseteq B$.*

Just as in the case of classical modal logic, the relational and the algebraic semantics of bilattice modal logic are interrelated via a Stone-type duality [12, Theorem 18]. In the case of bilattices, another essential ingredient is the so-called *twist-structure* representation. Let $\mathbf{A} = \langle A, \wedge, \vee, \sim, \Diamond_+, \Diamond_-, 0, 1 \rangle$ be a *bimodal Boolean algebra* [12, Definition 11], i.e. a structure such that $\langle A, \wedge, \vee, \sim, 0, 1 \rangle$ is a Boolean algebra and $\Diamond_+$ and $\Diamond_-$ are unary operators that preserve finite joins (no relation between the two is required). The dual operators $\Box_+$ and $\Box_-$ are defined in the usual way as $\Box_+ x := \sim \Diamond_+ \sim x$ and $\Box_- x := \sim \Diamond_- \sim x$. The *twist-structure over* $\mathbf{A}$ is the algebra $\mathbf{A}^{\bowtie} = \langle A \times A, \wedge, \vee, \supset, \neg, \Diamond, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ with operations given, for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in A \times A$, by:

$$
\begin{aligned}
\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle &:= \langle a_1 \wedge b_1,\ a_2 \vee b_2 \rangle \\
\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle &:= \langle a_1 \vee b_1,\ a_2 \wedge b_2 \rangle \\
\langle a_1, a_2 \rangle \supset \langle b_1, b_2 \rangle &:= \langle \sim a_1 \vee b_1, a_1 \wedge b_2 \rangle \\
\neg \langle a_1, a_2 \rangle &:= \langle a_2,\ a_1 \rangle \\
\Diamond \langle a_1, a_2 \rangle &:= \langle \Diamond_+ a_1,\ \Box_+ a_2 \wedge \sim \Diamond_- a_1 \rangle \\
\mathsf{f} &:= \langle 0, 1 \rangle \\
\mathsf{t} &:= \langle 1, 0 \rangle \\
\bot &:= \langle 0, 0 \rangle \\
\top &:= \langle 1, 1 \rangle
\end{aligned}
$$

It is straightforward to check that any twist-structure is a modal bilattice. More interestingly, *any modal bilattice is isomorphic to a twist-structure* [12, Theorem 12]. This means that instead of working directly with modal bilattices, one can (and we will) without loss of generality focus only on twist-structures.

The twist-structure construction allows us to relate four-valued Kripke frames and modal bilattices via Jónsson-Tarski duality for classical modal logic. Given a modal bilattice $\mathbf{B}$ viewed as a twist-structure $\mathbf{A}^{\bowtie}$, we can consider the structure $\langle \mathbf{A}_{\bullet}, R_+, R_- \rangle$, where $\langle \mathbf{A}_{\bullet}, R_+ \rangle$ and $\langle \mathbf{A}_{\bullet}, R_- \rangle$ are the classical Kripke frames associated to the modal Boolean algebras $\langle A, \wedge, \vee, \sim, \Diamond_+, 0, 1 \rangle$ and $\langle A, \wedge, \vee, \sim, \Diamond_-, 0, 1 \rangle$ according to Jónsson-Tarski duality. The relations $R_+$ and $R_-$ can obviously be combined into one four-valued relation $R_4$ by letting, for instance, $R(w, w') = \mathsf{t}$ iff $\langle w, w' \rangle \in R_+ \cap R_-$, $R(w, w') = \top$ iff $\langle w, w' \rangle \in R_+ \backslash R_-$, $R(w, w') = \bot$ iff $\langle w, w' \rangle \in R_- \backslash R_+$ and $R(w, w') = \mathsf{f}$

iff $\langle w, w' \rangle \notin R_+ \cup R_-$. In this way[3] we obtain a four-valued Kripke frame $\langle \mathbf{A}_\bullet, R \rangle$. Conversely, every four-valued Kripke frame $F = \langle W, R \rangle$ can be viewed as a pair of Kripke frames $\langle W, R_+ \rangle$, $\langle W, R_- \rangle$ by defining $\langle w, w' \rangle \in R_+$ iff $R(w, w') \in \{\mathsf{t}, \top\}$ and $\langle w, w' \rangle \in R_-$ iff $R(w, w') \in \{\mathsf{t}, \bot\}$. Thus we obtain classical Kripke frames $F_+ = \langle W, R_+ \rangle$ and $F_- = \langle W, R_- \rangle$, to which one associates modal Boolean algebras $(F_+)^\bullet$ and $(F_-)^\bullet$ according to Jónsson-Tarski duality. Since $(F_+)^\bullet$ and $(F_-)^\bullet$ share the same carrier set, we actually have a bimodal Boolean algebra $F^\bullet$, from which a modal bilattice $(F^\bullet)^{\bowtie}$ can be obtained via the twist-structure construction. It is shown in [12] that the correspondence between four-valued Kripke frames and modal bilattices extends to Kripke models and algebraic models, which implies that the relational and the algebraic semantics for bilattice modal logic are indeed equivalent.

## 3 Pseudo-quotients on modal bilattices

When considering epistemic updates in the context of bilattice logic, we have to take into account that validity of a formula in our logic only depends on its "positive part". In fact, any two formulas $\varphi, \psi$ are logically equivalent if and only if, for every valuation $v \colon Fm \to \mathsf{FOUR}$, it holds that $\pi_1(v(\varphi)) = \pi_1(v(\psi))$, where $\pi_1$ denotes first component projection defined by the twist-structure representation of $\mathsf{FOUR}$ as $\{0,1\} \times \{0,1\}$. For instance, $\mathsf{t}$ and $\top$ (viewed as propositional constants) are both valid formulas (hence, logically equivalent) because $\pi_1(\mathsf{t}) = \pi_1(\top) = 1$. Thus, in particular, the public announcements of $\mathsf{t}$ or $\top$ should both be vacuous. This unusual feature, which depends only on the non-modal support of the logic, can be traced back to Belnap's proposal that derivations should preserve (only) positive evidence (see [4], [5] for a discussion of the intuitions justifying this choice). An alternative characterization of logical equivalence is the following: any two formulas $\varphi, \psi$ are logically equivalent if and only if $v({\sim}{\sim}\varphi) = v({\sim}{\sim}\psi)$ for any valuation $v$. This remark motivates the definition of pseudo-quotients that we are going to introduce below, but before we proceed let us make one more observation that may help avoiding misunderstandings.

The fact that logical equivalence (and hence validity) of a formula only depends on its positive part does not mean that the negative part does not play any role in bilattice logic. For instance, announcing the negation of $\mathsf{t}$ (that is, $\mathsf{f}$) does not have the same effect as announcing the negation of $\top$ (which is $\top$ itself): the latter announcement remains vacuous and thus does not produce any change in the original model, while the former, as in the classical case, makes the model collapse. This is due to an essential feature of bilattice negation, namely the fact that $\varphi$ and $\psi$ being logically equivalent does not entail that $\neg\varphi$ is equivalent to $\neg\psi$. As mentioned in [12, Section VIII], using the twist-structure representation it may indeed be possible to embed bilattice

[3] Although it is obviously possible to combine the information conveyed by $R_+$ and $R_-$ in many alternative ways, the work in [12] indicates that the one suggested above is, at least from a technical point of view, the most suitable one.

modal logic in classical (bi)modal logic, but this is not as straightforward as it may seem, for in order to account for the negative part too one would need to translate formulas of bilattice logic into *pairs* of formulas of classical logic.

Given a modal bilattice $\mathbf{B}$ and an element $a \in B$, we define a relation $\equiv_a$ by the following prescription: for all $b, c \in B$,

$$b \equiv_a c \qquad \text{iff} \qquad b \wedge \sim\sim a = c \wedge \sim\sim a.$$

This definition is adapted from (and can indeed be seen as a special case of) that of [14][4]. The only difference is that, as noted above, here we need to consider only the positive part of $a \in B$, hence the term $\sim\sim a$. We are now going to see that the above-defined relation is indeed a congruence of the non-modal reduct of any modal bilattice.

**Lemma 3.1 ([16], Lemma 2.1)** *Let $\mathbf{A}^{\bowtie}$ be a twist-structure over a Boolean algebra $\mathbf{A}$. Then, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle, \langle c_1, c_2\rangle \in A \times A$,*

$$\langle b_1, b_2\rangle \equiv_{\langle a_1, a_2\rangle} \langle c_1, c_2\rangle \qquad \textit{iff} \qquad b_1 \equiv_{a_1} c_1 \quad \textit{and} \quad b_2 \equiv_{a_1} c_2$$

*where $\equiv_{a_1}$ is defined as in [14, Section 3.2], i.e., $x \equiv_{a_1} y$ iff $x \wedge a_1 = y \wedge a_1$.*

Notice that in the preceding lemma, as mentioned above, the negative part $a_2$ of the pair $\langle a_1, a_2\rangle$ does not play any role. The following result is a straightforward consequence.

**Fact 3.2 ([16], Fact 2.2)** *For any modal bilattice $\mathbf{B}$ and any $a \in B$, the relation $\equiv_a$ is a congruence of the non-modal reduct of $\mathbf{B}$.*

As happened in [14], our relation $\equiv_a$ is in general not compatible with the modal operator(s). The next step is thus to find a suitable definition for modal operators on the pseudo-quotient. We begin with the following observation (cf. [14, Fact 6]).

**Fact 3.3** *Let $\mathbf{B}$ be a modal bilattice and $a \in B$. Then*

*(i)* $[b \wedge \sim\sim a] = [b]$ *for every $b \in B$. Hence, for every $b \in B$, there exists a unique $c \in B$ such that $c \in [b]_a$ and $c \leq_t \sim\sim a$.*

*(ii)* $[b] \leq_t [c]$ *iff* $b \wedge \sim\sim a \leq_t c \wedge \sim\sim a$ *for all $b, c \in B$.*

**Proof.** Essentially the same as [14, Fact 6], replacing $a$ by $\sim\sim a$. □

Item (i) of Fact 3.3 implies that for each equivalence class modulo $\equiv_a$ we can choose a canonical representative, namely the unique element in the given class that is below $\sim\sim a$ in the truth order. Hence we can define an (injective) map $i' = i'_a \colon \mathbf{B}^a \to \mathbf{B}$ given, for every $[b] \in B^a$, by $i'[b] := b \wedge \sim\sim a$. Notice, moreover, that the composition $\pi \cdot i'$ is the identity on $\mathbf{B}^a$.

[4] Notice however that, while the equation $\neg\neg x = x$ is valid in any bilattice, in general it is not the case that $\sim\sim x = x$. This explains why our definition does not coincide with that of [14], and is also the reason why, even if two formulas $\varphi$ and $\psi$ are logically equivalent (hence $v(\sim\sim\varphi) = v(\sim\sim\psi)$ for any valuation $v$), it may well happen that $v(\varphi) \neq v(\psi)$.

At this point we are ready to introduce modal operator(s) on the pseudo-quotient. Given $a, b \in \mathbf{B}$, we let

$$\Diamond^a[b] := [\Diamond(b \wedge \sim\sim a)] = [\Diamond(b \wedge \sim\sim a) \wedge \sim\sim a]$$

The dual operator is defined as $\Box^a[b] := \neg\Diamond^a\neg[b]$. Using Fact 3.2 and the identities of modal bilattices, it is easy to check that, in keeping with [14, Section 3.3.2], we have

$$\Box^a[b] = [\Box(a \supset b)] = [a \supset \Box(a \supset b)].$$

This could thus be taken as an alternative but equivalent definition.

The following result shows that our definition indeed suits our purpose (cf. [14, Fact 10]).

**Fact 3.4 ([16], Fact 2.4)** *For every modal bilattice* $\mathbf{B}$ *and all* $a, b, c \in B$*:*

*(i)* $\Diamond^a[\mathsf{f}] = [\mathsf{f}]$

*(ii)* $\Diamond^a([b] \vee [c]) = \Diamond^a[b] \vee \Diamond^a[c]$

*(iii)* $\Box^a([b] \supset [\bot]) = \Diamond^a[b] \supset [\bot]$

*(iv) Hence,* $(\mathbf{B}^a, \Diamond^a)$ *is a modal bilattice.*

The following lemma relates the pseudo-quotient construction and the twist-structure representation of modal bilattices. This will be used in Section 5.2.

**Lemma 3.5** *Let* $\mathbf{A}^{\bowtie}$ *be a modal twist-structure over a bimodal Boolean algebra* $\mathbf{A}$*, and let* $\langle a_1, a_2\rangle \in A \times A$*. Then* $(\mathbf{A}^{\bowtie})^{\langle a_1, a_2\rangle} \cong (\mathbf{A}^{a_1})^{\bowtie}$.

## 4 Axiomatization fo BPAL

Our calculus for bilattice public announcement logic is defined over the language $\langle \wedge, \vee, \supset, \neg, \Diamond, \langle\alpha\rangle, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$, where $\alpha \in Fm$. Derived connectives $\langle \sim, \Box, \otimes, \oplus, \to, *, \leftrightarrow\rangle$ are introduced as before. Moreover, we let $[\alpha]\varphi := \neg\langle\alpha\rangle\neg\varphi$. BPAL is axiomatically defined by the axioms and rules of the above-mentioned (local) calculus for bilattice modal logic [12] augmented with the following axioms:

| | | |
|---|---|---|
| **Interaction with logical constants** | $\langle\alpha\rangle\mathsf{f} \leftrightarrow \mathsf{f}$ | $\langle\alpha\rangle\mathsf{t} \leftrightarrow \sim\sim\alpha$ |
| | $\langle\alpha\rangle\top \leftrightarrow (\alpha \wedge \top)$ | $\langle\alpha\rangle\bot \leftrightarrow \neg(\alpha \supset \bot)$ |
| **Interaction with** $\wedge$ | $\langle\alpha\rangle(\varphi \wedge \psi) \leftrightarrow (\langle\alpha\rangle\varphi \wedge \langle\alpha\rangle\psi)$ | |
| **Interaction with** $\vee$ | $\langle\alpha\rangle(\varphi \vee \psi) \leftrightarrow (\langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi)$ | |
| **Interaction with** $\supset$ | $\langle\alpha\rangle(\varphi \supset \psi) \leftrightarrow (\sim\sim\alpha \wedge (\langle\alpha\rangle\varphi \supset \langle\alpha\rangle\psi))$ | |
| **Interaction with** $\neg$ | $\langle\alpha\rangle\neg\varphi \leftrightarrow (\sim\sim\alpha \wedge \neg\langle\alpha\rangle\varphi)$ | |
| **Interaction with** $\Diamond$ | $\langle\alpha\rangle\Diamond\varphi \leftrightarrow (\sim\sim\alpha \wedge \Diamond\langle\alpha\rangle\varphi)$ | |
| **Preservation of facts** | $\langle\alpha\rangle p \leftrightarrow (\sim\sim\alpha \wedge p)$ | |

where $\varphi, \psi, \alpha$ are arbitrary formulas, while $p$ is a propositional variable. We observe that, using the rules and axioms of the non-modal basis of BPAL, it is easy to establish that the following formulas are derivable:

| | | |
|---|---|---|
| **Interaction with logical constants** | $[\alpha]\mathsf{f} \leftrightarrow {\sim}\alpha$ | $[\alpha]\mathsf{t} \leftrightarrow \mathsf{t}$ |
| | $[\alpha]\top \leftrightarrow \alpha \supset \top$ | $[\alpha]\bot \leftrightarrow (\alpha \supset \bot)$ |
| **Interaction with** $\wedge$ | $[\alpha](\varphi \wedge \psi) \leftrightarrow ([\alpha]\varphi \wedge [\alpha]\psi)$ | |
| **Interaction with** $\vee$ | $[\alpha](\varphi \vee \psi) \leftrightarrow (\alpha \supset ([\alpha]\varphi \vee [\alpha]\varphi))$ | |
| **Interaction with** $\supset$ | $[\alpha](\varphi \supset \psi) \leftrightarrow (\langle\alpha\rangle\varphi \supset \langle\alpha\rangle\psi)$ | |
| **Interaction with** $\neg$ | $[\alpha]\neg\varphi \leftrightarrow \neg\langle\alpha\rangle\varphi$ | |
| **Interaction with** $\Diamond$ | $[\alpha]\Diamond\varphi \leftrightarrow (\alpha \supset \Diamond\langle\alpha\rangle\varphi)$ | |
| **Interaction with** $\Box$ | $[\alpha]\Box\varphi \leftrightarrow (\alpha \supset \Box[\alpha]\varphi)$ | |
| **Preservation of facts** | $[\alpha]p \leftrightarrow (\alpha \supset p)$ | |

# 5 Algebraic and relational models of BPAL

In this section we introduce two kinds of semantics that will be proven to be (equivalent and) complete with respect to the calculus introduced in Section 4. The first kind is the algebraic semantics for BPAL, which we define as indicated by the algebraic analysis of pseudo-quotients on modal bilattices developed in Section 3. The second kind is the relational semantics based on Kripke models. We are then going to use duality to see that the two semantics are indeed equivalent.

## 5.1 Algebraic semantics

We define an *algebraic model* as a tuple $M = (\mathbf{B}, v)$ where $\mathbf{B}$ is a modal bilattice and $v\colon Var \to B$. The *extension* map $[\![\cdot]\!]_M : Fm \to \mathbf{B}$ is defined as follows:

$$
\begin{aligned}
[\![p]\!]_M &:= v(p) && \\
[\![c]\!]_M &:= c^{\mathbf{B}} && \text{for } c \in \{\mathsf{f}, \mathsf{t}, \bot, \top\}\\
[\![\circ\varphi]\!]_M &:= \circ^{\mathbf{B}}[\![\varphi]\!]_M && \text{for } \circ \in \{\neg, \Diamond\}\\
[\![\varphi \bullet \psi]\!]_M &:= [\![\varphi]\!]_M \bullet^{\mathbf{B}} [\![\psi]\!]_M && \text{for } \bullet \in \{\wedge, \vee, \supset\}\\
[\![\langle\alpha\rangle\varphi]\!]_M &:= {\sim}{\sim}[\![\alpha]\!]_M \wedge^{\mathbf{B}} i'([\![\varphi]\!]_{M^\alpha}) &&\\
[\![[\alpha]\varphi]\!]_M &:= [\![\alpha]\!]_M \supset^{\mathbf{B}} i'([\![\varphi]\!]_{M^\alpha}) &&
\end{aligned}
$$

where $M^\alpha = (\mathbf{B}^\alpha, v^\alpha)$ is given by $\mathbf{B}^\alpha = \mathbf{B}^{[\![\alpha]\!]_M}$ and $v^\alpha = \pi \circ v\colon Var \to \mathbf{B}^\alpha$. That is, $[\![p]\!]_{M^\alpha} = v^\alpha(p) = \pi(v(p)) = \pi([\![p]\!]_M)$ for every $p \in Var$.

We define $\Gamma \vDash_{BPAL} \varphi$ iff, for every algebraic model $M = (\mathbf{B}, F, v)$, it holds that $[\![\gamma]\!] \in F$ for all $\gamma \in \Gamma$ implies $[\![\varphi]\!] \in F$. We will see in the next section that the calculus introduced in Section 4 is sound and complete with respect to the semantics provided by the above-defined algebraic models. We are now going to use duality theory and algebraic semantics to introduce a relational semantics for BPAL.

## 5.2 Relational semantics and duality

Consider a four-valued Kripke frame $\mathcal{F}$. For simplicity we view the four-valued accessibility relation $R$ as split into two standard relations, so we let $\mathcal{F} =$

$\langle W, R_+, R_- \rangle$ and let $s \subseteq W$. We define $\mathcal{F}^s = \langle W^s, R_+^s, R_-^s \rangle$, *the subframe of* $\mathcal{F}$ *relativized to* $s$, as follows (cf. [14, Definition 19]): $W^s = s$, $R_+^s = R_+ \cap (s \times s)$ and $R_-^s = R_- \cap (s \times s)$. Given a four-valued Kripke model $M = \langle W, R_+, R_-, v \rangle$ and $\alpha \in Fm$, we define $v_+(\alpha) := \{w \in W : v(\alpha, w) \in \{\mathsf{t}, \top\}\}$. Analogously, we can define $v_-(\alpha) := \{w \in W : v(\alpha, w) \in \{\mathsf{f}, \top\}\}$ but notice that $v_-(\alpha) = v_+(\neg\alpha)$. The submodel $M^\alpha = \langle W^\alpha, R_+^\alpha, R_-^\alpha, v^\alpha \rangle$ is then defined as follows. $W^\alpha := v_+(\alpha)$, $R_+^\alpha := R^+ \cap (W^\alpha \times W^\alpha)$, $R_-^\alpha := R^- \cap (W^\alpha \times W^\alpha)$, and for all $p \in Var$ and $w \in W^\alpha$,

$$v^\alpha(p, w) = \begin{cases} \mathsf{t} \text{ iff } w \in v(p) \text{ and } w \notin v(\neg p) \\ \top \text{ iff } w \in v(p) \text{ and } w \in v(\neg p) \\ \bot \text{ iff } w \notin v(p) \text{ and } w \notin v(\neg p) \\ \mathsf{f} \text{ iff } w \notin v(p) \text{ and } w \in v(\neg p) \end{cases}$$

Extending $v^\alpha$ to arbitrary formulas in the usual way, we can introduce a notion of satisfaction for BPAL formulas of type $\langle\alpha\rangle\varphi$ as follows:

$$M, w \vDash \langle\alpha\rangle\varphi \quad \text{iff} \quad M, w \vDash \alpha \text{ and } M^\alpha, w \vDash \varphi.$$

Noticing that $M, w \vDash \alpha$ iff $M, w \vDash {\sim}{\sim}\alpha$, one easily sees that the above definition is in keeping with the algebraic one given in the preceding section.

In order to prove equivalence between the algebraic and the relational semantics for BPAL, we consider *complex algebras* of four-valued Kripke frames as defined in [12]. For any four-valued Kripke frame $\mathcal{F} = \langle W, R_+, R_- \rangle$, following Jónsson-Tarski duality for classical modal logic, we can construct the complex algebras of the two frames $\langle W, R_+ \rangle$ and $\langle W, R_- \rangle$, which are the structures $\langle P(W), \cap, \cup, \sim, \Diamond_+ \rangle$ and $\langle P(W), \cap, \cup, \sim, \Diamond_- \rangle$, where $\sim$ is the Boolean complement operation and $\Diamond_+ U := R_+^{-1}[U]$, $\Diamond_- U := R_-^{-1}[U]$ for all $U \subseteq W$. These are not only modal Boolean algebras, they are also *perfect* (see below). The structure $\langle P(W), \cap, \cup, \sim, \Diamond_+, \Diamond_- \rangle$ is thus a bimodal Boolean algebra. We can then apply the twist-structure construction introduced in Section 2 to obtain a modal bilattice. We define the *complex algebra* of $\mathcal{F} = \langle W, R_+, R_- \rangle$ as the twist-structure $\mathcal{F}^\bullet = \langle P(W), \cap, \cup, \sim, \Diamond_+, \Diamond_- \rangle^{\bowtie}$.

Given a four-valued Kripke model $M = \langle \mathcal{F}, v \rangle$, we can define a valuation $v^\bullet : Var \to \mathcal{F}^\bullet$, for every $p \in Var$, as $v^\bullet(p) := \langle v_+(p), v_+(\neg p) \rangle$, where $v_+(p) := \{w \in W : v(p, w) \in \{\mathsf{t}, \top\}\}$ as before. We then extend $v^\bullet$ homomorphically to any formula $\varphi$ in the language of bilattice modal logic and we set $\langle \mathcal{F}^\bullet, v^\bullet \rangle \vDash \varphi$ iff $v^\bullet \geq_t \top^{\mathcal{F}^\bullet}$. The following result follows from the duality developed in [12].

**Proposition 5.1** *For every four-valued Kripke model* $M = \langle \mathcal{F}, v \rangle$ *and every formula* $\varphi$ *of bilattice modal logic,* $M \vDash \varphi$ *iff* $\langle \mathcal{F}^\bullet, v^\bullet \rangle \vDash \varphi$.

The proof of the following proposition is analogous to the classical case (cf. [14], Proposition 5]).

**Proposition 5.2** *Let* $M = \langle \mathcal{F}, v \rangle$ *be a four-valued Kripke model,* $\alpha$ *a BPAL formula and* $\mathbf{B} = \mathcal{F}^\bullet$ *the complex algebra of* $\mathcal{F}$*. Let* $M^\alpha = \langle W^\alpha, R_+^\alpha, R_-^\alpha, v^\alpha \rangle$ *be defined as above and denote* $a := \langle v_+(\alpha), v_+(\neg\alpha) \rangle \in B$*. Then the complex* *algebra* $\mathbf{B}^\alpha$ *of* $\langle W^\alpha, R_+^\alpha, R_-^\alpha\rangle$ *can be identified up to modal bilattice isomorphism with* $(\mathbf{B}/Ker(\pi), \Diamond^\alpha)$*, where* $\pi : \mathbf{B} \to \mathbf{B}^\alpha$ *is defined by* $\pi(b) = b \wedge \sim\sim a$*, and* $\Diamond^\alpha[b]_{Ker(\pi)} = [\Diamond^{\mathbf{B}}(b \wedge \sim\sim a)]_{Ker(\pi)}$ *for all* $b \in B$*. The isomorphism* $\mu\colon (\mathbf{B}/Ker(\pi), \Diamond^\alpha) \to \mathbf{B}^\alpha$ *is defined by* $\mu[\langle X_1, X_2\rangle] := \langle X_1 \cap v_+(\alpha), X_2 \cap v_+(\alpha)\rangle$ *for all* $X_1, X_2 \subseteq W$*.*

Recall that, given a modal bilattice $\mathbf{B}$ with associated pseudo-quotient $\mathbf{B}^a$, we define the map $i'\colon \mathbf{B}^a \to \mathbf{B}$ by $i'[b] := b \wedge \sim\sim a$ for all $[b] \in B^a$. For any four-valued Kripke frame $\mathcal{F} = \langle W, R_+, R_-\rangle$ and $s \subseteq W$, we also have an injective map from $\mathcal{F}^s$ to $\mathcal{F}$ given by the inclusion $j\colon W^s \to W$. For a pair $\langle Y_1, Y_2\rangle \in W^s \times W^s$, we let $i\langle Y_1, Y_2\rangle := \langle j(Y_1), j(Y_2)\rangle$. It is easy to check that the map $\nu\colon \mathbf{B}^\alpha \to (\mathbf{B}/Ker(\pi), \Diamond^\alpha)$ defined by $\nu\langle Y_1, Y_2\rangle := [i(\langle Y_1, Y_2\rangle)]_{Ker(\pi)}$ is the inverse of the map $\mu$ of Proposition 5.2. Using this, the following proposition can be proved similarly to [14], Proposition 7].

**Proposition 5.3** *If* $\mathbf{B} = \mathcal{F}^\bullet$ *for some four-valued Kripke frame* $\mathcal{F}$ *and* $a = \langle a_1, a_2\rangle \in \mathbf{B}$*, then* $i'(c) = i(\mu(c))$ *for every* $c \in \mathbf{B}^a$*, where* $\mu : \mathbf{B}^a \to (\mathcal{F}^{a_1})^\bullet$ *is the modal bilattice isomorphism identifying the two algebras. It follows that* $i(c) = i'(\nu(c))$ *for every* $c \in (\mathcal{F}^{a_1})^\bullet$*, where* $\nu\colon (\mathcal{F}^{a_1})^\bullet \to \mathbf{B}^a$ *is the inverse of* $\mu$*.*

In light of the above results, we are going to take a closer look at the modal bilattices that arise as complex algebras of Kripke frames. As we will see, these are the *perfect modal bilattices*.

In general, a lattice $\langle L, \wedge, \vee, 0, 1\rangle$ endowed with a modal operator $\Diamond$ is *perfect* when it is: (i) complete, (ii) completely distributive (infinitary $\bigwedge$ distributes over infinitary $\bigvee$), (iii) completely $\wedge$-generated by its completely $\wedge$-prime members, and (iv) when $\Diamond$ preserves infinitary $\bigvee$. Property (iii) means the following. An element $x \in L$ is *completely* $\wedge$*-prime* if $y \neq 1$ and, for every $S \subseteq L$ such that $\bigwedge S \leq x$, there is $s \in S$ such that $s \leq x$. Dually, $y \in L$ is *completely* $\vee$*-prime* if $y \neq 0$ and, whenever $y \leq \bigvee S$ for some $S \subseteq L$, there is $s \in S$ such that $y \leq s$. We say that $L$ is completely $\wedge$-generated (resp., completely $\vee$-generated) by $S \subseteq L$ if for every $x \in L$ there is $S' \subseteq S$ such that $x = \bigwedge S'$ (resp., $x = \bigvee S'$). In the context of distributive lattices, the two properties are equivalent.

It is well-known that a Boolean algebra $\mathbf{A}$ is perfect if and only if $\mathbf{A}$ is complete as a lattice and *atomic*. The latter means that $\mathbf{A}$ is completely $\vee$-generated by the set of its *atoms* $\mathsf{At}(\mathbf{A})$, defined as follows:

$$\mathsf{At}(\mathbf{A}) := \{x \in A : x \neq 0 \text{ and, for all } y \in A,\ y < x \text{ implies } y = 0\}.$$

We define a *perfect bimodal Boolean algebra* as a bimodal Boolean algebra $(\mathbf{A}, \Diamond_+, \Diamond_-)$ such that $(\mathbf{A}, \Diamond_+)$ and $(\mathbf{A}, \Diamond_-)$ are both perfect modal Boolean algebras, i.e., $\mathbf{A}$ is a complete atomic Boolean algebra and, moreover, both $\Diamond_+$ and $\Diamond_-$ preserve arbitrary joins. It follows from duality for classical modal logic that $\langle P(W), \cap, \cup, \sim, \Diamond_+, \Diamond_-\rangle$ is a perfect bimodal Boolean algebra. We are going to see that twist-structures over perfect bimodal Boolean algebras are exactly the algebraic objects that correspond via duality to four-valued Kripke frames. We say that a modal bilattice $\mathbf{B} = \langle B, \wedge, \vee, \supset, \neg, \Diamond, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$

is *perfect* when (i) $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t}\rangle$ is a perfect lattice and (ii) $\Diamond$ preserves $\bigvee$. The following result, which is easily proved, shows an alternative condition that we could have taken as our definition of perfect modal bilattices.

**Fact 5.4** *A modal bilattice* $\mathbf{B}$ *is perfect if and only if* $\mathbf{B} = \mathbf{A}^{\bowtie}$ *with* $\mathbf{A}$ *a perfect bimodal Boolean algebra.*

It follows from the above remarks that to each four-valued Kripke frame $\mathcal{F}$ corresponds a perfect modal bilattice $\mathcal{F}^{\bullet}$. We are now going to see that, conversely, to each perfect modal bilattice $\mathbf{B}$ we can associate a four-valued Kripke frame $\mathbf{B}_{\bullet}$.

We can assume without loss of generality $\mathbf{B} = \mathbf{A}^{\bowtie}$, where $\mathbf{A} = \langle A, \wedge, \vee, \sim, \Diamond_+, \Diamond_-, 0, 1\rangle$ is a bimodal Boolean algebra. We take $\mathsf{At}(\mathbf{A})$ as the set of points of our Kripke frame, on which we define relations $R_+$ and $R_-$ given, for all $x, y \in \mathsf{At}(\mathbf{A})$, by:

$$xR_+y \quad \text{iff} \quad x \leq \Diamond_+ y, \qquad\qquad xR_-y \quad \text{iff} \quad x \leq \Diamond_- y.$$

Thus, we define the *prime structure of* $\mathbf{B}$ as the four-valued Kripke frame $\mathbf{B}_{\bullet} = \langle \mathsf{At}(\mathbf{A}), R_+, R_-\rangle$. The following results summarizes the duality between perfect modal bilattices and four-valued Kripke frames (cf. [14, Proposition 18]).

**Proposition 5.5** *For every four-valued Kripke frame* $\mathcal{F}$ *and every perfect modal bilattice* $\mathbf{B}$*, we have* $\mathcal{F} \cong (\mathcal{F}^{\bullet})_{\bullet}$ *and* $\mathbf{B} \cong (\mathbf{B}_{\bullet})^{\bullet}$.

The correspondence of objects established by the preceding proposition extends to morphisms and can thus be formulated as a categorical duality. We will not pursue this here, but we are going to see how the correspondence sketched above allows us to translate epistemic updates from the algebraic into the relational setting.

Given a perfect modal bilattice $\mathbf{B} = \mathbf{A}^{\bowtie}$ and $a = \langle x, x'\rangle \in B$, we let $\bar{a} := \{y \in \mathsf{At}(\mathbf{A}) : y \leq x\}$. Thus, the subframe $(\mathbf{B}_{\bullet})^{\bar{a}}$ of the prime structure $\mathbf{B}_{\bullet}$ is $\langle \bar{a}, R_+ \cap (\bar{a} \times \bar{a}), R_- \cap (\bar{a} \times \bar{a})\rangle$. We then have the following.

**Proposition 5.6** *For every perfect modal bilattice* $\mathbf{B}$ *and every* $a \in B$*, we have* $(\mathbf{B}^a)_{\bullet} \cong (\mathbf{B}_{\bullet})^{\bar{a}}$.

Rephrasing a remark in [14, Section 4.3], we can say that the identification between the two relational structures above shows that the mechanism of epistemic updates for public announcements is essentially unchanged when moving from the classical to an intuitionistic and even to a bilattice setting.

Joining the above results, it is easy to see that the definition of satisfaction for formulas of type $\langle\alpha\rangle\varphi$,

$$M, w \vDash \langle\alpha\rangle\varphi \quad \text{iff} \quad M, w \vDash \alpha \text{ and } M^{\alpha}, w \vDash \varphi$$

can be rewritten as follows: $w \in v_+(\langle\alpha\rangle\varphi)$ iff $\exists w' \in W^{\alpha}$ such that $j(w') = w \in v_+(\alpha)$ and $w' \in v_+^{\alpha}(\varphi)$. Since the map $j : W^{\alpha} \hookrightarrow W$ is injective, we have $w' \in v_+^{\alpha}(\varphi)$ iff $w = j(w') \in j(v_+^{\alpha}(\varphi))$.

Hence we have $w \in v_+(\langle\alpha\rangle\varphi)$ iff $w \in v_+(\alpha) \cap j(v_+^\alpha(\varphi))$, i.e. $v_+(\langle\alpha\rangle\varphi) = v_+(\alpha) \cap j(v_+^\alpha(\varphi))$.

Since $v_+(\alpha) = v_+({\sim}{\sim}\alpha)$ for any $\alpha \in Fm$ and any valuation $v$ and, as observed earlier, satisfaction of a formula in bilattice modal logic only depends, for each valuation $v$, on its "positive part" $v_+(\alpha)$, we have that the result of Proposition 5.1 indeed extends to any BPAL formula.

**Proposition 5.7** *For every four-valued Kripke model $M = \langle \mathcal{F}, v\rangle$ and every formula $\varphi$ of BPAL, $M \vDash \varphi$ iff $\langle \mathcal{F}^\bullet, v^\bullet\rangle \vDash \varphi$.*

## 6 Soundness and completeness

The following lemmas are needed to establish that the calculus of Section 4 is sound with respect to the above-introduced algebraic semantics (cf. [14, Lemmas 29-34]).

**Lemma 6.1 ([16], Lemma 4.1)** *Let $M = (\mathbf{B}, v)$ be an algebraic model and $\varphi$ a formula such that $[\![\varphi]\!]_{M^\alpha} = \pi([\![\varphi]\!]_M)$ for any $\alpha \in Fm$. Then $[\![\langle\alpha\rangle\varphi]\!]_M = {\sim}{\sim}[\![\alpha]\!]_M \wedge [\![\varphi]\!]_M$ and $[\![[\alpha]\varphi]\!]_M = [\![\alpha]\!]_M \supset [\![\varphi]\!]_M$.*

**Fact 6.2 ([16], Lemma 4.2)** *Let $\mathbf{B}$ be modal bilattice, $a \in B$, and let $i' =: \mathbf{B}^a \to \mathbf{B}$ be given, for every $[b] \in B^a$, by $i'[b] := b \wedge {\sim}{\sim}a$. Then, for every $[b], [c] \in B^a$,*

*(i)* $i'([b] \wedge [c]) = i'[b] \wedge i'[c]$

*(ii)* $i'([b] \vee [c]) = i'[b] \vee i'[c]$

*(iii)* $i'([b] \supset [c]) = {\sim}{\sim}a \wedge (i'[b] \supset i'[c])$

*(iv)* $i'(\neg[b]) = {\sim}{\sim}a \wedge \neg i'[b]$

*(v)* $i'(\Diamond^a[b]) = {\sim}{\sim}a \wedge \Diamond(i'[b]) = {\sim}{\sim}a \wedge \Diamond({\sim}{\sim}a \wedge i'[b])$

*(vi)* $i'(\Box^a[b]) = {\sim}{\sim}a \wedge \Box(a \supset i'[b])$.

**Lemma 6.3 ([16], Lemma 4.3)** *For any algebraic model $M = (\mathbf{B}, v)$ with underlying modal bilattice $\mathbf{B} = \langle B, \wedge, \vee, \supset, \neg, \Diamond, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$ and for and all formulas $\alpha, \varphi, \psi \in Fm$,*

*(i)* $[\![\langle\alpha\rangle(\varphi \wedge \psi)]\!]_M = [\![\langle\alpha\rangle\varphi]\!]_M \wedge [\![\langle\alpha\rangle\psi]\!]_M$

*(ii)* $[\![\langle\alpha\rangle(\varphi \vee \psi)]\!]_M = [\![\langle\alpha\rangle\varphi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M$

*(iii)* $[\![\langle\alpha\rangle(\varphi \supset \psi)]\!]_M = {\sim}{\sim}[\![\alpha]\!]_M \wedge ([\![\langle\alpha\rangle\varphi]\!]_M \supset [\![\langle\alpha\rangle\psi]\!]_M)$

*(iv)* $[\![\langle\alpha\rangle\neg\varphi]\!]_M = {\sim}{\sim}[\![\alpha]\!]_M \wedge \neg[\![\langle\alpha\rangle\varphi]\!]_M$

*(v)* $[\![[\alpha]\varphi]\!]_M = [\![\neg\langle\alpha\rangle\neg\varphi]\!]_M$

*(vi)* $[\![\langle\alpha\rangle\Diamond\varphi]\!]_M = {\sim}{\sim}[\![\alpha]\!]_M \wedge \Diamond[\![\langle\alpha\rangle\varphi]\!]_M$

*(vii)* $[\![\langle\alpha\rangle\Box\varphi]\!]_M = {\sim}{\sim}[\![\alpha]\!]_M \wedge \Box[\![[\alpha]\varphi]\!]_M$.

Item (v) of the preceding lemma shows that the choice of considering the formula $[\alpha]\varphi$ as an abbreviation for $\neg\langle\alpha\rangle\neg\varphi$ is indeed sound. The following result easily follows from Lemma 6.3.

**Fact 6.4** *For any algebraic model* $M = (\mathbf{B}, v)$ *with underlying modal bilattice* $\mathbf{B} = \langle B, \wedge, \vee, \supset, \neg, \Diamond, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ *and for all formulas* $\alpha, \varphi, \psi \in Fm$,

*(i)* $[\![[\alpha](\varphi \wedge \psi)]\!]_M = [\![[\alpha]\varphi]\!]_M \wedge [\![[\alpha]\psi]\!]_M$

*(ii)* $[\![[\alpha](\varphi \vee \psi)]\!]_M = [\![\alpha]\!]_M \supset ([\![\langle\alpha\rangle\varphi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M)$

*(iii)* $[\![[\alpha](\varphi \supset \psi)]\!]_M = [\![\langle\alpha\rangle\varphi]\!]_M \supset [\![\langle\alpha\rangle\psi]\!]_M$

*(iv)* $[\![[\alpha]\neg\varphi]\!]_M = \neg[\![\langle\alpha\rangle\varphi]\!]_M$

*(v)* $[\![[\alpha]\Diamond\varphi]\!]_M = [\![\alpha]\!]_M \supset \Diamond[\![\langle\alpha\rangle\varphi]\!]_M$

*(vi)* $[\![[\alpha]\Box\varphi]\!]_M = [\![\alpha]\!]_M \supset \Box[\![[\alpha]\varphi]\!]_M$.

We are now ready to state the announced completeness result.

**Theorem 6.5** *The calculus for BPAL is sound and complete with respect to algebraic and relational models.*

As a potential direction for future work, we would like here to mention the possibility of extending BPAL in order to define a bilattice version of the logic of epistemic actions and knowledge of [2], along the line, e.g., of [13] which extends this logic to an intuitionistic setting. Another interesting development would be to formalize in BPAL a concrete example of multi-agent reasoning, such as the muddy children puzzle (see [14, Section 5]): this may prove useful in order to better appreciate the potentiality and limits of our new formalism.

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## Appendix

**Proof of Lemma 3.2.** It is sufficient to check that the statement holds in a twist-structure $\mathbf{B} = \mathbf{A}^{\bowtie}$. Assume $\langle b_1, b_2\rangle \equiv_{\langle a_1,a_2\rangle} \langle c_1, c_2\rangle$ and $\langle d_1, d_2\rangle \equiv_{\langle a_1,a_2\rangle} \langle e_1, e_2\rangle$. By Lemma 3.1, this is equivalent to $b_1 \equiv_{a_1} c_1$, $b_2 \equiv_{a_1} c_2$, $d_1 \equiv_{a_1} e_1$, $d_2 \equiv_{a_1} e_2$. Since $\equiv_{a_1}$ is a congruence of the Boolean algebra $\mathbf{A}$, we have, for instance, $\sim b_1 \vee d_1 \equiv_{a_1} \sim c_1 \vee e_1$ and $b_1 \wedge d_2 \equiv_{a_1} c_1 \wedge e_2$. By Lemma 3.1 again, this means that $\langle b_1, b_2\rangle \supset \langle d_1, d_2\rangle \equiv_{\langle a_1,a_2\rangle} \langle c_1, c_2\rangle \supset \langle e_1, e_2\rangle$. Compatibility with all the other bilattice operations can be shown in a similar way.

**Proof of Lemma 3.5.** We claim that the map $h\colon (\mathbf{A}^{\bowtie})^{\langle a_1,a_2\rangle} \to (\mathbf{A}^{a_1})^{\bowtie}$ defined, for all $\langle b_1, b_2\rangle \in \mathbf{A}^{\bowtie}$, by $h([\langle b_1, b_2\rangle]_{\langle a_1,a_2\rangle}) := \langle [b_1]_{a_1}, [b_2]_{a_1}\rangle$ is a modal bilattice isomorphism. Surjectivity of $h$ is immediate. To prove that $h$ is one-to-one, assume $[\langle b_1, b_2\rangle]_{\langle a_1,a_2\rangle} \neq [\langle c_1, c_2\rangle]_{\langle a_1,a_2\rangle}$. This means that $\langle b_1, b_2\rangle \wedge \sim\sim\langle a_1, a_2\rangle = \langle b_1 \wedge a_1, b_2 \vee \sim a_1\rangle \neq \langle c_1 \wedge a_1, c_2 \vee \sim a_1\rangle = \langle c_1, c_2\rangle \wedge \sim\sim\langle a_1, a_2\rangle$. Notice that $b_2 \vee \sim a_1 = c_2 \vee \sim a_1$ iff $\sim(b_2 \vee \sim a_1) = \sim b_2 \wedge \sim\sim a_1 = \sim c_2 \wedge \sim\sim a_1 = \sim(c_2 \vee \sim a_1)$ iff $[\sim b_2]_{a_1} = [\sim c_2]_{a_1}$ iff $[b_2]_{a_1} = [c_2]_{a_1}$. Thus we have either $[b_1]_{a_1} \neq [c_1]_{a_1}$ or $[b_2]_{a_1} \neq [c_2]_{a_1}$. Hence $\langle [b_1]_{a_1}, [b_2]_{a_1}\rangle \neq \langle [c_1]_{a_1}, [c_2]_{a_1}\rangle$, as required. Thus $h$ is a bijection. Moreover, using Fact 3.2 and [14, Fact 7.4], one can check that, for instance,

$$
\begin{aligned}
&h([\langle b_1, b_2\rangle]_{\langle a_1,a_2\rangle} \supset [\langle c_1, c_2\rangle]_{\langle a_1,a_2\rangle}) = \\
&= h([\langle b_1, b_2\rangle \supset \langle c_1, c_2\rangle]_{\langle a_1,a_2\rangle}) \\
&= h([\langle \sim b_1 \vee c_1, b_1 \wedge c_2\rangle]_{\langle a_1,a_2\rangle}) \\
&= \langle [\sim b_1 \vee c_1]_{a_1}, [b_1 \wedge c_2]_{a_1}\rangle \\
&= \langle \sim[b_1]_{a_1} \vee [c_1]_{a_1}, [b_1]_{a_1} \wedge [c_2]_{a_1}\rangle \\
&= \langle [b_1]_{a_1}, [b_2]_{a_1}\rangle \supset \langle [c_1]_{a_1}, [c_2]_{a_1}\rangle \\
&= h([\langle b_1, b_2\rangle]_{\langle a_1,a_2\rangle}) \supset h([\langle c_1, c_2\rangle]_{\langle a_1,a_2\rangle}).
\end{aligned}
$$

The cases of the other non-modal connectives are similar, so we omit the proof. We can use Fact 3.2 and [14, Fact 7.4] to check that the modal operator is also preserved:

$$
\begin{aligned}
&h(\Diamond^{\langle a_1,a_2\rangle}[\langle b_1,b_2\rangle]_{\langle a_1,a_2\rangle}) = \\
&= h([\Diamond(\langle b_1,b_2\rangle \wedge \sim\sim\langle a_1,a_2\rangle)]_{\langle a_1,a_2\rangle}) \\
&= h([\Diamond\langle b_1 \wedge a_1, b_2 \vee \sim a_1\rangle]_{\langle a_1,a_2\rangle}) \\
&= h([\langle \Diamond_+(b_1 \wedge a_1), \Box_+(b_2 \vee \sim a_1) \wedge \Box_- \sim(b_1 \wedge a_1)\rangle]_{\langle a_1,a_2\rangle}) \\
&= \langle [\Diamond_+(b_1 \wedge a_1)]_{a_1}, [\Box_+(b_2 \vee \sim a_1) \wedge \Box_- \sim(b_1 \wedge a_1)]_{a_1}\rangle \\
&= \langle [\Diamond_+(b_1 \wedge a_1)]_{a_1}, [\Box_+(b_2 \vee \sim a_1)]_{a_1} \wedge [\Box_- \sim(b_1 \wedge a_1)]_{a_1}\rangle \\
&= \langle [\Diamond_+(b_1 \wedge a_1)]_{a_1}, [\Box_+(\sim a_1 \vee b_2)]_{a_1} \wedge [\Box_-(\sim a_1 \vee \sim b_1)]_{a_1}\rangle \\
&= \langle \Diamond_+[b_1]_{a_1}, \Box_+[b_2]_{a_1} \wedge \Box_-[\sim b_1]_{a_1}\rangle \\
&= \langle \Diamond_+[b_1]_{a_1}, \Box_+[b_2]_{a_1} \wedge \Box_- \sim[b_1]_{a_1}\rangle \\
&= \Diamond\langle [b_1]_{a_1}, [b_2]_{a_1}\rangle \\
&= \Diamond(h([\langle b_1,b_2\rangle]_{\langle a_1,a_2\rangle})).
\end{aligned}
$$

**Proof of Proposition 5.5.** $\mathcal{F} \cong (\mathcal{F}^\bullet)_\bullet$ is the first claim. We have $(\mathcal{F}^\bullet)_\bullet = \langle \mathsf{At}(P(W)), R'_+, R'_-\rangle$. Clearly the map $\epsilon\colon \mathcal{F} \to (\mathcal{F}^\bullet)_\bullet$ given by $\epsilon(x) := \{x\}$ for all $x \in W$ is a bijection. Moreover, for all $x, y \in W$, we have

$$
\begin{array}{llllll}
\epsilon(x)R'_+\epsilon(y) & \text{iff} & \{x\}R'_+\{y\} & & \text{iff} & \{x\} \subseteq \Diamond_+\{y\} \\
 & \text{iff} & \{x\} \subseteq R_+^{-1}[\{y\}] & & \text{iff} & x \in R_+^{-1}[\{y\}] \\
 & \text{iff} & xR_+y. & & &
\end{array}
$$

Similarly one proves that $\epsilon(x)R'_-\epsilon(y)$ if and only if $xR_-y$. We now prove $\mathbf{B} \cong (\mathbf{B}_\bullet)^\bullet$ for $\mathbf{B} = \mathbf{A}^{\bowtie}$. Consider the map $\eta\colon \mathbf{B} \to (\mathbf{B}_\bullet)^\bullet$ defined, for all $\langle a_1, a_2\rangle \in A \times A = B$, by $\eta\langle a_1, a_2\rangle := \langle \iota(a_1), \iota(a_2)\rangle$, where $\iota\colon \mathbf{A} \to P(\mathsf{At}(\mathbf{A}))$ is given by $\iota(a) := \{b \in \mathsf{At}(\mathbf{A}) : b \leq a\}$. It easily follows from [14, Proposition 18] that $\iota$ is a Boolean algebra isomorphism and, moreover, $\iota(\Diamond_+ a) = R_+^{-1}[\iota(a)]$ and $\iota(\Diamond_- a) = R_-^{-1}[\iota(a)]$ for all $a \in A$. That is, $\iota$ is an isomorphism of bimodal Boolean algebras. It is then straightforward to check that $\eta$ is a modal bilattice isomorphism. For instance, we have

$$
\begin{aligned}
\eta(\Diamond\langle a_1,a_2\rangle) &= \eta\langle \Diamond_+ a_1, \Box_+ a_2 \wedge \sim \Diamond_- a_1\rangle \\
&= \langle \iota(\Diamond_+ a_1), \iota(\Box_+ a_2 \wedge \sim \Diamond_- a_1)\rangle \\
&= \langle \iota(\Diamond_+ a_1), \iota(\Box_+ a_2) \wedge \sim \iota(\Diamond_- a_1)\rangle \\
&= \langle \Diamond_+(\iota(a_1)), \Box_+(\iota(a_2)) \wedge \sim \Diamond_-(\iota(a_1))\rangle \\
&= \Diamond\langle \iota(a_1), \iota(a_2)\rangle \\
&= \Diamond(\eta\langle a_1,a_2\rangle).
\end{aligned}
$$

**Proof of Proposition 5.6.** Let $\mathbf{B} = \mathbf{A}^{\bowtie}$ and $a = \langle x, x'\rangle \in A \times A$. In this case, by Lemma 3.5, we have $\mathbf{B}^a \cong (\mathbf{A}^x)^{\bowtie}$. We are thus going to define an isomorphism $\kappa\colon ((\mathbf{A}^x)^{\bowtie})_\bullet \to (((\mathbf{A}^{\bowtie})_\bullet)^{\bar{a}}$ given by $\kappa([y]_x) := y \wedge x$ for all $y \in A$ with $[y]_x \in \mathsf{At}(\mathbf{A}^x)$. By [14, Fact 20] we have $\kappa([y]_x) \in \mathsf{At}(\mathbf{A})$, and it is easy to see that $\kappa$ is a bijection. It remains to check $\kappa$ preserves $R_+$ and $R_-$. For all $y, z \in A$ such that $[y]_x, [z]_x \in \mathsf{At}(\mathbf{A}^x)$,

$$
\begin{array}{llll}
[y]_x R_+ [z]_x & \text{iff} & [y]_x \leq \Diamond_+^x [z]_x & \\
& \text{iff} & [y]_x \leq [\Diamond_+(z \wedge x)]_x & \\
& \text{iff} & y \wedge x \leq \Diamond_+(z \wedge x) \wedge x & [14, \text{Fact } 6.2] \\
& \text{iff} & y \wedge x \leq \Diamond_+(z \wedge x) & \\
& \text{iff} & (y \wedge x) R_+ (z \wedge x) & \\
& \text{iff} & \kappa([y]_x) R_+ \kappa([z]_x). &
\end{array}
$$

**Proof of Lemma 6.1.** Concerning the first statement:

$$
\begin{aligned}
[\![\langle\alpha\rangle\varphi]\!]_M &= {\sim}{\sim}[\![\alpha]\!]_M \wedge i'([\![\varphi]\!]_{M^\alpha}) = {\sim}{\sim}[\![\alpha]\!]_M \wedge i'(\pi([\![\varphi]\!]_M)) = \\
&= {\sim}{\sim}[\![\alpha]\!]_M \wedge ([\![\varphi]\!]_M \wedge [\![{\sim}{\sim}\alpha]\!]_M) = {\sim}{\sim}[\![\alpha]\!]_M \wedge [\![\varphi]\!]_M.
\end{aligned}
$$

Concerning the second:

$$
\begin{aligned}
[\![[\alpha]\varphi]\!]_M &= [\![\alpha]\!]_M \supset i'([\![\varphi]\!]_{M^\alpha}) & \\
&= [\![\alpha]\!]_M \supset i'(\pi([\![\varphi]\!]_M)) & \\
&= [\![\alpha]\!]_M \supset ([\![\varphi]\!]_M \wedge [\![{\sim}{\sim}\alpha]\!]_M) & \\
&= ([\![\alpha]\!]_M \supset [\![\varphi]\!]_M) \wedge ([\![\alpha]\!]_M \supset [\![{\sim}{\sim}\alpha]\!]_M) & (1) \\
&= ([\![\alpha]\!]_M \supset [\![\varphi]\!]_M) \wedge ([\![\alpha]\!]_M \supset {\sim}{\sim}[\![\alpha]\!]_M) & \\
&= ([\![\alpha]\!]_M \supset [\![\varphi]\!]_M) \wedge \mathsf{t} & \mathsf{t} = x \supset {\sim}{\sim}x \\
&= [\![\alpha]\!]_M \supset [\![\varphi]\!]_M. & x \leq_t \mathsf{t}
\end{aligned}
$$

Here (1) holds because the equation $x \supset (y \wedge z) = (x \supset y) \wedge (x \supset z)$ is satisfied by every modal bilattice.

**Proof of Fact 6.2.**

(i)

$$
\begin{aligned}
i'([b] \wedge [c]) &= i'([b \wedge c]) & \text{Fact 3.2} \\
&= (b \wedge c) \wedge {\sim}{\sim}a & \\
&= (b \wedge {\sim}{\sim}a) \wedge (c \wedge {\sim}{\sim}a) & \\
&= i'[b] \wedge i'[c]. &
\end{aligned}
$$

(ii)

$$
\begin{aligned}
i'([b] \vee [c]) &= i'([b \vee c]) & \text{Fact 3.2} \\
&= (b \vee c) \wedge {\sim}{\sim}a & \\
&= (b \wedge {\sim}{\sim}a) \vee (c \wedge {\sim}{\sim}a) & \text{distributivity} \\
&= i'[b] \vee i'[c]. &
\end{aligned}
$$

(iii) We are going to use Fact 3.2 together with the following identities:

$$
\begin{aligned}
{\sim}{\sim}x \wedge (y \supset z) &= {\sim}{\sim}x \wedge ((y \wedge {\sim}{\sim}x) \supset z) \\
\mathsf{t} &= (x \wedge y) \supset {\sim}{\sim}y \\
(x \supset y) \wedge (x \supset z) &= x \supset (y \wedge z)
\end{aligned}
$$

which are valid in any modal bilattice. We have:

$$
\begin{aligned}
i'([b] \supset [c]) &= i'[b \supset c] \\
&= \sim\sim a \wedge (b \supset c) = \sim\sim a \wedge ((b \wedge \sim\sim a) \supset c) = \\
&= \sim\sim a \wedge (((b \wedge \sim\sim a) \supset c) \wedge \mathsf{t}) \\
&= \sim\sim a \wedge (((b \wedge \sim\sim a) \supset c) \wedge ((b \wedge \sim\sim a) \supset \sim\sim a)) \\
&= \sim\sim a \wedge ((b \wedge \sim\sim a) \supset (c \wedge \sim\sim a)) \\
&= \sim\sim a \wedge (i'[b] \supset i'[c]).
\end{aligned}
$$

(iv)

$$
\begin{aligned}
i'(\neg[b]) &= i'([\neg b]) = \sim\sim a \wedge \neg b = && \text{Fact 3.2} \\
&= (\sim\sim a \wedge \neg b) \vee \mathsf{f} && \mathsf{f} \leq_t x \\
&= (\sim\sim a \wedge \neg b) \vee (\sim\sim a \wedge \neg\sim\sim a) && \mathsf{f} = \sim\sim x \wedge \neg\sim\sim x \\
&= \sim\sim a \wedge (\neg b \vee \neg\sim\sim a) && \text{distributivity} \\
&= \sim\sim a \wedge \neg(b \wedge \sim\sim a) && \text{De Morgan law} \\
&= \sim\sim a \wedge \neg i'[b].
\end{aligned}
$$

(v) Straightforward, because we have on the one hand $i'(\Diamond^a[b]) = i'[\Diamond(b \wedge \sim\sim a)] = \sim\sim a \wedge \Diamond(b \wedge \sim\sim a) = \sim\sim a \wedge \Diamond(i'[b])$, and on the other $i'(\Diamond^a[b]) = i'[\Diamond(b \wedge \sim\sim a)] = i'[\Diamond(b \wedge \sim\sim a) \wedge \sim\sim a] = \sim\sim a \wedge \sim\sim a \wedge \Diamond(b \wedge \sim\sim a) = \sim\sim a \wedge \Diamond(i'[b])$.

**Proof of Theorem 6.5.** Soundness of the preservation of facts and logical constants axioms follow from Lemma 6.1. For the remaining axioms we only need to invoke Lemma 6.3. The proof of completeness is similar to those for classical and intuitionistic PAL [14, Theorem 22] and follows from the reducibility of BPAL to the bilattice modal logic of [12] via reduction axioms. By the deduction-detachment theorem, we can without loss of generality limit ourselves to single formulas. Let $\varphi$ be a valid BPAL formula. Consider some innermost occurrence of a dynamic modality in $\varphi$. Hence, the subformula $\psi$ having that occurrence labeling the root of its generation tree has the form $\langle\alpha\rangle\psi'$ for some formula $\psi'$ in the static language. The distribution axioms make it possible to equivalently transform $\psi$ by pushing the dynamic modality down the generation tree, through the static connectives, until it attaches to a proposition letter or to a constant symbol. Here the dynamic modality disappears by applying the appropriate 'preservation of facts' or 'interaction with constant' axiom. The process is repeated for all dynamic modalities of $\varphi$, so as to obtain a formula $\varphi'$ which is provably equivalent to $\varphi$. Since $\varphi$ is valid by assumption, and since the process preserves provable equivalence, by soundness we can conclude that $\varphi'$ is valid. By Theorem 2.2, we can conclude that $\varphi'$ is provable in bilattice modal logic and thus in BPAL. This, together with the provable equivalence of $\varphi$ and $\varphi'$, concludes the proof.