Implicative twist-structures

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Abstract. The twist-structure construction is used to represent algebras related to non-classical logics (e.g., Nelson algebras, bilattices) as a special kind of power of better-known algebraic structures (distributive lattices, Heyting algebras). We study a specific type of twist-structure (called *implicative twist-structure*) obtained as a power of a generalized Boolean algebra, focusing on the implication-negation fragment of the usual algebraic language of twist-structures. We prove that implicative twist-structures form a variety which is semisimple, congruence-distributive, finitely generated and has equationally definable principal congruences. We characterize the congruences of each algebra in the variety in terms of the congruences of the associated generalized Boolean algebra. We classify and axiomatize the subvarieties of implicative twist-structures. We define a corresponding logic and prove that it is algebraizable with respect to our variety.

1. Introduction

The twist-structure construction is a convenient way to represent algebras related to non-classical logics as a special kind of power of some other algebraic structure. The usefulness of this representation lies mainly in the fact that it allows us to investigate and solve logical, topological and algebraic problems concerning relatively esoteric classes of algebras by using results on better-known structures, such as Heyting or Boolean algebras.

For instance, it is well-known that *Nelson lattices*, the algebraic counterpart of Nelson logic [24], can be represented as twist-structures over (i.e., special powers of) Heyting algebras [33, 31]. The more recent [25] generalizes the result on Nelson lattices obtaining a twist-structure representation for *N4-lattices*, the algebraic semantics of the paraconsistent version of Nelson logic [2]. The twist-structure construction has also been applied to the study of

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residuated lattices [32, 11, 12] and of algebras used as semantics for modal paraconsistent logics [27, 26, 30, 28, 20].

In [7, 19] extensions of the twist-structure construction have been used to obtain a convenient representation for several classes of *bilattices*: algebras originally introduced in computer science to unify a variety of inference systems, which turned out to have an independent interest both from a logical and an algebraic point of view [22, 23, 29, 8, 7].

The present work introduces and studies a specific type of twist-structure construction and the corresponding logic, restricting our attention to a reduced subset of the usual algebraic language of twist-structures, corresponding, on a logical level, to negation and implication. We try to show that the structures thus constructed have an independent interest, both from a logical and a purely algebraic point of view, showing that most of the known results on general twist-structures can be obtained even when working within such a reduced fragment of the language.

Some of the results of the present paper can also be found, although in a slightly different guise, in the dissertation [29]. It is proved there that the bilattice logic of Arieli and Avron [3] is algebraizable (in the sense of [5]) with respect to a variety of algebras introduced in [29] under the name of *implicative bilattices*. The implicative twist-structures that we are going to define in the next section correspond, on a logical level, to the negation-implication fragment of the Arieli-Avron logic and, on an algebraic level, to the negation-implication subreducts of implicative bilattices.

The paper is organized as follows.

In Section 2 we introduce the concrete construction that allows us to build what we call an *implicative twist-structure* as a special power of a distributive lattice satisfying certain additional properties (a *classical implicative lattice*). We consider some especially interesting examples of algebras obtained through this construction and fix the notation that will be used throughout the paper.

In Section 3 we introduce through an equational presentation an abstract class of algebras, called *$\mathcal{I}$-algebras*, and we prove (Theorem 3.8) that these correspond precisely to the implicative twist-structures of Section 2. We end the section with a brief discussion of a problem that is still open: namely, whether the representation given by Theorem 3.8 can be improved to obtain a characterization result similar to the ones that are known for Nelson and N4-lattices.

Section 4 studies the variety of $\mathcal{I}$-algebras from the point of view of universal algebra. The main result (Theorem 4.7) is a characterization of the congruences of each algebra in the variety in terms of the congruences of its associated classical implicative lattice. This allows us to prove that $\mathcal{I}$-algebras form an arithmetical variety that is generated by a single finite algebra. We also use the main result to axiomatize the subvarieties of $\mathcal{I}$-algebras, showing that for some of them it is actually possible to improve the representation result of Theorem 3.8 as indicated above. We end the section by mentioning the problem of classifying the subquasivarieties of $\mathcal{I}$-algebras, which is still open.

In Section 5 we introduce a Hilbert-style calculus as a syntactic counterpart of our algebraic structures. We prove that the logic defined by our calculus is algebraizable (in the sense of [5]) with respect to the class of $\mathcal{I}$-algebras and give syntactic presentations for its axiomatic extensions, which correspond to subvarieties of $\mathcal{I}$-algebras.

In the concluding Section 6 we briefly discuss some topics that in our opinion deserve further investigation.

2. The implicative twist-structure construction

By a *classical implicative lattice* we mean an algebra $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that $\langle L, \sqcap, \sqcup, 1\rangle$ is a lattice (whose order we denote by $\leq$) with top element 1 that satisfies the following properties: for all $x, y, z \in L$,

$$x \sqcap y \leq z \quad \text{if and only if} \quad y \leq x\backslash z, \tag{R}$$

$$(x\backslash y)\backslash x = x. \tag{P}$$

Property (R) is usually called *residuation*, while (P) is known in logical contexts as *Peirce's law*.

The name "classical implicative lattices", which can be found in [16], is motivated by the fact that this class of algebras is the algebraic semantics of the negation-free fragment of classical propositional logic. The name *generalized Boolean algebras* [1] can also be found in the literature, as classical implicative lattices coincide in fact with 0-free subreducts of Boolean algebras.

It is well-known [18] that (R) can be expressed by equations, therefore classical implicative lattices form a variety. It is also easy to prove that (R) implies that the lattice reduct $\langle L, \sqcap, \sqcup\rangle$ of any classical implicative lattice $\mathbf{L}$ is distributive. Let us also note that the reduct $\langle L, \backslash, 1\rangle$ forms what is known as a *Tarski algebra* [34], which we will define formally in the next section. For now let us just notice that any Tarski algebra is a join semilattice in which the join $\sqcup$ is given by

$$x \sqcup y = (x\backslash y)\backslash y$$

and that the semilattice order $\leq$ can be defined by

$$x \leq y \quad \text{if and only if} \quad x\backslash y = 1.$$

In the case of classical implicative lattices, this implies that the join $\sqcup$ need not be included in the set of primitive operations because it can be explicitly defined as shown above. Likewise, the constant 1 can be omitted from the signature because the following conditions hold in any classical implicative lattice (and in any Taski algebra):

$$x \leq x \backslash x \qquad\qquad x \backslash x = y \backslash y,$$

that is, one can simply define $1 := x \backslash x$.

We are now ready to introduce our construction.

Definition 2.1. Let $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1 \rangle$ be a classical implicative lattice. The *full implicative twist-structure over* $\mathbf{L}$ is the algebra $\mathbf{L}^{\bowtie} = \langle L \times L, \supset, \neg \rangle$ with operations defined, for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L \times L$, as follows:

$$\langle a_1, a_2 \rangle \supset \langle b_1, b_2 \rangle := \langle a_1 \backslash b_1, a_1 \sqcap b_2 \rangle,$$
$$\neg \langle a_1, a_2 \rangle := \langle a_2,\ a_1 \rangle .$$

An *implicative twist-structure over* $\mathbf{L}$ is a subalgebra $\mathbf{A}$ (with respect to the language $\{\supset, \neg\}$) of the full twist-structure $\mathbf{L}^{\bowtie}$ satisfying the property that $\pi_1(A) = L$, where $\pi_1(A) = \{a_1 \in L : \langle a_1, a_2 \rangle \in A$ for some $a_2 \in L\}$. We write $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$ to mean that $\mathbf{A}$ is a twist-structure over $\mathbf{L}$.

Consistently with the logical interpretation of the algebraic operations of N4-lattices (and also bilattices), we think of the $\supset$ operation as an *implication* and of $\neg$ as a *negation*. Notice that negation is involutive, i.e., $\neg\neg = Id_A$. The technical condition that $\pi_1(A) = L$ is meant to ensure that the relation between an implicative twist-structure and its associated classical implicative lattice is in some sense canonical (i.e., that $\mathbf{A}$ is somehow maximal inside $\mathbf{L}^{\bowtie}$). This will become more clear later, when we will start with an abstractly-defined class of algebras and will prove that they coincide with the implicative twist-structures defined above. Notice also that, because of the presence of negation, it holds that $\pi_1(A) = \pi_2(A)$, where $\pi_2(A) = \{a_2 \in L : \langle a_1, a_2 \rangle \in A$ for some $a_1 \in L\}$.

As mentioned above, our implicative twist-structures can be seen as a special case of the twist-structures of [25], the only differences being that (i) we admit only two algebraic operations in the language (corresponding, on a logical level, to implication and negation) and (ii) that we assume that the underlying lattice satisfies Peirce's law. Formally, this can be expressed by saying that our implicative twist-structures coincide with the $\{\supset, \neg\}$-subreducts of the *implicative bilattices* of [29, 7, 9] and also with the $\{\supset, \neg\}$-subreducts of the subvariety of N4-lattices satisfying the following identity:

$$((x \supset y) \supset x) \supset x = x \supset x.$$

On any implicative twist-structure $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$ it is possible to define several (pre-)order relations by using the order $\leq$ of the associated lattice $\mathbf{L}$. Following the existing theory (and notation) of twist-structures, we focus our attention on the pre-order $\preceq$ defined as follows: for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in A \subseteq L \times L$,

$$\langle a_1, a_2 \rangle \preceq \langle b_1, b_2 \rangle \quad \text{if and only if} \quad a_1 \leq b_1.$$

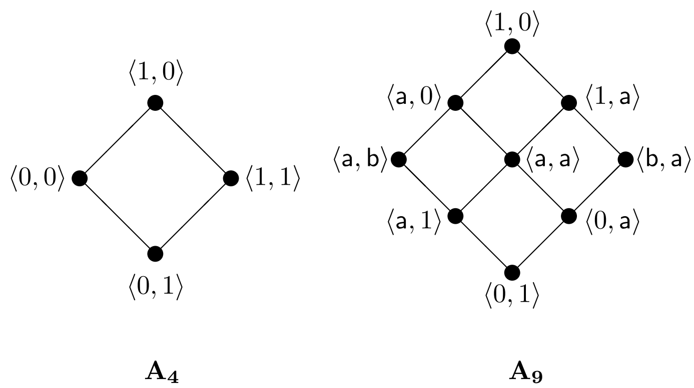

Figure 1. Examples of implicative twist-structures

It is easy to check that $\preceq$ is in fact a pre-order and also that, for all $a, b \in A$, the following holds:

$$a \preceq b \quad \text{if and only if} \quad a \supset b = (a \supset b) \supset (a \supset b). \tag{2.1}$$

It is also obvious that the following conditions are equivalent:

(i) $a = b$,
(ii) $a \preceq b$, $b \preceq a$, $\neg a \preceq \neg b$ and $\neg b \preceq \neg a$.

Using $\preceq$ we can further define the following relations, which are readily seen to be partial orders:

$$\leq_1 := \{\langle a, b\rangle \in A \times A : a \preceq b \text{ and } \neg b \preceq \neg a\},$$
$$\leq_2 := \{\langle a, b\rangle \in A \times A : a \preceq b \text{ and } \neg a \preceq \neg b\}.$$

Component-wise, we have that, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L \times L$,

$$\langle a_1, a_2\rangle \leq_1 \langle b_1, b_2\rangle \quad \text{iff} \quad a_1 \leq b_1 \text{ and } b_2 \leq a_2,$$
$$\langle a_1, a_2\rangle \leq_2 \langle b_1, b_2\rangle \quad \text{iff} \quad a_1 \leq b_1 \text{ and } a_2 \leq b_2.$$

The minimum and maximum elements corresponding to $\leq_1$ (which need not belong to an arbitrary non-full implicative twist-structure) are then, respectively, $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$, where 0 and 1 denote the minimum and maximum elements of **L**. Similarly, the minimum and maximum elements of $\leq_2$ are $\langle 0, 0\rangle$ and $\langle 1, 1\rangle$. Notice that the negation operator is anti-monotonic w.r.t. $\leq_1$ and monotonic w.r.t. $\leq_2$. In fact, if our implicative twist-structure happens to be (the reduct of) an N4-lattice, then $\leq_1$ is its natural lattice order. Likewise, if the implicative twist-structure is (the reduct of) an implicative bilattice in the sense of [29], then $\leq_1$ corresponds to the so-called *truth order* and $\leq_2$ to the so-called *knowledge order* of the bilattice.

Figure 1 shows some interesting examples of implicative twist-structures (the poset structure represented in the Hasse diagram is the one corresponding to the $\leq_1$ order defined above). The algebra $\mathbf{A_4}$ is the (full) implicative twist-structure over the two-element classical implicative lattice $\mathbf{L_2} = \langle \{0,1\}, \sqcap, \sqcup, \backslash, 1 \rangle$ which is the $\{\sqcap, \sqcup, \backslash, 1\}$-reduct of the two-element Boolean algebra. $\mathbf{A_4}$ has three non-trivial subalgebras, which we denote by $\mathbf{A_3^+}$, $\mathbf{A_3^-}$ and $\mathbf{A_2}$, corresponding to universes $A_3^+ = \{\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle\}$, $A_3^- = \{\langle 0,0\rangle, \langle 1,0\rangle, \langle 0,1\rangle\}$, $A_2 = \{\langle 1,0\rangle, \langle 0,1\rangle\}$, and a trivial one having as universe $\{\langle 1,1\rangle\}$. Notice that, while the trivial subalgebra is in fact an implicative twist-structure over the one-element classical implicative lattice, $\mathbf{A_3^+}$, $\mathbf{A_3^-}$ and $\mathbf{A_2}$ are all implicative twist-structures over $\mathbf{L_2}$ because they meet the condition that $\pi_1(A_3^+) = \pi_1(A_3^-) = \pi_1(A_2) = \{0,1\}$. As we will see in a later section, $\mathbf{A_4}$, $\mathbf{A_3^+}$ and $\mathbf{A_3^-}$ play an important role within the class of all implicative twist-structures. Let us also note that $\mathbf{A_2}$ is isomorphic to the two-element Boolean algebra (where the operation $\supset$ is the Boolean implication and $\neg$ is the Boolean complementation).

The algebra $\mathbf{A_9}$ is another example that will turn out to be especially interesting for us. It is an implicative twist-structure over the four-element classical implicative lattice $\mathbf{L_4} = \langle \{0, \mathsf{a}, \mathsf{b}, 1\}, \sqcap, \sqcup, \backslash, 1 \rangle$ which is the $\{\sqcap, \sqcup, \backslash, 1\}$-reduct of the four-element Boolean algebra. Notice that the eight-element set $A_9 - \{\langle \mathsf{a}, \mathsf{a}\rangle\}$ is the universe of a subalgebra of $\mathbf{A_9}$, which we will denote by $\mathbf{A_8}$. It is easy to check that $\mathbf{A_8}$ is generated inside $(\mathbf{L_4})^{\bowtie}$ by the set $\{\langle 1, \mathsf{a}\rangle, \langle \mathsf{a}, 0\rangle\}$. Notice also that in all the above-mentioned examples except $\mathbf{A_8}$, the partial order $\leq_1$ is in fact a lattice order, i.e., any finite set has a least upper bound and a greatest lower bound. This is not true in $\mathbf{A_8}$ because for instance $\langle 1, \mathsf{a}\rangle$ and $\langle \mathsf{a}, 0\rangle$ do not have a greatest lower bound (which is precisely $\langle \mathsf{a}, \mathsf{a}\rangle$ in $\mathbf{A_9}$). On the other hand, in the above examples the partial order $\leq_2$ never forms a lattice except in $\mathbf{A_4}$.

Let us introduce some more abbreviations that will be convenient. Given an implicative twist-structure $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$ and elements $a, b \in A$, we define

$$\begin{aligned} a * b &:= \neg(a \supset \neg b), \\ a \to b &:= (a \supset b) * (\neg b \supset \neg a), \\ a \leftrightarrow b &:= (a \to b) * (b \to a). \end{aligned}$$

Viewing the elements of $\mathbf{A}$ as pairs $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L \times L$ and applying the definitions, we have that

$$\langle a_1, a_2\rangle * \langle b_1, b_2\rangle = \langle a_1 \sqcap b_1, a_1 \backslash b_2\rangle.$$

From our point of view the key feature of the above-defined operation is that it allows us to recover the meet of the underlying lattice, if only in the first component. For instance, it is easy to check that the condition that $a_1 = 1$ can be expressed as follows:

$$\langle a_1, a_2\rangle = \langle a_1, a_2\rangle \supset \langle a_1, a_2\rangle.$$

This is so because, applying the definitions, we have that

$$\langle a_1, a_2\rangle \supset \langle a_1, a_2\rangle = \langle a_1\backslash a_1, a_1 \sqcap a_2\rangle = \langle 1, a_1 \sqcap a_2\rangle.$$

Now, if we want to express the two conditions that $a_1 = 1$ and $b_1 = 1$ together using only one equation, we can do it as follows:

$$\langle a_1, a_2\rangle * \langle b_1, b_2\rangle = (\langle a_1, a_2\rangle * \langle b_1, b_2\rangle) \supset (\langle a_1, a_2\rangle * \langle b_1, b_2\rangle).$$

For analogous reasons we are mainly interested in the behaviour of the operations $\rightarrow$ and $\leftrightarrow$ in the first component of each pair. According to the above definitions, these are, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L \times L$,

$$\pi_1(\langle a_1, a_2\rangle \rightarrow \langle b_1, b_2\rangle) = (a_1\backslash b_1) \sqcap (b_2\backslash a_2),$$
$$\pi_1(\langle a_1, a_2\rangle \leftrightarrow \langle b_1, b_2\rangle) = (a_1\backslash b_1) \sqcap (b_2\backslash a_2) \sqcap (b_1\backslash a_1) \sqcap (a_2\backslash b_2).$$

We see then that, for all $a, b \in A$, the following equivalences hold:

$$a \leq_1 b \quad \text{if and only if} \quad a \rightarrow b = (a \rightarrow b) \supset (a \rightarrow b)$$

and

$$a = b \quad \text{if and only if} \quad a \leftrightarrow b = (a \leftrightarrow b) \supset (a \leftrightarrow b).$$

The above properties suggest that, from a logical point of view, the operation $\rightarrow$ behaves somehow like an implication connective (alternative to the other implication $\supset$) and that $\leftrightarrow$ behaves like an equivalence connective.

We end the section by showing how, given an implicative twist-structure $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$, it is possible to recover the underlying classical implicative lattice $\mathbf{L}$. This will give us hints on how to proceed in order to obtain an abstract characterization of implicative twist-structures.

From the pre-order $\preceq$ introduced above we can define an equivalence relation in the usual way: for all $a, b \in A$,

$$a \equiv b \quad \text{if and only if} \quad a \preceq b \;\text{ and }\; b \preceq a.$$

On ordered pairs $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L \times L$, we see that

$$\langle a_1, a_2\rangle \equiv \langle b_1, b_2\rangle \quad \text{if and only if} \quad a_1 = b_1.$$

It is therefore easy to check that the relation $\equiv$ is compatible with the operations $\supset$ and $*$ but it is not compatible with any other operation of the implicative twist-structure. We can thus consider the quotient $\langle A/{\equiv}, *, \supset\rangle$. We note that $\langle A/{\equiv}, *\rangle$ is a meet semilattice and that $\langle *, \supset\rangle$ form a residuated pair in $A/{\equiv}$. Moreover, $\langle A/{\equiv}, \supset\rangle$ is a Tarski algebra whose maximum element 1 is given by the equivalence class of $a \supset a$ for any $a \in A$. As mentioned before, this implies that we can define a join operation on $A/{\equiv}$ as follows: for all equivalence classes $[a], [b] \in A/{\equiv}$,

$$[a] \sqcup [b] = ([a] \supset [b]) \supset [b].$$

Putting all these observations together, one can check that the order associated with the meet semilattice $\langle A/{\equiv}, *\rangle$ and the one associated with the join semilattice $\langle A/{\equiv}, \sqcup\rangle$ are actually the same, i.e., $\langle A/{\equiv}, *, \sqcup, \supset, 1\rangle$ is indeed a classical implicative lattice which is isomorphic to **L**. As we will see in the next section, the important point is that the construction sketched above can be carried out even if we start from an abstract definition of implicative twist-structures, as long as we impose certain (equational) properties on the class of algebras we are defining (for instance, that the relation $\preceq$ defined as in (2.1) be a pre-order, etc.).

3. Abstract twist-structures

We are now going to introduce a variety of algebras that will be proven to correspond precisely to the implicative twist-structures defined in the previous section. We will be working with algebras $\mathbf{A} = \langle A, \supset, \neg \rangle$ of type $\langle 2, 1\rangle$ and will adopt the same abbreviations used in the previous section, that is, we define, for all $a, b \in A$,

$$\begin{aligned} a * b &:= \neg(a \supset \neg b), \\ a \to b &:= (a \supset b) * (\neg b \supset \neg a), \\ a \leftrightarrow b &:= (a \to b) * (b \to a), \\ a \preceq b &\Leftrightarrow a \supset b = (a \supset b) \supset (a \supset b), \\ a \equiv b &\Leftrightarrow a \preceq b \ \text{ and } \ b \preceq a. \end{aligned}$$

Let us also introduce the following abbreviation: for any element $a \in A$, we write $E(a)$ as a shorthand for $a = a \supset a$ and we let

$$E(\mathbf{A}) := \{a \in A : a = a \supset a\}.$$

Using this notation, we have that $a \preceq b$ if and only if $E(a \supset b)$ if and only if $a \supset b \in E(\mathbf{A})$.

We are now ready to introduce our definition of abstract implicative twist-structures.

Definition 3.1. An $\mathcal{I}$-*algebra* is an algebra $\mathbf{A} = \langle A, \supset, \neg \rangle$ satisfying the following equations:

$$(x \supset x) \supset y = y, \tag{I1}$$

$$x \supset (y \supset z) = (x \supset y) \supset (x \supset z) = y \supset (x \supset z), \tag{I2}$$

$$((x \supset y) \supset x) \supset x = x \supset x, \tag{I3}$$

$$x \supset (y \supset z) = (x * y) \supset z, \tag{I4}$$

$$\neg\neg x = x, \tag{I5}$$

$$(x \leftrightarrow y) \supset x = (x \leftrightarrow y) \supset y. \tag{I6}$$

We denote by $\mathcal{I}$-Alg the variety of $\mathcal{I}$-algebras.

We should point out that the name $\mathcal{I}$-*algebra* has already been used in the literature (for instance in [17] and, in a different sense yet, in [21]); these algebraic structures are not related to ours.

Let us first check that the proposed axiomatization is sound, in the sense that the implicative twist-structures defined in the previous section actually satisfy the above axioms. The following known properties of classical implicative lattices will be useful (Cf. [29, Proposition 5.1.1]).

Proposition 3.2. *In a classical implicative lattice* $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1 \rangle$, *for all* $a, b, c \in L$*:*

(i) $a \backslash a = 1$
(ii) $1 \backslash a = a$
(iii) $a \backslash (b \backslash c) = (a \backslash b) \backslash (a \backslash c) = (a \sqcap b) \backslash c$
(iv) $a \sqcap (a \backslash b) = a \sqcap b$
(v) $(a \backslash b) \backslash a = a$
(vi) $((a \backslash b) \sqcap (b \backslash a)) \backslash a = ((a \backslash b) \sqcap (b \backslash a)) \backslash b$
(vii) $a \sqcup (a \backslash b) = 1$
(viii) $a \sqcap (b \backslash a) = a$
(ix) $a \backslash (b \sqcap c) = (a \backslash b) \sqcap (a \backslash c)$.

Proposition 3.3. *Any implicative twist-structure is an* $\mathcal{I}$*-algebra.*

Proof. We are going to check that (I1) to (I6) are satisfied by any twist-structure $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$, where $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1 \rangle$. Let $a_1, a_2, b_1, b_2, c_1, c_2 \in L$.
(I1) Applying the definitions, we have that

$$\begin{aligned} (\langle a_1, a_2 \rangle \supset \langle a_1, a_2 \rangle) \supset \langle b_1, b_2 \rangle &= \langle (a_1 \backslash a_1) \backslash b_1, (a_1 \backslash a_1) \sqcap b_2 \rangle \\ &= \langle 1 \backslash b_1, 1 \sqcap b_2 \rangle && \text{by Prop. 3.2 (i)} \\ &= \langle b_1, b_2 \rangle && \text{by Prop. 3.2 (ii).} \end{aligned}$$

(I2) Let us first prove that $x \supset (y \supset z) = y \supset (x \supset z)$ holds:

$$\begin{aligned} & \langle a_1, a_2 \rangle \supset (\langle b_1, b_2 \rangle \supset \langle c_1, c_2 \rangle) \\ = & \langle a_1 \backslash (b_1 \backslash c_1), a_1 \sqcap b_1 \sqcap c_2 \rangle \\ = & \langle b_1 \backslash (a_1 \backslash c_1), b_1 \sqcap a_1 \sqcap c_2 \rangle && \text{by Prop. 3.2 (iii)} \\ = & \langle b_1, b_2 \rangle \supset (\langle a_1, a_2 \rangle \supset \langle c_1, c_2 \rangle). \end{aligned}$$

As to the remaining equation, applying Proposition 3.2 (iii) and (iv), we have:

$$\begin{aligned} & (\langle a_1, a_2 \rangle \supset \langle b_1, b_2 \rangle) \supset (\langle a_1, a_2 \rangle \supset \langle c_1, c_2 \rangle) \\ = & \langle (a_1 \backslash b_1) \backslash (a_1 \backslash c_1), (a_1 \backslash b_1) \sqcap a_1 \sqcap c_2 \rangle \\ = & \langle a_1 \backslash (b_1 \backslash c_1), a_1 \sqcap b_1 \sqcap c_2 \rangle. \end{aligned}$$

(I3) We have that

$$\begin{aligned} & ((\langle a_1, a_2 \rangle \supset \langle b_1, b_2 \rangle) \supset \langle a_1, a_2 \rangle) \supset \langle a_1, a_2 \rangle \\ = & \langle ((a_1 \backslash b_1) \backslash a_1) \backslash a_1, ((a_1 \backslash b_1) \backslash a_1) \sqcap a_2 \rangle \\ = & \langle a_1 \backslash a_1, a_1 \sqcap a_2 \rangle && \text{by Prop. 3.2 (v)} \\ = & \langle a_1, a_2 \rangle \supset \langle a_1, a_2 \rangle. \end{aligned}$$

(I4). We have that

$$\begin{aligned} & \langle a_1, a_2\rangle \supset (\langle b_1, b_2\rangle \supset \langle c_1, c_2\rangle) \\ = {} & \langle a_1 \backslash (b_1 \backslash c_1), a_1 \sqcap b_1 \sqcap c_2\rangle \\ = {} & \langle (a_1 \sqcap b_1)\backslash c_1, a_1 \sqcap b_1 \sqcap c_2\rangle && \text{by Prop. 3.2 (iii)} \\ = {} & (\langle a_1, a_2\rangle * \langle b_1, b_2\rangle) \supset \langle c_1, c_2\rangle. \end{aligned}$$

(I5) Immediate.
(I6) We need to check that

$$(\langle a_1, a_2\rangle \leftrightarrow \langle b_1, b_2\rangle) \supset \langle a_1, a_2\rangle = (\langle a_1, a_2\rangle \leftrightarrow \langle b_1, b_2\rangle) \supset \langle b_1, b_2\rangle.$$

According to our definitions, we have that the first component of $\langle a_1, a_2\rangle \leftrightarrow \langle b_1, b_2\rangle$ is $(a_1\backslash b_1) \sqcap (b_2\backslash a_2) \sqcap (b_1\backslash a_1) \sqcap (a_2\backslash b_2)$. Therefore we need to check that

$$\begin{aligned} & ((a_1\backslash b_1) \sqcap (b_2\backslash a_2) \sqcap (b_1\backslash a_1) \sqcap (a_2\backslash b_2))\backslash a_1 \\ = {} & ((a_1\backslash b_1) \sqcap (b_2\backslash a_2) \sqcap (b_1\backslash a_1) \sqcap (a_2\backslash b_2))\backslash b_1 \end{aligned}$$

and

$$\begin{aligned} & (a_1\backslash b_1) \sqcap (b_2\backslash a_2) \sqcap (b_1\backslash a_1) \sqcap (a_2\backslash b_2) \sqcap a_2 \\ = {} & (a_1\backslash b_1) \sqcap (b_2\backslash a_2) \sqcap (b_1\backslash a_1) \sqcap (a_2\backslash b_2) \sqcap b_2. \end{aligned}$$

As to the former equality, we have that

$$\begin{aligned} & ((a_1\backslash b_1) \sqcap (b_2\backslash a_2) \sqcap (b_1\backslash a_1) \sqcap (a_2\backslash b_2))\backslash a_1 \\ = {} & ((a_2\backslash b_2) \sqcap (b_2\backslash a_2))\backslash(((a_1\backslash b_1) \sqcap (b_1\backslash a_1))\backslash a_1) && \text{by Prop. 3.2 (iii)} \\ = {} & ((a_2\backslash b_2) \sqcap (b_2\backslash a_2))\backslash(((a_1\backslash b_1) \sqcap (b_1\backslash a_1))\backslash b_1) && \text{by Prop. 3.2 (vi)} \\ = {} & ((a_1\backslash b_1) \sqcap (b_2\backslash a_2) \sqcap (b_1\backslash a_1) \sqcap (a_2\backslash b_2))\backslash b_1 && \text{by Prop. 3.2 (iii).} \end{aligned}$$

As to the latter, we have

$$\begin{aligned} & (a_1\backslash b_1) \sqcap (b_2\backslash a_2) \sqcap (b_1\backslash a_1) \sqcap (a_2\backslash b_2) \sqcap a_2 \\ = {} & (a_1\backslash b_1) \sqcap (b_1\backslash a_1) \sqcap a_2 \sqcap b_2 && \text{by Prop. 3.2 (iv)} \\ = {} & (a_1\backslash b_1) \sqcap (b_2\backslash a_2) \sqcap (b_1\backslash a_1) \sqcap (a_2\backslash b_2) \sqcap b_2 && \text{by Prop. 3.2 (iv).} \quad \Box \end{aligned}$$

We now turn to the issue of proving the converse of Proposition 3.3, i.e., that any $\mathcal{I}$-algebra can be represented as a twist-structure. The first step is to check that the relation $\equiv$ is an equivalence relation which is moreover compatible with the operations $\{\supset, *\}$. For this we will need some lemmas.

Let us start by stating some facts that follow immediately from Definition 3.1. First of all, notice that the set $E(\mathbf{A})$ can be alternatively defined as follows:

$$E(\mathbf{A}) \ := \ \{a \supset a : a \in A\}.$$

This is so because, if $a = a \supset a$, then obviously the element a satisfies the above condition. On the other hand, if $a = b \supset b$ for some other $b \in A$, then by (I1) we have $a \supset a = (b \supset b) \supset (b \supset b) = b \supset b = a$.

Lemma 3.4. *Let $\mathbf{A}$ be an $\mathcal{I}$-algebra and $a, b, c, d \in A$. Then,*

(i) $a \equiv b$ *if and only if* $a \supset e = b \supset e$ *for all* $e \in A$,
(ii) if $a \equiv b$ *and* $c \equiv d$, *then* $(a \supset c) \equiv (b \supset d)$ *and* $(a * c) \equiv (b * d)$.

Proof. (i) Assume $a \supset b = (a \supset b) \supset (a \supset b)$ and $b \supset a = (b \supset a) \supset (b \supset a)$. Let $e \in A$. Then,

$$\begin{aligned} a \supset e &= (a \supset b) \supset (a \supset e) && \text{by hyp. and (I1)} \\ &= a \supset (b \supset e) && \text{by (I2)} \\ &= b \supset (a \supset e) && \text{by (I2)} \\ &= (b \supset a) \supset (b \supset e) && \text{by (I2)} \\ &= b \supset e && \text{by hyp. and (I1).} \end{aligned}$$

Conversely, assume $a \supset e = b \supset e$ for all $e \in A$. Then,

$$\begin{aligned} a \supset b &= b \supset b && \text{by hypothesis} \\ &= (b \supset b) \supset (b \supset b) && \text{by (I1)} \\ &= (a \supset b) \supset (a \supset b) && \text{by hypothesis.} \end{aligned}$$

By symmetry, we obtain $b \supset a = (b \supset a) \supset (b \supset a)$.
(ii) Assume $a \equiv b$ and $c \equiv d$. Then, by (i), we have that $a \supset e = b \supset e$ and $c \supset e = d \supset e$ for any element $e \in A$. Thus,

$$\begin{aligned} (a \supset c) \supset (b \supset d) &= (a \supset c) \supset (a \supset d) && \text{by hypothesis} \\ &= a \supset (c \supset d) && \text{by (I2)} \\ &= a \supset (d \supset d) && \text{by hypothesis} \\ &= (a \supset d) \supset (a \supset d) && \text{by (I2).} \end{aligned}$$

This means that $(a \supset c) \preceq (b \supset d)$ as desired. A symmetrical reasoning shows that $(b \supset d) \preceq (a \supset c)$ as well. So we conclude that $(a \supset c) \equiv (b \supset d)$. In order to prove that $(a * c) \equiv (b * d)$, just observe that, for any element $e \in A$, we have that

$$\begin{aligned} (a * c) \supset e &= a \supset (c \supset e) && \text{by (I4)} \\ &= b \supset (c \supset e) && \text{by hypothesis} \\ &= b \supset (d \supset e) && \text{by hypothesis} \\ &= (b * d) \supset e && \text{by (I4).} \end{aligned}$$

Then, applying (i) again we obtain the desired result. □

Proposition 3.5. *For any $\mathcal{I}$-algebra* $\mathbf{A}$, *the relation* $\equiv$ *defined as above is an equivalence relation which is moreover compatible with the operations* $*$ *and* $\supset$.

Proof. It is easy to check that $\equiv$ is an equivalence relation. In fact, $a \equiv a$ holds because $a \supset a = (a \supset a) \supset (a \supset a)$ is an instance of (I1). Symmetry of $\equiv$ follows immediately from the definition, while transitivity follows from Lemma 3.4 (i). Finally, compatibility with the operations $*$ and $\supset$ has been established in Lemma 3.4 (ii). □

Next we prove that the quotient algebra $\langle A/\equiv, \supset\rangle$ is a Tarski algebra. Let us recall the definition: a *Tarski algebra* [34] is an algebra $\langle B, \supset\rangle$ of type $\langle 2\rangle$ satisfying the following equations:

(T1) $(x \supset y) \supset x = x$,
(T2) $x \supset (y \supset z) = y \supset (x \supset z)$,
(T3) $(x \supset y) \supset y = (y \supset x) \supset x$.

Tarski algebras obviously form a variety. Given that the equation $x \supset x = y \supset y$ is valid in this variety, a constant $1 := x \supset x$ can be added to the algebraic language without loss of generality.

Proposition 3.6. *For any $\mathcal{I}$-algebra $\mathbf{A}$, the quotient algebra $\langle A/\equiv, \supset\rangle$ is a Tarski algebra.*

Proof. We have to check that $A/\equiv$ satisfies equations (T1) to (T3).
(T1) Let $a, b \in A$ and let $[a], [b] \in A/\equiv$ be the corresponding equivalence classes. We need to prove that $[(a \supset b) \supset a] = [a]$. This amounts to proving that $(a \supset b) \supset a \preceq a$ and $a \preceq (a \supset b) \supset a$. The former is an immediate consequence of (I1) and (I3). As to the latter, applying (I2) we have that

$$\begin{aligned} a \supset ((a \supset b) \supset a) &= (a \supset b) \supset (a \supset a) \\ &= ((a \supset b) \supset a) \supset ((a \supset b) \supset a) \end{aligned}$$

which means that $a \supset ((a \supset b) \supset a) \in E(\mathbf{A})$, i.e., $a \preceq (a \supset b) \supset a$.
(T2) Follows immediately from (I2).
(T3) Let us first notice that

$$(a \supset b) \supset ((b \supset a) \supset b) \preceq (b \supset a) \supset ((a \supset b) \supset a).$$

This holds because, applying repeatedly (I2), we have that

$$\begin{aligned} &((a \supset b) \supset ((b \supset a) \supset b)) \supset ((b \supset a) \supset ((a \supset b) \supset a)) \\ =\ &((a \supset b) \supset ((b \supset a) \supset b)) \supset ((a \supset b) \supset ((b \supset a) \supset a)) \\ =\ &(a \supset b) \supset (((b \supset a) \supset b)) \supset ((b \supset a) \supset a))) \\ =\ &(a \supset b) \supset ((b \supset a) \supset (b \supset a)) \\ =\ &((a \supset b) \supset (b \supset a)) \supset ((a \supset b) \supset (b \supset a)). \end{aligned}$$

By symmetry, it follows that

$$(b \supset a) \supset ((a \supset b) \supset a) \preceq (a \supset b) \supset ((b \supset a) \supset b)$$

i.e.,

$$[(a \supset b) \supset ((b \supset a) \supset b)] = [(b \supset a) \supset ((a \supset b) \supset a)].$$

Now, having already proved that (T1) holds, we can use it as follows:

$$\begin{aligned} [(a \supset b) \supset ((b \supset a) \supset b)] &= [a \supset b] \supset [(b \supset a) \supset b] \\ &= [a \supset b] \supset [b] && \text{by (T1)} \\ &= [(a \supset b) \supset b]. \end{aligned}$$

Symmetrically, we have that $[(b \supset a) \supset ((a \supset b) \supset a)] = [(b \supset a) \supset a]$ and this immediately implies the desired result. $\square$

By the previous proposition, to any $\mathcal{I}$-algebra $\mathbf{A}$ we can associate the algebra $\mathsf{L}(\mathbf{A}) = \langle A/{\equiv}, *, \sqcup, \supset, 1\rangle$ whose operations are defined, for all equivalence classes $[a], [b] \in A/{\equiv}$ corresponding to elements $a, b \in A$, as follows:

$$\begin{aligned}
[a] * [b] &:= [a * b] \\
[a] \supset [b] &:= [a \supset b] \\
[a] \sqcup [b] &:= ([a] \supset [b]) \supset [b] = [(a \supset b) \supset b] \\
1 &:= [a \supset a].
\end{aligned}$$

We also know that $\langle A/{\equiv}, \sqcup, 1\rangle$ is a join semilattice whose order $\leq$ is defined by $[a] \leq [b]$ if and only if $[a] \supset [b] = 1$ (notice that this condition is equivalent to $a \preceq b$). It remains to show that the $*$ operation is actually the meet corresponding to $\leq$ and that the pair $\langle *, \supset\rangle$ satisfies the residuation property (R).

Proposition 3.7. *For any $\mathcal{I}$-algebra $\mathbf{A}$, the algebra $\mathsf{L}(\mathbf{A})$ is a classical implicative lattice.*

Proof. As mentioned above, we have to prove that, for all $[a], [b] \in A/{\equiv}$ corresponding to elements $a, b \in A$, it holds that $[a] \leq [b]$ if and only if $[a] * [b] = [a]$. Assume then that $[a] \leq [b]$, i.e., $a \preceq b$. We need to prove that $a * b \preceq a$ and $a \preceq a * b$. The first one holds in general, for we have that

$$\begin{aligned}
(a * b) \supset a &= a \supset (b \supset a) && \text{by (I4)} \\
&= b \supset (a \supset a) && \text{by (I2)} \\
&= (b \supset a) \supset (b \supset a) && \text{by (I2).}
\end{aligned}$$

As to the second, using the fact that $a \preceq b$, we have that

$$\begin{aligned}
a \supset (a * b) &= (a \supset b) \supset (a \supset (a * b)) && \text{by (I1)} \\
&= a \supset (b \supset (a * b)) && \text{by (I2)} \\
&= (a * b) \supset (a * b) && \text{by (I4).}
\end{aligned}$$

Conversely, assume $a * b \preceq a$ and $a \preceq a * b$. Reasoning as above, we can show that $a * b \preceq b$ holds in general:

$$\begin{aligned}
(a * b) \supset b &= a \supset (b \supset b) && \text{by (I4)} \\
&= (a \supset b) \supset (a \supset b) && \text{by (I2).}
\end{aligned}$$

Then we have that $a \preceq a * b \preceq b$, so the result follows by transitivity of $\preceq$. Finally, the residuation properly (R) is easily proved, because we have that, for all $a, b, c \in A$,

$$\begin{aligned}
a * b \preceq c \quad &\text{iff} \quad a \preceq b \supset c && \text{by (I4)} \\
&\text{iff} \quad b \preceq a \supset c && \text{by (I2).}
\end{aligned}$$

$\square$

Our next aim is to prove that any $\mathcal{I}$-algebra $\mathbf{A}$ is isomorphic to an implicative twist-structure over $\mathsf{L}(\mathbf{A})$. In order to do this, we are going to show that $\mathbf{A}$ can be embedded into the full implicative twist-structure $(\mathsf{L}(\mathbf{A}))^{\bowtie}$ through the map $\iota \colon A \to A/{\equiv} \times A/{\equiv}$ defined as

$$\iota(a) \ := \ \langle [a], [\neg a] \rangle, \tag{3.1}$$

where $[a]$ denotes the equivalence class modulo $\equiv$ of $a \in A$.

Theorem 3.8. *Let* $\mathbf{A}$ *be an* $\mathcal{I}$*-algebra. Then:*

(i) the map $\iota \colon A \to A/{\equiv} \times A/{\equiv}$ *defined in* (3.1) *is an embedding of* $\mathbf{A}$ *into the full implicative twist-structure* $(\mathsf{L}(\mathbf{A}))^{\bowtie}$,

(ii) $\pi_1(\iota(A)) = A/{\equiv}$, *so* $\mathbf{A}$ *is isomorphic to an implicative twist-structure over* $\mathsf{L}(\mathbf{A})$.

Proof. (i) To check that ι is injective, assume $\iota(a) = \iota(b)$ for some $a, b \in A$. That is, $[a] = [b]$ and $[\neg a] = [\neg b]$, with means that

$$a \supset b, \ b \supset a, \ \neg a \supset \neg b, \ \neg b \supset \neg a \in E(\mathbf{A}).$$

This implies that $a \to b \in E(\mathbf{A})$ because we have that

$$\begin{aligned} a \to b &= (a \supset b) * (\neg b \supset \neg a) \\ &= \neg((a \supset b) \supset \neg(\neg b \supset \neg a)) \\ &= \neg\neg(\neg b \supset \neg a) && \text{by assumptions and (I1)} \\ &= \neg b \supset \neg a && \text{by (I5).} \end{aligned}$$

The same reasoning shows that $b \to a \in E(\mathbf{A})$, which implies that $a \leftrightarrow b \in E(\mathbf{A})$ because

$$\begin{aligned} (a \to b) * (b \to a) &= \neg((a \to b) \supset \neg(b \to a)) \\ &= \neg\neg(b \to a) && \text{by assumptions and (I1)} \\ &= b \to a \in E(\mathbf{A}) && \text{by (I5).} \end{aligned}$$

Then, by (I1) and (I6), we have that

$$a = (a \leftrightarrow b) \supset a = (a \leftrightarrow b) \supset b = b.$$

Thus, ι is injective. It is also clear that $\iota(\neg a) = \neg\iota(a)$ for all $a \in A$. It remains to check that $\iota(a \supset b) = \iota(a) \supset \iota(b)$. We have that

$$\begin{aligned} \iota(a \supset b) &= \langle [a \supset b], [\neg(a \supset b)] \rangle \\ &= \langle [a \supset b], [\neg(a \supset \neg\neg b)] \rangle && \text{by (I5)} \\ &= \langle [a \supset b], [a * \neg b] \rangle \\ &= \langle [a] \supset [b], [a] * [\neg b] \rangle \\ &= \langle [a], [\neg a] \rangle \supset \langle [b], [\neg b] \rangle \\ &= \iota(a) \supset \iota(b). \end{aligned}$$

(ii) Obvious, because $A/{\equiv}$ is obtained as a quotient from A. □

Theorem 3.8 tells us that $\mathcal{I}$-algebras coincide, up to isomorphism, with the implicative twist-structures introduced in Section 2. We know therefore that we can view any $\mathcal{I}$-algebra $\mathbf{A}$ as a subalgebra of the full implicative twist-structure $(\mathsf{L}(\mathbf{A}))^{\bowtie}$ and from now on we will often make use of this result.

Let us start with a simple application. As mentioned in the previous section, any full twist-structure $\mathbf{L}^{\bowtie}$ has four natural bounds corresponding to the two partial orders $\leq_1$ and $\leq_2$. Adopting the notation used for bilattices in [29, 9, 7], let us denote:

$$\mathsf{f} := \langle 0, 1\rangle, \qquad \mathsf{t} := \langle 1, 0\rangle,$$
$$\bot := \langle 0, 0\rangle, \qquad \top := \langle 1, 1\rangle.$$

Using the representation given in Theorem 3.8, it is easy to check that an $\mathcal{I}$-algebra can have either just one of the above bounds (in which case it needs to be $\top$) or two (f, t), three ($\mathsf{f}, \mathsf{t}, \bot$ or $\mathsf{f}, \mathsf{t}, \top$), or all four. If we want to add any of these as constants to the algebraic language of $\mathcal{I}$-algebras, it is sufficient to expand the axiomatization of Definition 3.1 with the appropriate equations shown below:

$$(\top) \qquad \top = \neg\top \ \text{ and } \ \top \supset x = x,$$
$$(\bot) \qquad \bot = \neg\bot \ \text{ and } \ \bot \supset x = \bot \supset \bot,$$
$$(\mathsf{t}) \qquad x \supset \mathsf{t} = \mathsf{t},$$
$$(\mathsf{f}) \qquad \mathsf{f} = \neg\, \mathsf{t}.$$

A natural question to ask is whether it is possible to characterize precisely the subsets of a given full implicative twist-structure $\mathbf{L}^{\bowtie}$ that are universes of implicative twist-structures over $\mathbf{L}$. The corresponding problem for N4-lattices has been solved in [25, Theorem 3.1], but in our case we only know of a sufficient condition, as we are going to see.

Let us start by observing that, if $A, B \subseteq L \times L$ are universes of implicative twist-structures $\mathbf{A}, \mathbf{B} \subseteq \mathbf{L}^{\bowtie}$, then $A \cup B$ is also the universe of an implicative twist-structure over $\mathbf{L}$. To see this, notice that $A \cup B$ is obviously closed under the $\neg$ operation. Moreover, if $\langle a_1, a_2\rangle \in A$ and $\langle b_1, b_2\rangle \in B$, then $\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle \in B$. This holds because $\pi_1(B) = L$, therefore there must be some $c \in L$ such that $\langle a_1, c\rangle \in B$ and we have that

$$\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle = \langle a_1 \backslash b_1, a_1 \sqcap b_2\rangle = \langle a_1, c\rangle \supset \langle b_1, b_2\rangle \in B.$$

Thus, $A \cup B$ is closed under $\supset$ as well and obviously $\pi_1(A \cup B) = L$, i.e., $A \cup B$ is in fact the universe of an implicative twist-structure over $\mathbf{L}$. This reasoning obviously extends to arbitrary unions of subsets $A_i \subseteq L \times L$ corresponding to implicative twist-structures $\mathbf{A}_i \subseteq \mathbf{L}^{\bowtie}$.

It is easy to check that a sufficient condition for a subset of $L \times L$ to be the universe of an implicative twist-structure is the following (Cf. [25, Proposition 3.3]). Let $U, D \subseteq L$ be, respectively, a non-empty up-set and a non-empty down-set with respect to the lattice order of the classical implicative lattice $\mathbf{L}$.

Then the set

$$A \;:=\; \{\langle a_1, a_2\rangle \in L \times L : a_1 \sqcup a_2 \in U,\; a_1 \sqcap a_2 \in D\} \tag{3.2}$$

is the universe of an implicative twist-structure. Let us check that A is closed under the algebraic operations. The case of $\neg$ is immediate. Let us then assume that $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A$ and check that $\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle = \langle a_1 \backslash b_1, a_1 \sqcap b_2\rangle \in A$. This amounts to proving that $(a_1 \backslash b_1) \sqcup (a_1 \sqcap b_2) \in U$ and $(a_1 \backslash b_1) \sqcap a_1 \sqcap b_2 \in D$. As to the former, we have that

$$\begin{aligned}
(a_1 \backslash b_1) \sqcup (a_1 \sqcap b_2) &= ((a_1 \backslash b_1) \sqcup a_1) \sqcap ((a_1 \backslash b_1) \sqcup b_2) && \text{by distributivity}\\
&= 1 \sqcap ((a_1 \backslash b_1) \sqcup b_2) && \text{by Prop. 3.2 (vii)}\\
&= (a_1 \backslash b_1) \sqcup b_2 \\
&\geq b_1 \sqcup b_2 && \text{by Prop. 3.2 (viii).}
\end{aligned}$$

By assumption, $b_1 \sqcup b_2 \in U$ and U is an up-set, so we are done. As to the latter, applying by Proposition 3.2 (iv), we have

$$(a_1 \backslash b_1) \sqcap a_1 \sqcap b_2 = a_1 \sqcap b_1 \sqcap b_2 \leq b_1 \sqcap b_2.$$

Again, by assumption, $b_1 \sqcap b_2 \in D$ and D is a down-set, so we are done. A is then the universe of an implicative twist-structure. In general, it might happen that $\pi_1(A) \subsetneq L$, but it is easy to prove that $\pi_1(A)$ is the universe of a subalgebra of $\mathbf{L}$, which means that A is the universe of an implicative twist-structure over some classical implicative lattice having as universe $\pi_1(A)$.

Conversely, given an arbitrary implicative twist-structure $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$, we can define

$$\begin{aligned}
U(\mathbf{A}) \;&:=\; \{a_1 \sqcup a_2 : \langle a_1, a_2\rangle \in A\},\\
D(\mathbf{A}) \;&:=\; \{a_1 \sqcap a_2 : \langle a_1, a_2\rangle \in A\}.
\end{aligned}$$

It is easy to check that $U(\mathbf{A})$ is an up-set and $D(\mathbf{A})$ is a down-set of $\mathbf{L}$. In fact, suppose $a \in U(\mathbf{A})$, i.e., $a = a_1 \sqcup a_2$ for some $\langle a_1, a_2\rangle \in A$, and $a \leq b$ for some $b \in L$. Since $\pi_1(A) = \pi_2(A) = L$, we may assume that there is $c \in L$ such that $\langle b, c\rangle \in A$. Then we have that

$$\begin{aligned}
(\langle b, c\rangle \supset \langle a_1, a_2\rangle) \supset \langle a_1, a_2\rangle &= \langle b \sqcup a_1, (b \backslash a_1) \sqcap a_2\rangle\\
&= \langle b, (b \backslash a_1) \sqcap a_2\rangle.
\end{aligned}$$

By assumption $b \geq a_2$, so $b \sqcup ((b \backslash a_1) \sqcap a_2) = b$ and we conclude that $b \in U(\mathbf{A})$. A similar reasoning shows that $D(\mathbf{A})$ is a down-set. Given $a = a_1 \sqcap a_2 \in D(\mathbf{A})$ for some $\langle a_1, a_2\rangle \in A$ and $b \leq a$, we consider $\langle b, c\rangle \in A$ and check that

$$\langle b, c\rangle \supset \langle a_1, a_2\rangle = \langle b \backslash a_1, b \sqcap a_2\rangle = \langle 1, b\rangle \in A.$$

Since $1 \sqcap b = b$, we conclude that $b \in D(\mathbf{A})$.

Let us further notice that in any implicative twist-structure $\mathbf{A}$ it holds that, for all $\langle a_1, a_2\rangle \in A$,

$$\begin{aligned}(\langle a_1, a_2\rangle \supset \neg\langle a_1, a_2\rangle) \supset \neg\langle a_1, a_2\rangle &= (\langle a_1, a_2\rangle \supset \langle a_2, a_1\rangle) \supset \langle a_2, a_1\rangle \\ &= \langle a_1 \backslash a_2, a_1\rangle \supset \langle a_2, a_1\rangle \\ &= \langle (a_1\backslash a_2)\backslash a_2, (a_1\backslash a_2) \sqcap a_1\rangle \\ &= \langle a_1 \sqcup a_2, a_1 \sqcap a_2\rangle.\end{aligned}$$

This means that, defining

$$I(\mathbf{A}) \ := \ \{(\langle a_1, a_2\rangle \supset \neg\langle a_1, a_2\rangle) \supset \neg\langle a_1, a_2\rangle : \langle a_1, a_2\rangle \in A\}$$

we have

$$I(\mathbf{A}) = \{\langle a_1 \sqcup a_2, a_1 \sqcap a_2\rangle : \langle a_1, a_2\rangle \in A\},$$

which means that $\pi_1(I(\mathbf{A})) = U(\mathbf{A})$ and $\pi_2(I(\mathbf{A})) = D(\mathbf{A})$. It is also easy to check that

$$I(\mathbf{A}) = \{\langle a_1, a_2\rangle \in A : a_1 \geq a_2\}.$$

Thus, given an arbitrary implicative twist-structure $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$, we can construct the up-set $U(\mathbf{A}) \subseteq L$ and the down-set $D(\mathbf{A}) \subseteq L$ and then apply (3.2) to define

$$A' \ := \ \{\langle a_1, a_2\rangle \in L \times L : a_1 \sqcup a_2 \in U(\mathbf{A}),\ a_1 \sqcap a_2 \in D(\mathbf{A})\}. \tag{3.3}$$

It is easy to see that $A \subseteq A'$. If we could prove that $A' \subseteq A$, then we would have shown that any implicative twist-structure can be constructed in this way. As mentioned above, this is false in general, as it can happen that $A \subsetneq A'$. For a counterexample, consider the implicative twist-structure $\mathbf{A}_8 \subseteq \mathbf{L_4}^{\bowtie}$ shown in Figure 1. The universe of $\mathbf{A}_8$ is

$$A_8 = \{\langle 1, \mathsf{a}\rangle, \langle \mathsf{a}, 0\rangle, \langle \mathsf{a}, \mathsf{b}\rangle, \langle 1, 0\rangle, \langle \mathsf{a}, 1\rangle, \langle 0, \mathsf{a}\rangle, \langle \mathsf{b}, \mathsf{a}\rangle, \langle 0, 1\rangle\}.$$

This means that $U(\mathbf{A_8}) = \{a, 1\}$ and $D(\mathbf{A_8}) = \{0, a\}$. According to the definition given in (3.2), we have that $\langle a, a\rangle \in A'_8$, but $\langle a, a\rangle \notin A_8$. That is, $A_8 \subsetneq A'_8$.

Drawing inspiration from this counterexample, we can try to refine the construction of (3.2) as follows. Given a classical implicative lattice $\mathbf{L}$, let $\{\langle U_i, D_i\rangle : i \in I\}$ be a family of pairs such that $U_i \subseteq L$ is an up-set and $D_i \subseteq L$ is a down-set of $\mathbf{L}$ for each $i \in I$. Define

$$A \ := \ \bigcup_{i\in I}\{\langle a_1, a_2\rangle \in L \times L : a_1 \sqcup a_2 \in U_i,\ a_1 \sqcap a_2 \in D_i\}. \tag{3.4}$$

Notice that, consistently with the observation made above about the union of universes of implicative twist-structures being itself the universe of an implicative twist-structure, an arbitrary union of sets satisfying (3.4) will again satisfy (3.4).

Let us check that A is closed under the algebraic operations of implicative twist-structures. As before, the case of $\neg$ is trivial. Now assume that

$\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A$, i.e., $a_1 \sqcup a_2 \in U_j$, $a_1 \sqcap a_2 \in D_j$, $b_1 \sqcup b_2 \in U_k$, $b_1 \sqcap b_2 \in D_k$ for some $j, k \in I$. Repeating the reasoning applied before, we have that

$$(a_1 \backslash b_1) \sqcup (a_1 \sqcap b_2) = (a_1 \backslash b_1) \sqcup b_2 \geq b_1 \sqcup b_2 \in U_k$$

and

$$(a_1 \backslash b_1) \sqcap a_1 \sqcap b_2 = a_1 \sqcap b_1 \sqcap b_2 \leq b_1 \sqcap b_2 \in D_k$$

which implies that $\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle \in A$.

The implicative twist-structure $\mathbf{A}_8$ of our counterexample can actually be obtained through the construction defined in (3.4) by taking the two pairs $U_1 = \{\mathsf{a}, 1\}$, $D_1 = \{0\}$ and $U_2 = \{1\}$, $D_2 = \{\mathsf{a}, 0\}$. The set A_8 is obtained as the union

$$\{\langle \mathsf{a}, 0\rangle, \langle 0, \mathsf{a}\rangle, \langle \mathsf{a}, \mathsf{b}\rangle, \langle \mathsf{b}, \mathsf{a}\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle\} \cup \{\langle \mathsf{a}, 1\rangle, \langle 1, \mathsf{a}\rangle, \langle \mathsf{a}, \mathsf{b}\rangle, \langle \mathsf{b}, \mathsf{a}\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle\}$$

in which the first set corresponds to the pair $\langle U_1, D_1\rangle$ and the second one to $\langle U_2, D_2\rangle$. We see then that element $\langle \mathsf{a}, \mathsf{a}\rangle$ is ruled out as it does not satisfy the condition for belonging to either of the above sets.

As mentioned before, although we have seen that (3.4) provides a way to construct implicative twist-structures, the question whether all implicative twist-structures can be obtained in this way is still open.

4. Universal algebraic properties

In this section we study the class of implicative twist-structures (the variety of $\mathcal{I}$-algebras) from a universal algebraic point of view. We start by looking at congruences.

The first result we prove is that $\mathcal{I}$-algebras have equationally definable principle congruences. Recall that a variety of algebras is said to have *equationally definable principal congruences* (abbreviated EDPC) if there is a finite set Σ of equations of the form $t(x, y, z, u) = t'(x, y, z, u)$ such that, for any algebra $\mathbf{A}$ in the variety and for all elements $a, b, c, d \in A$, it holds that $\langle c, d\rangle \in \Theta(a, b)$ if and only if $t(a, b, c, d) = t'(a, b, c, d)$ for all equations in Σ. EDPC is a rather strong property: in particular it implies congruence-distributivity and the congruence extension property [4, Theorem 1.2].

Lemma 4.1. *Let* $\mathbf{A}$ *be an* $\mathcal{I}$*-algebra and* $a, b, c, d, e \in A$*. Then:*

(i) if $a \supset b = a \supset c$*, then* $a \supset \neg b = a \supset \neg c$*,*

(ii) if $a \supset b = a \supset c$ *and* $a \supset d = a \supset e$*, then* $a \supset (b \supset d) = a \supset (c \supset e)$*.*

Proof. (i) We view our $\mathcal{I}$-algebra as an implicative twist-structure $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$ whose elements are pairs $a = \langle a_1, a_2\rangle$, $b = \langle b_1, b_2\rangle$ etc. Our assumption is then that $\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle = \langle a_1, a_2\rangle \supset \langle c_1, c_2\rangle$, which means that $a_1 \backslash b_1 = a_1 \backslash c_1$ and $a_1 \sqcap b_2 = a_1 \sqcap c_2$. From the former equality we obtain $a_1 \sqcap (a_1 \backslash b_1) = a_1 \sqcap (a_1 \backslash c_1)$. By Proposition 3.2 (iv), we have $a_1 \sqcap (a_1 \backslash b_1) = a_1 \sqcap b_1$ and $a_1 \sqcap (a_1 \backslash c_1) = a_1 \sqcap c_1$. Hence, $a_1 \sqcap b_1 = a_1 \sqcap c_1$. From $a_1 \sqcap b_2 = a_1 \sqcap c_2$ we obtain $a_1 \backslash (a_1 \sqcap b_2) = a_1 \backslash (a_1 \sqcap c_2)$. Applying Proposition 3.2 (ix), we have

$a_1\backslash(a_1 \sqcap b_2) = (a_1\backslash a_1) \sqcap (a_1\backslash b_2) = a_1\backslash b_2$ and $a_1\backslash(a_1 \sqcap c_2) = (a_1\backslash a_1) \sqcap (a_1\backslash c_2) = a_1\backslash c_2$. Hence, $a_1\backslash b_2 = a_1\backslash c_2$. These together mean that, as desired,

$$\langle a_1, a_2\rangle \supset \neg\langle b_1, b_2\rangle = \langle a_1\backslash b_2, a_1 \sqcap b_1\rangle = \langle a_1\backslash c_2, a_1 \sqcap c_1\rangle = \langle a_1, a_2\rangle \supset \neg\langle c_1, c_2\rangle.$$

(ii) Straightforward, given that

$$\begin{aligned} a \supset (b \supset d) &= (a \supset b) \supset (a \supset d) && \text{by (I2)} \\ &= (a \supset c) \supset (a \supset e) && \text{by assumptions} \\ &= a \supset (c \supset e) && \text{by (I2).} \end{aligned}$$ □

Theorem 4.2. *The variety of $\mathcal{I}$-algebras has EDPC.*

Proof. Given an $\mathcal{I}$-algebra $\mathbf{A}$ and $a, b \in A$, let us denote by $\Theta(a, b)$ the congruence generated by the pair $\langle a, b\rangle$. Define

$$\theta := \{\langle c, d\rangle \in A \times A : (a \leftrightarrow b) \supset c = (a \leftrightarrow b) \supset d\}.$$

We will prove that $\theta = \Theta(a, b)$. Clearly θ is an equivalence relation and, by (I6), we have that $\langle a, b\rangle \in \theta$. Lemma 4.1 implies that θ is a congruence of $\mathbf{A}$. Hence we have that $\Theta(a, b) \subseteq \theta$.
To prove the other inclusion, assume $\langle c, d\rangle \in \theta$. Notice that $\langle a, b\rangle \in \Theta(a, b)$ implies that $\langle a \leftrightarrow a, a \leftrightarrow b\rangle \in \Theta(a, b)$. Then we also have that $\langle (a \leftrightarrow a) \supset c, (a \leftrightarrow b) \supset c\rangle, \langle (a \leftrightarrow a) \supset d, (a \leftrightarrow b) \supset d\rangle \in \Theta(a, b)$. By assumption $(a \leftrightarrow b) \supset c = (a \leftrightarrow b) \supset d$, so by transitivity of $\Theta(a, b)$ we obtain $\langle (a \leftrightarrow a) \supset c, (a \leftrightarrow a) \supset d\rangle \in \Theta(a, b)$. But $(a \leftrightarrow a) \supset c = (\neg a \supset \neg a) \supset c = c$ and similarly $(a \leftrightarrow a) \supset d = d$, as can easily be checked in any implicative twist-structure. Hence, we conclude that $\langle c, d\rangle \in \Theta(a, b)$, i.e., $\theta \subseteq \Theta(a, b)$. □

Corollary 4.3. *The variety of $\mathcal{I}$-algebras is congruence-distributive and enjoys the congruence-extension property.*

The proof of Theorem 4.2 actually shows that $(x \leftrightarrow y) \supset z$ is a *ternary deductive term* in the sense of [6, Definition 2.1]. This implies EDPC [6, Corollary 2.5], but is even stronger. For instance, it follows that $\mathcal{I}$-algebras have 3-permutable congruences [6, Theorem 2.9] and enjoy a stronger form of the congruence-extension property [6, Lemma 2.11].

We proceed to obtain further information on congruences of $\mathcal{I}$-algebras. Our next aim is to prove that the lattice $\mathrm{Con}(\mathbf{A})$ of congruences of any $\mathcal{I}$-algebra $\mathbf{A} \subseteq (\mathsf{L}(\mathbf{A}))^{\bowtie}$ is isomorphic to the lattice $\mathrm{Con}(\mathsf{L}(\mathbf{A}))$ of congruences of the classical implicative lattice $\mathsf{L}(\mathbf{A})$.

For this we define a map $H\colon \mathrm{Con}(\mathbf{A}) \to \mathrm{Con}(\mathsf{L}(\mathbf{A}))$ in the following way. For any $\theta \in \mathrm{Con}(\mathbf{A})$, let

$$H(\theta) := \{\langle [a], [b]\rangle : \langle a \supset b, (a \supset b) \supset (a \supset b)\rangle, \langle b \supset a, (b \supset a) \supset (b \supset a)\rangle \in \theta\}.$$

where $[a], [b] \in \mathsf{L}(\mathbf{A})$ are equivalence classes modulo $\equiv$ corresponding to elements $a, b \in A$.

Let us start by checking that $H(\theta) \in \mathrm{Con}(\mathsf{L}(\mathbf{A}))$. We will often make use of the following fact.

Lemma 4.4. *Let* $\mathbf{A}$ *be an* $\mathcal{I}$*-algebra,* $a, b \in A$ *and* $\theta \in \mathrm{Con}(\mathbf{A})$*. The following conditions are equivalent:*

(i) $\langle a \supset b, (a \supset b) \supset (a \supset b)\rangle,\ \langle b \supset a, (b \supset a) \supset (b \supset a)\rangle \in \theta$*,*
(ii) $\langle a \supset b, c\rangle,\ \langle b \supset a, d\rangle \in \theta$ *for some* $c, d \in E(\mathbf{A})$*,*
(iii) $\langle a \supset c, b \supset c\rangle \in \theta$ *for all* $c \in A$*.*

Proof. Obviously (i) implies (ii), given that $(a \supset b) \supset (a \supset b), (b \supset a) \supset (b \supset a) \in E(\mathbf{A})$. The converse implication is also easy, because (ii) implies that $\langle (a \supset b) \supset (a \supset b), c \supset (a \supset b)\rangle \in \theta$ and, by (I1), we have that $c \supset (a \supset b) = (a \supset b)$. A symmetrical reasoning shows that (ii) implies that $\langle b \supset a, (b \supset a) \supset (b \supset a)\rangle \in \theta$. So (i) and (ii) are equivalent. We can show that (i) is equivalent to (iii) as follows. Notice that (i) means that the quotient algebra $\mathbf{A}/\theta$ (which is an $\mathcal{I}$-algebra too, because the class of $\mathcal{I}$-algebras is a variety, hence closed under the operation of taking quotients, i.e., homomorphic images) satisfies the identities $[a \supset b]_\theta = [(a \supset b) \supset (a \supset b)]_\theta$ and $[b \supset a]_\theta = [(b \supset a) \supset (b \supset a)]_\theta$. This means that $[a]_\theta \preceq [b]_\theta$ and $[b]_\theta \preceq [a]_\theta$. By Lemma 3.4 (i), these two conditions are equivalent to $[a]_\theta \supset [c]_\theta = [b]_\theta \supset [c]_\theta$ for all $[c] \in \mathbf{A}/\theta$, which means that $\langle a \supset c, b \supset c\rangle \in \theta$ for all $c \in A$. □

Proposition 4.5. *Let* $\mathbf{A} \subseteq (\mathsf{L}(\mathbf{A}))^{\bowtie}$ *be an* $\mathcal{I}$*-algebra and* $\theta \in \mathrm{Con}(\mathbf{A})$*. Then* $H(\theta) \in \mathrm{Con}(\mathsf{L}(\mathbf{A}))$*.*

Proof. In order to check that the map H is well-defined, assume $\langle [a], [b]\rangle \in H(\theta)$ and $[a] = [a']$ and $[b] = [b']$ for some $a', b' \in A$. We have to prove that $\langle [a'], [b']\rangle \in H(\theta)$, i.e. (using Lemma 4.4), that $\langle a' \supset b', c\rangle, \langle b' \supset a', d\rangle \in \theta$ for some $c, d \in E(\mathbf{A})$. By assumption, we have that $[a] \supset [b] = [a \supset b] = [a' \supset b'] = [a'] \supset [b']$. By definition, this means that $(a \supset b) \supset (a' \supset b') \in E(\mathbf{A})$. By assumption, $\langle a \supset b, (a \supset b) \supset (a \supset b)\rangle \in \theta$. Then

$$(a \supset b) \supset (a' \supset b')\ \theta\ ((a \supset b) \supset (a \supset b)) \supset (a' \supset b') = a' \supset b,$$

where the last equality holds because of (I1). By Lemma 4.4, taking $c = (a \supset b) \supset (a' \supset b')$, we obtain that $\langle a' \supset b', c\rangle \in \theta$. By symmetry, the same reasoning allows to prove that $\langle b' \supset a', d\rangle \in \theta$, where $d = (b \supset a) \supset (b' \supset a')$. Now we have to check that $H(\theta)$ is actually a congruence of the classical implicative lattice $\mathsf{L}(\mathbf{A})$. Symmetry of $H(\theta)$ is obvious. Reflexivity follows immediately from the fact that, by (I1), we have that $(a \supset a) \supset (a \supset a) = a \supset a$ for all $a \in A$. Transitivity can easily be proved using the equivalence between (i) and (iii) of Lemma 4.4. Next we prove that $H(\theta)$ is compatible with the algebraic operations of the classical implicative lattice $\mathsf{L}(\mathbf{A})$. Assume $\langle [a], [b]\rangle, \langle [c], [d]\rangle \in H(\theta)$. Let us check that $\langle [a] \supset [c], [b] \supset [d]\rangle \in H(\theta)$, i.e., that $\langle [a \supset c], [b \supset d]\rangle \in H(\theta)$, i.e., that there are elements $a', b' \in E(\mathbf{A})$ such that $\langle (a \supset c) \supset (b \supset d), a'\rangle, \langle (b \supset d) \supset (a \supset c), b'\rangle \in \theta$. Reasoning as in the proof of Lemma 4.4, we notice that for instance $\langle [a], [b]\rangle \in H(\theta)$ implies that $[a \supset b]_\theta, [b \supset a]_\theta \in E(\mathbf{A}/\theta)$, where $\mathbf{A}/\theta$ is the usual quotient of $\mathbf{A}$ by the congruence θ. Given that the relation $\equiv\ \subseteq \mathbf{A}/\theta \times \mathbf{A}/\theta$ is compatible with the operation $\supset$ in $\mathbf{A}/\theta$, we know that $[a \supset b]_\theta, [b \supset a]_\theta, [c \supset d]_\theta, [d \supset c]_\theta \in E(\mathbf{A}/\theta)$ implies that $[(a \supset c) \supset (b \supset d)]_\theta, [(b \supset d) \supset (a \supset c)]_\theta \in E(\mathbf{A}/\theta)$. This means that $[(a \supset c) \supset (b \supset d)]_\theta = [a']_\theta$ for some $a' \in A$ such that $[a']_\theta \in E(\mathbf{A}/\theta)$, i.e., such that $[a']_\theta = [a']_\theta \supset [a']_\theta = [a' \supset a']_\theta$. Thus, we have that $[(a \supset c) \supset (b \supset d)]_\theta = [a']_\theta = [a' \supset a']_\theta$, i.e., $\langle (a \supset c) \supset (b \supset d), a' \rangle \in \theta$ as required. A symmetrical argument shows that $\langle (b \supset d) \supset (a \supset c), b' \rangle \in \theta$ for some $b' \in E(\mathbf{A})$. Compatibility with the operation $\sqcup$ of $\mathsf{L}(\mathbf{A})$ follows immediately from the previous case because $[a] \sqcup [b] = ([a] \supset [b]) \supset [b]$ for all $a, b \in A$. To prove compatibility with the operation $*$ of $\mathsf{L}(\mathbf{A})$ recall that, in any $\mathcal{I}$-algebra, it holds that $[a] = [b]$ and $[c] = [d]$ imply $[a * c] = [b * d]$. We can then apply the same strategy as above to conclude that $[(a * c) \supset (b * d)]_\theta, [(b * d) \supset (a * c)]_\theta \in E(\mathbf{A}/\theta)$, which means that $\langle [a * c], [b * d] \rangle = \langle [a] * [c], [b] * [d] \rangle \in H(\theta)$. □

We claim that the map $H\colon \mathrm{Con}(\mathbf{A}) \to \mathrm{Con}(\mathsf{L}(\mathbf{A}))$ has an inverse

$$H^{-1}\colon \mathrm{Con}(\mathsf{L}(\mathbf{A})) \to \mathrm{Con}(\mathbf{A})$$

given, for all $\eta \in \mathrm{Con}(\mathsf{L}(\mathbf{A}))$, by

$$H^{-1}(\eta) := \{\langle a, b \rangle \in A \times A : \langle [a], [b] \rangle, \langle [\neg a], [\neg b] \rangle \in \eta\}. \tag{4.1}$$

Let us check that $H^{-1}(\eta)$ is actually a congruence of $\mathbf{A}$. Compatibility with the $\neg$ operation easily follows from (I5). As to the implication operation, we need to prove that $\langle [a \supset c], [b \supset d] \rangle, \langle [\neg(a \supset c)], [\neg(b \supset d)] \rangle \in \eta$ whenever $\langle [a], [b] \rangle, \langle [\neg a], [\neg b] \rangle, \langle [c], [d] \rangle, \langle [\neg c], [\neg d] \rangle \in \eta$. Given that $[a] \supset [c] = [a \supset c]$ and $[b] \supset [d] = [b \supset d]$, the assumptions immediately imply that $\langle [a \supset c], [b \supset d] \rangle \in \eta$. To see that $\langle [\neg(a \supset c)], [\neg(b \supset d)] \rangle \in \eta$, notice that, by (I5), we have that $[\neg(a \supset c)] = [\neg(a \supset \neg\neg c)] = [a * \neg c] = [a] * [\neg c]$. Similarly, we have that $[\neg(b \supset d)] = [b] * [\neg d]$. Then compatibility of η with the operation $*$ immediately implies the desired result.

Next we check that H and H^{-1} are mutually inverse.

Proposition 4.6. *Let $\mathbf{A} \subseteq (\mathsf{L}(\mathbf{A}))^{\bowtie}$ be an $\mathcal{I}$-algebra, $\theta \in \mathrm{Con}(\mathbf{A})$ and $\eta \in \mathrm{Con}(\mathsf{L}(\mathbf{A}))$. Then $H^{-1}(H(\theta)) = \theta$ and $H(H^{-1}(\eta)) = \eta$.*

Proof. Let $\theta \in \mathrm{Con}(\mathbf{A})$. By definition, $\langle a, b \rangle \in H^{-1}(H(\theta))$ means that $\langle [a], [b] \rangle$, $\langle [\neg a], [\neg b] \rangle \in H(\theta)$. By Lemma 4.4, this means that there are elements $c, d, c', d' \in E(\mathbf{A})$ such that $\langle a \supset b, c \rangle, \langle b \supset a, d \rangle, \langle \neg a \supset \neg b, c' \rangle, \langle \neg b \supset \neg a, d' \rangle \in \theta$. If this happens, then in the quotient $\mathcal{I}$-algebra $\mathbf{A}/\theta$ we have that $a \supset b, b \supset a, \neg a \supset \neg b, \neg b \supset \neg a \in E(\mathbf{A}/\theta)$. As can be easily checked in any implicative twist-structure, these last conditions imply that $a \leftrightarrow b \in E(\mathbf{A}/\theta)$. Applying (I6) we have then that $[a]_\theta = [b]_\theta$. This means that $\langle a, b \rangle \in \theta$, i.e., $H^{-1}(H(\theta)) \subseteq \theta$.
Conversely, if $\langle a, b \rangle \in \theta$, then $\langle a \supset b, b \supset b \rangle \in \theta$ as well and $b \supset b \in E(\mathbf{A})$. Similarly, we have that $\langle a \supset a, b \supset a \rangle \in \theta$ and $a \supset a \in E(\mathbf{A})$. Given that $\langle a, b \rangle \in \theta$ implies $\langle \neg a, \neg b \rangle \in \theta$, the same reasoning yields $\langle \neg a \supset \neg b, \neg b \supset \neg b \rangle, \langle \neg a \supset \neg a, \neg b \supset \neg a \rangle \in \theta$. Then, by Lemma 4.4, we have that $\langle a, b \rangle \in H^{-1}(H(\theta))$,

i.e., $\theta \subseteq H^{-1}(H(\theta))$. Hence, $\theta = H^{-1}(H(\theta))$.
Assume $\eta \in \mathrm{Con}(\mathsf{L}(\mathbf{A}))$. By Lemma 4.4 we have that $\langle [a],[b] \rangle \in H(H^{-1}(\eta))$ if and only if there are $c,d \in E(\mathbf{A})$ such that $\langle a \supset b, (a \supset b) \supset (a \supset b) \rangle, \langle b \supset a, (b \supset a) \supset (b \supset a) \rangle \in H^{-1}(\eta)$. This means that $\langle [a \supset b], [(a \supset b) \supset (a \supset b)] \rangle, \langle [b \supset a], [(b \supset a) \supset (b \supset a)] \rangle, \langle [\neg(a \supset b)], [\neg((a \supset b) \supset (a \supset b))] \rangle, \langle [\neg(b \supset a)][\neg((b \supset a) \supset (b \supset a))] \rangle \in \eta$. Since $\mathsf{L}(\mathbf{A})$ is a classical implicative lattice, we have that $[(a \supset b) \supset (a \supset b)] = [(b \supset a) \supset (b \supset a)] = 1$. That is, in the quotient $\mathsf{L}(\mathbf{A})/\eta$, which is also a classical implicative lattice because this class of algebras is a variety, we have that $[[a \supset b]]_\eta = [[b \supset a]]_\eta = [1]_\eta$. Hence, $[[a]]_\eta = [[b]]_\eta$, which means that $\langle [a],[b] \rangle \in \eta$. Thus, $H(H^{-1}(\eta)) \subseteq \eta$.
Conversely, if $\langle [a],[b] \rangle \in \eta$, then $\langle [a] \supset [b], [b] \supset [b] \rangle$ as well. Since $[a] \supset [b] = [a \supset b]$ and $[b] \supset [b] = [b \supset b] = 1$, we can conclude that $\langle [a \supset b], [(a \supset b) \supset (a \supset b)] \rangle \in \eta$. Similarly, we obtain $\langle [b \supset a], [(b \supset a) \supset (b \supset a)] \rangle \in \eta$. To see that $\langle [\neg(a \supset b)], [\neg((a \supset b) \supset (a \supset b))] \rangle \in \eta$, notice that $\langle [a],[b] \rangle \in \eta$ implies $\langle [a]*[a], [a]*[b] \rangle = \langle [a], [a]*[b] \rangle \in \eta$, which implies $\langle [a]*[\neg b], [a]*[b]*[\neg b] \rangle \in \eta$. On the one hand, by (I5) we have that $[a]*[\neg b] = [a * \neg b] = [\neg(a \supset \neg\neg b)] = [\neg(a \supset b)]$. On the other hand, since we are in a classical implicative lattice, we have that

$$
\begin{aligned}
[a]*[b]*[\neg b] &= ([a] \supset [b]) * [a] * [\neg b] && \text{by Prop. 3.2 (iv)} \\
&= [(a \supset b)] * [a * \neg b] \\
&= [(a \supset b)] * [\neg(a \supset \neg\neg b)] \\
&= [(a \supset b)] * [\neg(a \supset b)] && \text{by (I5)} \\
&= [(a \supset b) * (\neg(a \supset b))] \\
&= [\neg((a \supset b) \supset \neg\neg(a \supset b))] \\
&= [\neg((a \supset b) \supset (a \supset b))] && \text{by (I5).}
\end{aligned}
$$

Therefore we can conclude that $\langle [a]*[\neg b], [a]*[b]*[\neg b] \rangle = \langle [\neg(a \supset b)], [\neg((a \supset b) \supset (a \supset b))] \rangle \in \eta$ as required. A symmetric argument allows to prove that $\langle [\neg(b \supset a)][\neg((b \supset a) \supset (b \supset a))] \rangle \in \eta$. Hence, we conclude that $\eta \subseteq H(H^{-1}(\eta))$ and therefore $\eta = H(H^{-1}(\eta))$. □

From the previous propositions we easily obtain the result announced earlier.

Theorem 4.7. *For any $\mathcal{I}$-algebra $\mathbf{A} \subseteq (\mathsf{L}(\mathbf{A}))^{\bowtie}$, the maps H and H^{-1} defined above establish a lattice isomorphism between* $\mathrm{Con}(\mathbf{A})$ *and* $\mathrm{Con}(\mathsf{L}(\mathbf{A}))$.

Proof. It suffices to note that both H and H^{-1} are clearly order-preserving by definition and, by Proposition 4.6, they are inverse of one another. □

The above theorem tells us that, in order to obtain information on lattice-theoretical properties of the congruence lattice of any $\mathcal{I}$-algebra $\mathbf{A} \subseteq (\mathsf{L}(\mathbf{A}))^{\bowtie}$, it is sufficient to look at the congruence lattice of $\mathsf{L}(\mathbf{A})$. In particular, it provides us with an alternative proof that $\mathcal{I}$-Alg is a congruence-distributive variety, since it is well-known that classical implicative lattices are congruence-distributive.

We now also know that the subdirectly irreducible $\mathcal{I}$-algebras are precisely those $\mathbf{A} \subseteq (\mathsf{L}(\mathbf{A}))^{\bowtie}$ such that $\mathsf{L}(\mathbf{A})$ is a subdirectly irreducible classical implicative lattice. Now the only subdirectly irreducible classical implicative lattice is the two-element one $\mathbf{L_2}$ (see, for instance, [29, Proposition 5.1.10]). It follows that the only subdirectly irreducible $\mathcal{I}$-algebras are those which can be embedded into the full implicative twist-structure $(\mathbf{L_2})^{\bowtie}$ shown in Figure 1. These are $\mathbf{A_4} = \mathbf{L_2}^{\bowtie}$, $\mathbf{A_3^+}$, $\mathbf{A_3^-}$ and $\mathbf{A_2}$. Notice that all these algebras are in fact simple, which implies that the variety of $\mathcal{I}$-algebras is semisimple [10, Lemma IV.12.2]. This, together with Theorem 4.2, implies that the variety of $\mathcal{I}$-algebras is filtral [4]. The following result is also straightforward.

Theorem 4.8. *The variety of $\mathcal{I}$-algebras is generated by its four-element member $\mathbf{A_4}$.*

Proof. By the above considerations, we have that the variety of $\mathcal{I}$-algebras is generated by $\{\mathbf{A_4}, \mathbf{A_3^+}, \mathbf{A_3^-}, \mathbf{A_2}\}$. But $\mathbf{A_3^+}, \mathbf{A_3^-}$ and $\mathbf{A_2}$ are subalgebras of $\mathbf{A_4}$, therefore we have that $\{\mathbf{A_3^+}, \mathbf{A_3^-}, \mathbf{A_2}\} \subseteq \mathsf{HSP}(\mathbf{A_4})$, from which the result easily follows. □

Theorem 4.8 implies that $\mathcal{I}$-Alg is locally finite [10, Theorem II.10.16]. It is also easy to prove that the quasivariety generated by $\mathbf{A_4}$ coincides with the class of all $\mathcal{I}$-algebras. This follows from the fact that all the subdirectly irreducible members of $\mathsf{V}(\mathbf{A_4})$ are isomorphic to subalgebras of $\mathbf{A_4}$ (see, for instance, [15, Theorem 3.6]).

As mentioned in Section 2, our implicative twist-structure construction can be viewed as a specialization of those used in [25, 29, 7] to construct N4-lattices and various classes of bilattices. On the other hand, the techniques we have introduced here are in a certain sense more general than those used in the above-mentioned works, because we have made use of a more reduced algebraic language. In this respect, the message of the present paper is that many interesting results can be proven even if we restrict ourselves to the negation-implication language. For instance, since any implicative bilattice has an $\mathcal{I}$-algebra reduct, we see that the twist-structure representation of implicative bilattices of [29, 7] can now be obtained as a corollary of our Theorem 3.8. A similar reasoning also applies to the main logical results contained in Section 5 (e.g., Theorems 5.2 and 5.7). Obviously the correspondence of congruences stated in Theorem 4.7 need not be preserved by language expansions, but it is interesting to note that analogous results have been proven, using different techniques, about N4-lattices [25, Corollary 4.3], various classes of bilattices [8, Propositions 3.8 and 3.13], [7, Theorems 4.3 and 4.13] and even twist-structures over residuated lattices [28, Theorem 5.3]. The following remark presents an example of an important universal algebraic property which is indeed lost when confine ourselves to the negation-implication language.

Remark 4.9. N4-lattices and implicative bilattices are both arithmetical varieties, i.e., congruence-distributive as well as congruence-permutable. The same is true of classical implicative lattices, and we may wonder whether the isomorphism established by Theorem 4.7 allows us to transfer this result to $\mathcal{I}$-algebras. The answer is negative, as shown by the following reasoning. Recall that $\mathcal{I}$-Alg is congruence-distributive, finitely-generated and semisimple. If it was also congruence-permutable (hence arithmetical), then it would be directly representable [10, Theorem IV.13.8], and this would imply that every finite $\mathcal{I}$-algebra $\mathbf{A}$ is congruence-uniform [10, Theorem IV.13.4]. This means [10, Definition IV.13.3] that, for any congruence θ of $\mathbf{A}$ and any $a, b \in A$, we have $|[a]_\theta| = |[b]_\theta|$. However, the algebra $\mathbf{A_8}$ shown in Figure 1 is not congruence-uniform. To see this, consider the congruence θ corresponding to the following partition: $\{\langle 1,0\rangle, \langle \mathsf{a},0\rangle, \langle \mathsf{a},\mathsf{b}\rangle\}$, $\{\langle 1,\mathsf{a}\rangle, \langle \mathsf{a},1\rangle\}$, $\{\langle 0,1\rangle, \langle \mathsf{b},\mathsf{a}\rangle, \langle 0,\mathsf{a}\rangle\}$. Then clearly $|[\langle 1,0\rangle]_\theta| \neq |[\langle 1,\mathsf{a}\rangle]_\theta|$. Taking into account what we observed above, we conclude that congruences of $\mathcal{I}$-algebras are 3-permutable yet not permutable.

We are now in a position to address the question of classifying the subvarieties of $\mathcal{I}$-algebras. Given that $\mathbf{A_2}$ is a subalgebra of both $\mathbf{A_3^+}$ and $\mathbf{A_3^-}$, it is easy to see that there are at most four non-trivial proper subvarieties to consider, namely $\mathsf{V}(\{\mathbf{A_3^+}, \mathbf{A_3^-}\})$, $\mathsf{V}(\mathbf{A_3^+})$, $\mathsf{V}(\mathbf{A_3^-})$ and $\mathsf{V}(\mathbf{A_2})$. The next proposition provides equational presentations for them and proves that they are all distinct (their distinctness can alternatively be confirmed using Jónsson's Theorem [10, Corollary IV.6.10]).

Proposition 4.10. *The subvarieties of $\mathcal{I}$-algebras may be axiomatized by adding the following equations to* (I1)-(I6)*:*

$\mathsf{V}(\{\mathbf{A_3^+}, \mathbf{A_3^-}\})$	$\neg(x \supset x) \preceq (y \supset \neg y) \supset \neg y$
$\mathsf{V}(\mathbf{A_3^+})$	$(\neg x \supset x) \supset x = x \supset x$
$\mathsf{V}(\mathbf{A_3^-})$	$x \supset x = y \supset y$
$\mathsf{V}(\mathbf{A_2})$	$x \supset y = \neg y \supset \neg x.$

Proof. It will be not only convenient but also instructive to check the above equations in an implicative twist-structure $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$. The first one, $\neg(x \supset x) \preceq (y \supset \neg y) \supset \neg y$, means that for arbitrary elements $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A \subseteq L \times L$,

$$a_1 \sqcap a_2 \leq b_1 \sqcup b_2.$$

It is easy to check that this condition is satisfied in $\mathbf{A_3^+}$, $\mathbf{A_3^-}$ and $\mathbf{A_2}$ but not in $\mathbf{A_4}$ because we can take $\langle a_1, a_2\rangle = \langle 1,1\rangle$ and $\langle b_1, b_2\rangle = \langle 0,0\rangle$. Notice also that $\mathsf{V}(\{\mathbf{A_3^+}, \mathbf{A_3^-}\}) = \mathsf{V}(\{\mathbf{A_3^+}, \mathbf{A_3^-}, \mathbf{A_2}\})$, because $\mathbf{A_2}$ is a subalgebra of both $\mathbf{A_3^+}$ and $\mathbf{A_3^-}$. The same reasoning implies that $\mathsf{V}(\mathbf{A_3^+}) = \mathsf{V}(\{\mathbf{A_3^+}, \mathbf{A_2}\})$ and $\mathsf{V}(\mathbf{A_3^-}) = \mathsf{V}(\{\mathbf{A_3^-}, \mathbf{A_2}\})$.

The second equation, $(\neg x \supset x) \supset x = x \supset x$, means that, for any $\langle a_1, a_2\rangle \in A$,

$$\langle (a_2 \backslash a_1) \backslash a_1, (a_2 \backslash a_1) \sqcap a_1 \rangle = \langle a_1 \sqcup a_2, a_1 \sqcap a_2\rangle = \langle 1, a_1 \sqcap a_2\rangle,$$

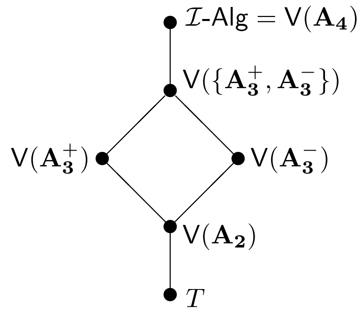

Figure 2. The lattice of subvarieties of $\mathcal{I}$-algebras

i.e., $a_1 \sqcup a_2 = 1$, where 1 is the maximum element of $\mathbf{L}$. Again, we can check that this is not true in $\mathbf{A_3^-}$ (thus, a fortiori, in $\mathbf{A_4}$) because of the presence of the element $\langle 0, 0\rangle$.
The equation $x \supset x = y \supset y$ plays a symmetrical role to the previous one. In fact, it means that, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A$,

$$a_1 \sqcap a_2 = b_1 \sqcap b_2.$$

But this means that $a_1 \sqcap a_2 \leq c$ for any $c \in L$, i.e., $a_1 \sqcap a_2$ must be the minimum element of the classical implicative lattice $\mathbf{L}$. That is, $\mathbf{L}$ must be (the reduct of) a Boolean algebra. This obviously fails in $\mathbf{A_3^+}$ because of the presence of the element $\langle 1, 1\rangle$.
Finally, if $\mathbf{A} \in \mathsf{V}(\mathbf{A_2}) \subseteq \mathsf{V}(\mathbf{A_3^+}) \cap \mathsf{V}(\mathbf{A_3^-})$, then, for any element $\langle a_1, a_2\rangle \in A$,

$$a_1 \sqcup a_2 = 1 \qquad \text{and} \qquad a_1 \sqcap a_2 = 0.$$

This means that a_1 and a_2 must be Boolean complements of each other in $\mathbf{L}$. It is easy to see that this condition implies that the last equation $x \supset y = \neg y \supset \neg x$ is satisfied, as it means that, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A$,

$$\langle a_1 \backslash b_1, a_1 \sqcap b_2\rangle = \langle b_2 \backslash a_2, a_2 \sqcap b_2\rangle,$$

that is, $a_1 \backslash b_1 = b_2 \backslash a_2$. Conversely, it is sufficient to check that $x \supset y = \neg y \supset \neg x$ fails both in $\mathbf{A_3^+}$ and in $\mathbf{A_3^-}$ to conclude that $\mathsf{V}(\mathbf{A_2})$ is in fact axiomatized by adding the equation $x \supset y = \neg y \supset \neg x$ to (I1)-(I6). □

Figure 2 shows the lattice of subvarieties of $\mathcal{I}$-algebras. While it is obvious that $\mathsf{V}(\mathbf{A_2})$ coincides with the variety of Boolean algebras (presented in the language of implication and negation), as far the author is aware, none of the other varieties introduced above has been previously studied in the literature.

We can now check that the construction introduced in (3.3) is actually sufficient for characterizing the implicative twist-structures corresponding to

$\mathsf{V}(\mathbf{A_3^+})$ and $\mathsf{V}(\mathbf{A_3^-})$. Let us recall the relevant definitions. Given a twist-structure $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$, we let

$$\begin{aligned} I(\mathbf{A}) &:= \{\langle a_1 \sqcup a_2, a_1 \sqcap a_2 \rangle : \langle a_1, a_2 \rangle \in A\}, \\ U(\mathbf{A}) &:= \pi_1(I(\mathbf{A})), \\ D(\mathbf{A}) &:= \pi_2(I(\mathbf{A})), \\ A' &:= \{\langle a_1, a_2 \rangle \in L \times L : a_1 \sqcup a_2 \in U(\mathbf{A}),\ a_1 \sqcap a_2 \in D(\mathbf{A})\}. \end{aligned}$$

It follows from what we have observed in the proof of Proposition 4.10 that:

- if $\mathbf{A} \in \mathsf{V}(\mathbf{A_3^+})$, then $U(\mathbf{A}) = \{1\}$ and $I(\mathbf{A}) = \{\langle 1, a_1 \sqcap a_2 \rangle : \langle a_1, a_2 \rangle \in A\}$,
- if $\mathbf{A} \in \mathsf{V}(\mathbf{A_3^-})$, then $D(\mathbf{A}) = \{0\}$ and $I(\mathbf{A}) = \{\langle a_1 \sqcup a_2, 0 \rangle : \langle a_1, a_2 \rangle \in A\}$

where 0 and 1 are the minimum and maximum elements of $\mathbf{L}$. Either of these additional conditions will enable us to prove that $A = A'$. We will need the following lemma.

Lemma 4.11. *Let $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$ be an implicative twist-structure. Then, for all $\langle a_1, a_2 \rangle \in L \times L$,*

$$\langle a_1, a_2 \rangle \in A \quad \textit{if and only if} \quad \langle a_1 \sqcup a_2, a_1 \sqcap a_2 \rangle \in I(\mathbf{A}).$$

Proof. (i) The rightward direction follows from the definition of $I(\mathbf{A})$. To prove the converse, assume $\langle a_1 \sqcup a_2, a_1 \sqcap a_2 \rangle \in I(\mathbf{A})$. Note that, by definition, we have that $I(\mathbf{A}) \subseteq A$, hence $\langle a_1 \sqcup a_2, a_1 \sqcap a_2 \rangle \in A$, which implies that $\neg\langle a_1 \sqcup a_2, a_1 \sqcap a_2 \rangle = \langle a_1 \sqcap a_2, a_1 \sqcup a_2 \rangle \in A$ as well. Moreover, since $\pi_1(A) = L$, there must be $b, c \in L$ such that $\langle a_1, b \rangle, \langle a_2, c \rangle \in A$. Then we have that $\langle a_1, b \rangle \supset \langle a_2, c \rangle = \langle a_1 \backslash a_2, a_1 \sqcap c \rangle \in A$. Hence we have that

$$\begin{aligned} & \langle a_1 \backslash a_2, a_1 \sqcap c \rangle \supset \langle a_1 \sqcap a_2, a_1 \sqcup a_2 \rangle \\ &= \langle (a_1 \backslash a_2) \backslash (a_1 \sqcap a_2), (a_1 \backslash a_2) \sqcap (a_1 \sqcup a_2) \rangle \\ &= \langle ((a_1 \backslash a_2) \backslash a_1) \sqcap ((a_1 \backslash a_2) \backslash a_2), (a_1 \backslash a_2) \sqcap (a_1 \sqcup a_2) \rangle && \text{by Prop. 3.2 (ix)} \\ &= \langle a_1 \sqcap (a_1 \sqcup a_2), (a_1 \backslash a_2) \sqcap (a_1 \sqcup a_2) \rangle && \text{Prop. 3.2 (v)} \\ &= \langle a_1, (a_1 \backslash a_2) \sqcap (a_1 \sqcup a_2) \rangle && \text{absorption} \\ &= \langle a_1, ((a_1 \backslash a_2) \sqcap a_1) \sqcup ((a_1 \backslash a_2) \sqcap a_2)) \rangle && \text{distributivity} \\ &= \langle a_1, (a_1 \sqcap a_2) \sqcup a_2 \rangle && \text{Prop. 3.2 (iv), (viii)} \\ &= \langle a_1, a_2 \rangle \in A && \text{absorption.} \end{aligned}$$

□

We are now able to prove the announced result.

Theorem 4.12. *For any implicative twist-structure $\mathbf{A} \subseteq \mathbf{L}^{\bowtie}$,*

(i) if $\mathbf{A} \in \mathsf{V}(\mathbf{A_3^+})$, then

$$A = \{\langle a_1, a_2 \rangle \in L \times L : a_1 \sqcup a_2 = 1 \textit{ and } a_1 \sqcap a_2 \in D(\mathbf{A})\},$$

(ii) if $\mathbf{A} \in \mathsf{V}(\mathbf{A_3^-})$, then

$$A = \{\langle a_1, a_2 \rangle \in L \times L : a_1 \sqcup a_2 \in U(\mathbf{A}) \textit{ and } a_1 \sqcap a_2 = 0\}.$$

Proof. In order to prove that $A = A'$, we will use Lemma 4.11, showing that $I(\mathbf{A}) = I(A')$, where

$$I(A') = \{\langle a_1, a_2 \rangle \in A' : a_1 \geq a_2\}.$$

As noted earlier, it is easy to see that $A \subseteq A'$. This implies that $I(\mathbf{A}) \subseteq I(A')$, therefore we only need to prove the other inclusion. Assume then $\langle a_1, a_2 \rangle \in I(A')$. This means that $a_1 \in U(\mathbf{A})$, $a_2 \in D(\mathbf{A})$ and $a_1 \geq a_2$. Now we reason by cases.
(i) If $\mathbf{A} \in \mathsf{V}(\mathbf{A_3^+})$, then $U(\mathbf{A}) = \{1\}$, therefore $a_1 = 1$. Moreover, $a_2 \in D(\mathbf{A}) = \pi_2(I(\mathbf{A}))$, which means that $\langle 1, a_2 \rangle = \langle a_1, a_2 \rangle \in I(\mathbf{A})$. Hence we conclude that $I(A') \subseteq I(\mathbf{A})$ as required.
(ii) If $\mathbf{A} \in \mathsf{V}(\mathbf{A_3^-})$, then $D(\mathbf{A}) = \{0\}$, so $a_2 = 0$. Moreover, $a_1 \in U(\mathbf{A}) = \pi_1(I(A))$, which means that $\langle a_1, 0 \rangle = \langle a_1, a_2 \rangle \in I(A)$, so again we are done. □

One consequence of Theorem 4.12 is that the variety $\mathsf{V}(\mathbf{A_3^-})$ coincides with the class of $\{\neg, \supset\}$-subreducts of Nelson lattices satisfying the identity:

$$((x \supset y) \supset x) \supset x = x \supset x.$$

Notice that we cannot hope to extend Theorem 4.12 to all the members of $\mathsf{V}(\{\mathbf{A_3^+}, \mathbf{A_3^-}\})$, because the $\mathcal{I}$-algebra $\mathbf{A_8}$ that was used as a counterexample already belongs to this variety.

To end the section, let us briefly discuss the topic of subquasivarieties of $\mathcal{I}$-algebras, which is particularly relevant from an algebraic logic point of view (see the next section). Although at present we are not able to give a classification of these quasivatieties, we will try to show that the topic is interesting enough to deserve further research.

As observed above, we know that $\mathsf{Q}(\mathbf{A_4}) = \mathsf{V}(\mathbf{A_4})$. The same reasoning shows that $\mathsf{Q}(\mathbf{A_3^+}) = \mathsf{V}(\mathbf{A_3^+})$ and $\mathsf{Q}(\mathbf{A_3^-}) = \mathsf{V}(\mathbf{A_3^-})$, and obviously $\mathsf{Q}(\mathbf{A_2}) = \mathsf{V}(\mathbf{A_2})$, as the latter is just the variety of Boolean algebras. We may wonder whether there are any quasivatieties of $\mathcal{I}$-algebras which are not varieties. This question is easily answered in the affirmative. Consider the following quasiequation:

$$x = \neg x \quad \Rightarrow \quad x = y \tag{4.2}$$

which corresponds to the requirement that the negation operator of the $\mathcal{I}$-algebra has no fixed points. In terms of the twist-structure representation, this means that $a_1 \neq a_2$ for all $\langle a_1, a_2 \rangle \in \mathbf{A}$. It is then easy to check that, of all the examples of $\mathcal{I}$-algebras considered so far, only $\mathbf{A_2}$ and $\mathbf{A_8}$ satisfy (4.2). This implies for instance that $\mathsf{Q}(\mathbf{A_8})$ cannot be a variety. If we denote by K the quasivariety of all $\mathcal{I}$-algebras that satisfy (4.2), then we have that $\mathsf{Q}(\mathbf{A_8}) \subseteq \mathsf{K} \cap \mathsf{V}(\{\mathbf{A_3^+}, \mathbf{A_3^-}\})$. By looking at the subalgebras of $\mathbf{A_8}$ it is equally easy to show that $\mathsf{K} \cap \mathsf{V}(\mathbf{A_3^+})$ and $\mathsf{K} \cap \mathsf{V}(\mathbf{A_3^-})$ are also proper (and distinct) quasivatieties.

5. Logics of implicative twist-structures

Given that the original motivation for studying implicative twist-structures arose in [29] from an algebraic logic analysis of the Arieli-Avron bilattice logic, it is interesting to look at the logical counterpart of our algebraic structures.

The language of the sentential logic we are going to introduce is the same as the algebraic signature of implicative twist-structures.

Definition 5.1. Let $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$ be the sentential logic defined through the Hilbert style calculus with axiom schemata:

$$p \supset (q \supset p), \tag{$\supset 1$}$$

$$(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)), \tag{$\supset 2$}$$

$$((p \supset q) \supset p) \supset p, \tag{$\supset 3$}$$

$$\neg(p \supset q) \supset p, \qquad \neg(p \supset q) \supset \neg q, \tag{$\neg \supset$}$$

$$p \supset (\neg q \supset \neg(p \supset q)), \tag{$\supset \neg$}$$

$$p \supset \neg\neg p, \qquad \neg\neg p \supset p, \tag{$\neg\neg$}$$

and with *modus ponens* as the only inference rule:

$$\frac{p \quad p \supset q}{q} \tag{MP}$$

It is well-known [35, Theorem 2.4.2] that any calculus having axioms ($\supset 1$) and ($\supset 2$) and (MP) as the only rule enjoys the classical Deduction-Detachment Theorem:

Theorem 5.2 (DDT)**.** *For all* $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$*,*

$$\Gamma, \varphi \vdash \psi \quad \textit{iff} \quad \Gamma \vdash \varphi \supset \psi.$$

We will sometimes use the DDT without notice in order to simplify our syntactical proofs in $\mathcal{L}$.

We are now going to prove that our logic $\mathcal{L}$ is algebraizable with respect to the variety of $\mathcal{I}$-algebras. For this we need to define a translation τ from formulas into equations in the language of $\mathcal{I}$-algebras and a translation ρ from equations into formulas. For $\varphi \in Fm$, we let

$$\tau(\varphi) \ := \ (\varphi = \varphi \supset \varphi)$$

and for $\Gamma \subseteq Fm$, we let $\tau(\Gamma) := \{\tau(\varphi) : \varphi \in \Gamma\}$. Conversely, given an equation $\varphi = \psi \in Eq$, we define

$$\rho(\varphi = \psi) \ := \ \{\varphi \supset \psi, \ \psi \supset \varphi, \ \neg\varphi \supset \neg\psi, \ \neg\psi \supset \neg\varphi\}.$$

We extend ρ to sets of equations in the same way as τ. It is not difficult to prove that the above-defined set of formulas $\rho(\varphi = \psi)$ is logically equivalent in $\mathcal{L}$ with $\{\varphi \to \psi, \ \psi \to \varphi\}$ and also with the single formula $\varphi \leftrightarrow \psi$, where as before we let

$$\begin{aligned} \varphi * \psi &:= \neg(\varphi \supset \neg\psi), \\ \varphi \to \psi &:= (\varphi \supset \psi) * (\neg\psi \supset \neg\varphi), \\ \varphi \leftrightarrow \psi &:= (\varphi \to \psi) * (\psi \to \varphi). \end{aligned}$$

Therefore, either of these sets can be alternatively used to define ρ. Let us also notice that, since the equation $x \supset x = x \to x$ is valid in any $\mathcal{I}$-algebra, we could also alternatively define $\tau(\varphi)$ as $\varphi = \varphi \to \varphi$.

Our next aim is to check that the calculus introduced in Definition 5.1 is indeed algebraizable. We will need the following lemma.

Proposition 5.3. *For all formulas $\varphi, \psi, \vartheta, \in Fm$,*

(i) $\{\varphi \supset \psi,\ \psi \supset \vartheta\} \vdash \varphi \supset \vartheta$,

(ii) $\vdash \varphi \supset \varphi$.

Proof. (i) By $(\supset 1)$ and MP we have $\psi \supset \vartheta \vdash \varphi \supset (\psi \supset \vartheta)$ and by $(\supset 2)$ we have $\vdash (\varphi \supset (\psi \supset \vartheta)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \vartheta))$. So, applying MP, we have $\psi \supset \vartheta \vdash (\varphi \supset \psi) \supset (\varphi \supset \vartheta)$. Hence, by MP, we obtain $\{\psi \supset \vartheta,\ \varphi \supset \psi\} \vdash (\varphi \supset \vartheta)$. (ii) $(\varphi \supset ((\psi \supset \varphi) \supset \varphi)) \supset ((\varphi \supset (\psi \supset \varphi)) \supset (\varphi \supset \varphi))$ is an instance of $(\supset 2)$, with $p = r = \varphi$ and $q = \psi \supset \varphi$. Moreover, $(\varphi \supset ((\psi \supset \varphi) \supset \varphi))$ and $(\varphi \supset (\psi \supset \varphi))$ are instances of $(\supset 1)$. Hence, applying MP twice, we obtain $\vdash \varphi \supset \varphi$. □

Proposition 5.4. *The logic $\mathcal{L}$ is algebraizable with defining equation $\varphi = \varphi \supset \varphi$ and equivalence formulas $\{\varphi \supset \psi, \psi \supset \varphi, \neg\varphi \supset \neg\psi, \neg\psi \supset \neg\varphi\}$.*

Proof. Using the intrinsic characterization given by Blok and Pigozzi [5, Theorem 4.7], it is sufficient to check that the following conditions hold: for all formulas $\varphi, \psi, \vartheta, \varphi_1, \varphi_2, \psi_1, \psi_2 \in Fm$,

(a) $\varphi \dashv\vdash \{\varphi \supset (\varphi \supset \varphi),\ (\varphi \supset \varphi) \supset \varphi,\ \neg\varphi \supset \neg(\varphi \supset \varphi),\ \neg(\varphi \supset \varphi) \supset \neg\varphi\}$,

(b) $\vdash \varphi \supset \varphi$,

(c) $\vdash \neg\varphi \supset \neg\varphi$,

(d) $\{\varphi \supset \psi,\ \psi \supset \varphi,\ \neg\varphi \supset \neg\psi,\ \neg\psi \supset \neg\varphi\} \vdash \{\varphi \supset \psi,\ \psi \supset \varphi,\ \neg\varphi \supset \neg\psi,\ \neg\psi \supset \neg\varphi\}$,

(e) $\{\varphi \supset \psi,\ \psi \supset \varphi,\ \neg\varphi \supset \neg\psi,\ \neg\psi \supset \neg\varphi,\ \psi \supset \vartheta, \vartheta \supset \psi,\ \neg\psi \supset \neg\vartheta,\ \neg\vartheta \supset \neg\psi\} \vdash \{\varphi \supset \vartheta,\ \vartheta \supset \varphi,\ \neg\varphi \supset \neg\vartheta,\ \neg\vartheta \supset \neg\varphi\}$,

(f) $\{\varphi \supset \psi,\ \psi \supset \varphi,\ \neg\varphi \supset \neg\psi,\ \neg\psi \supset \neg\varphi\} \vdash \{\neg\varphi \supset \neg\psi,\ \neg\psi \supset \neg\varphi,\ \neg\neg\varphi \supset \neg\neg\psi,\ \neg\neg\psi \supset \neg\neg\varphi\}$,

(g) $\{\varphi_1 \supset \psi_1,\ \psi_1 \supset \varphi_1,\ \neg\varphi_1 \supset \neg\psi_1,\ \neg\psi_1 \supset \neg\varphi_1,\ \varphi_2 \supset \psi_2,\ \psi_2 \supset \varphi_2,\ \neg\varphi_2 \supset \neg\psi_2,\ \neg\psi_2 \supset \neg\varphi_2\} \vdash \{(\varphi_1 \supset \varphi_2) \supset (\psi_1 \supset \psi_2),\ (\psi_1 \supset \psi_2) \supset (\varphi_1 \supset \varphi_2),\ \neg(\varphi_1 \supset \varphi_2) \supset \neg(\psi_1 \supset \psi_2),\ \neg(\psi_1 \supset \psi_2) \supset \neg(\varphi_1 \supset \varphi_2)\}$.

The non-trivial implications of (a) can be easily proved using $(\neg \supset)$. Items (b) and (c) follow immediately from Proposition 5.3 (ii), while (d) is trivial. Item (e) follows from Proposition 5.3 (i), while (f) can be easily proved using $(\neg\neg)$.

All the cases of (g) are also easily proved. Let us check just the last one as an example. Using the DDT, we will prove that

$$\{\psi_1 \supset \varphi_1,\ \neg\psi_2 \supset \neg\varphi_2,\ \neg(\psi_1 \supset \psi_2)\} \vdash \neg(\varphi_1 \supset \varphi_2).$$

By $(\neg \supset)$, we have that $\neg(\psi_1 \supset \psi_2) \vdash \{\psi_1,\ \neg\psi_2$. By transitivity of $\vdash$ and MP, we obtain then $\{\psi_1 \supset \varphi_1,\ \neg\psi_2 \supset \neg\varphi_2,\ \neg(\psi_1 \supset \psi_2)\} \vdash \{\varphi_1,\ \neg\varphi_2\}$ By $(\supset \neg)$, we have $\{\varphi_1,\ \neg\varphi_2\} \vdash \neg(\varphi_1 \supset \varphi_2)$, so applying again transitivity of $\vdash$ we obtain the desired result. □

We are now going to see that the equivalent algebraic semantics of our logic $\mathcal{L}$ is precisely the variety of $\mathcal{I}$-algebras. We will need to prove some more syntactical properties of our calculus.

Proposition 5.5. *For all formulas $\varphi, \psi, \vartheta, \in Fm$,*

(i) $\vdash ((\varphi \supset \varphi) \supset \psi) \supset \psi$,
(ii) $\vdash \varphi \supset ((\psi \supset \psi) \supset \varphi)$,
(iii) $\vdash \neg((\varphi \supset \varphi) \supset \psi) \supset \neg\psi$,
(iv) $\vdash \neg\psi \supset \neg((\varphi \supset \varphi) \supset \psi)$,
(v) $\vdash (\varphi \supset (\psi \supset \vartheta)) \supset (\psi \supset (\varphi \supset \vartheta))$,
(vi) $\vdash \neg(\varphi \supset (\psi \supset \vartheta)) \supset \neg(\psi \supset (\varphi \supset \vartheta))$,
(vii) $\vdash ((\varphi \supset \psi) \supset (\varphi \supset \vartheta)) \supset (\varphi \supset (\psi \supset \vartheta))$,
(viii) $\vdash \neg(\varphi \supset (\psi \supset \vartheta)) \supset \neg((\varphi \supset \psi) \supset (\varphi \supset \vartheta))$,
(ix) $\vdash \neg((\varphi \supset \psi) \supset (\varphi \supset \vartheta)) \supset \neg(\varphi \supset (\psi \supset \vartheta))$,
(x) $\vdash ((\varphi \supset (\neg\psi \supset \vartheta)) \supset (\neg(\varphi \supset \psi) \supset \vartheta)$,
(xi) $\vdash (\neg(\varphi \supset \psi) \supset \vartheta) \supset ((\varphi \supset (\neg\psi \supset \vartheta))$,
(xii) $\vdash \neg((\varphi \supset (\neg\psi \supset \vartheta)) \supset \neg(\neg(\varphi \supset \psi) \supset \vartheta)$,
(xiii) $\vdash \neg(\neg(\varphi \supset \psi) \supset \vartheta) \supset \neg((\varphi \supset (\neg\psi \supset \vartheta))$,
(xiv) $\vdash ((\varphi \leftrightarrow \psi) \supset \varphi) \supset ((\varphi \leftrightarrow \psi) \supset \psi)$,
(xv) $\vdash \neg((\varphi \leftrightarrow \psi) \supset \varphi) \supset \neg((\varphi \leftrightarrow \psi) \supset \psi)$,
(xvi) $\neg(\varphi \supset \varphi) \dashv\vdash \neg(((\varphi \supset \psi) \supset \varphi) \supset \varphi)$.

Proof. (i) Easy, applying Proposition 5.3 (ii) and MP.
(ii) It is an instance of $(\supset 2)$.
(iii) It is an instance of $(\neg \supset)$.
(iv) Note that $(\varphi \supset \varphi) \supset (\neg\psi \supset \neg((\varphi \supset \varphi) \supset \psi))$ is an instance of $(\supset \neg)$. Then, applying Proposition 5.3 (ii) and MP, we easily obtain the result.
(v) Easy, since, using the DDT, it boils down to proving that $\varphi \supset (\psi \supset \vartheta), \psi, \varphi \vdash \vartheta$.
(vi) By $(\neg \supset)$, we have that $\neg(\varphi \supset (\psi \supset \vartheta)) \vdash \{\varphi,\ \neg(\psi \supset \vartheta)\}$. Using $(\neg \supset)$ again we obtain $\neg(\psi \supset \vartheta) \vdash \{\psi,\ \neg\vartheta\}$. Hence, $\neg(\varphi \supset (\psi \supset \vartheta)) \vdash \{\varphi,\ \psi,\ \neg\vartheta\}$. By $(\supset \neg)$, we have that $\varphi, \neg\vartheta \vdash \neg(\varphi \supset \vartheta)$ and $\psi, \neg(\varphi \supset \vartheta) \vdash \neg(\psi \supset (\varphi \supset \vartheta))$. Then, applying transitivity of $\vdash$, we obtain $\neg(\varphi \supset (\psi \supset \vartheta)) \vdash \neg(\psi \supset (\varphi \supset \vartheta))$ as required.
(vii) All we have to prove is that $(\varphi \supset \psi) \supset (\varphi \supset \vartheta), \varphi, \psi \vdash \vartheta$. Since $\psi \vdash \varphi \supset \psi$, this is easily obtained by MP.

(viii) We will prove that $\neg(\psi \supset (\varphi \supset \vartheta)) \vdash \neg((\varphi \supset \psi) \supset (\varphi \supset \vartheta))$, so the result will follow by (vi). By $(\neg \supset)$, we have that $\neg(\psi \supset (\varphi \supset \vartheta)) \vdash \psi$ and $\neg(\psi \supset (\varphi \supset \vartheta)) \vdash \neg(\varphi \supset \vartheta)$. Since $\psi \vdash \varphi \supset \psi$, this implies that $\neg(\psi \supset (\varphi \supset \vartheta)) \vdash \varphi \supset \psi$. By $(\supset \neg)$, we have $\varphi \supset \psi, \neg(\varphi \supset \vartheta) \vdash \neg((\varphi \supset \psi) \supset (\varphi \supset \vartheta))$. Hence the result easily follows by transitivity of the derivability relation.
(ix) By $(\neg \supset)$, we have that $\neg((\varphi \supset \psi) \supset (\varphi \supset \vartheta)) \vdash \{\varphi \supset \psi,\ \neg(\varphi \supset \vartheta)\}$. Similarly we obtain $\neg(\varphi \supset \vartheta) \vdash \{\varphi,\ \neg\vartheta\}$. Applying transitivity of $\vdash$ and MP, we conclude that $\neg((\varphi \supset \psi) \supset (\varphi \supset \vartheta)) \vdash \{\psi,\ \neg\vartheta\}$. By $(\supset \neg)$, we have that $\psi, \neg\vartheta \vdash \neg(\psi \supset \vartheta)$ and $\{\neg(\psi \supset \vartheta),\ \varphi\} \vdash \neg(\varphi \supset (\psi \supset \vartheta))$. Hence, using again transitivity of $\vdash$ we obtain the desired result.
(x) By $(\neg \supset)$, we have that $\neg(\varphi \supset \psi) \vdash \{\varphi,\ \neg\psi\}$. Applying the transitivity of $\vdash$ and twice MP, we have that $\{(\varphi \supset (\neg\psi \supset \vartheta),\ \neg(\varphi \supset \psi)\} \vdash \vartheta$, from which the result easily follows.
(xi) It is sufficient to prove that $\neg(\varphi \supset \psi) \supset \vartheta, \varphi, \neg\psi \vdash \vartheta$. To see this, note that, by $(\supset \neg)$, we have that $\{\varphi,\ \neg\psi\} \vdash \neg(\varphi \supset \psi)$. Hence, modus ponens yields the desired result.
(xii) By $(\neg \supset)$, we have that $\neg((\varphi \supset (\neg\psi \supset \vartheta)) \vdash \{\varphi,\ \neg(\neg\psi \supset \vartheta)\}$. Applying monotonicity and $(\neg \supset)$ again, we have that $\{\varphi,\ \neg(\neg\psi \supset \vartheta)\} \vdash \{\varphi,\ \neg\psi,\ \neg\vartheta\}$. By $(\supset \neg)$, we have that $\varphi, \neg\psi \vdash \neg(\varphi \supset \psi)$ and $\neg(\varphi \supset \psi), \neg\vartheta \vdash \neg(\neg(\varphi \supset \psi) \supset \vartheta)$. Hence, transitivity of $\vdash$ yields the desired result.
(xiii) By $(\neg \supset)$, we have that $\neg(\neg(\varphi \supset \psi) \supset \vartheta) \vdash \{\neg(\varphi \supset \psi),\ \neg\vartheta\}$ and $\neg(\varphi \supset \psi) \vdash \{\varphi,\ \neg\psi\}$. Hence, $\neg(\neg(\varphi \supset \psi) \supset \vartheta) \vdash \{\varphi,\ \neg\psi,\ \neg\vartheta\}$. Now, using $(\supset \neg)$, we obtain $\neg\psi, \neg\vartheta \vdash \neg(\neg\psi \supset \vartheta)$ and $\varphi, \neg(\neg\psi \supset \vartheta) \vdash \neg((\varphi \supset (\neg\psi \supset \vartheta))$. Then by transitivity of $\vdash$ the result easily follows.
(xiv) Let us observe that, for all $\varphi, \psi \in Fm$, it holds that $\{\varphi, \psi\} \dashv\vdash \varphi * \psi$. The rightward direction holds because, by $(\neg\neg)$, we have that $\{\varphi, \psi\} \dashv\vdash \{\varphi, \neg\neg\psi\}$ and, by $(\supset \neg)$, we have that $\{\varphi, \neg\neg\psi\} \vdash \neg(\varphi \supset \neg\psi) = \varphi * \psi$. Conversely, by $(\neg \supset)$, we have that $\varphi * \psi \vdash \{\varphi, \neg\neg\psi\}$ and from this the result easily follows. This means that

$$\varphi \leftrightarrow \psi \dashv\vdash \{\varphi \to \psi, \psi \to \varphi\} \dashv\vdash \{\varphi \supset \psi, \psi \supset \varphi, \neg\varphi \supset \neg\psi, \neg\psi \supset \neg\varphi\}.$$

Then, given that, by MP, $(\varphi \leftrightarrow \psi) \supset \varphi, \varphi \leftrightarrow \psi \vdash \varphi$, it is easy to see that $(\varphi \leftrightarrow \psi) \supset \varphi, \varphi \leftrightarrow \psi \vdash \psi$. The desired result follows then by the DDT.
(xv) By $(\neg \supset)$, we have that $\neg((\varphi \leftrightarrow \psi) \supset \varphi) \vdash \{\varphi \leftrightarrow \psi, \neg\varphi\}$. Reasoning as in the proof of (xiv), we obtain $\{\varphi \leftrightarrow \psi, \neg\varphi\} \vdash \{\varphi \leftrightarrow \psi, \neg\psi\}$. Applying $(\neg \supset)$, we have that $\{\varphi \leftrightarrow \psi, \neg\psi\} \vdash \neg((\varphi \leftrightarrow \psi) \supset \psi)$. Hence, by transitivity of $\vdash$ we obtain $\neg((\varphi \leftrightarrow \psi) \supset \varphi) \vdash \neg((\varphi \leftrightarrow \psi) \supset \psi)$, so the result follows by the DDT.
(xvi) By $(\neg \supset)$ and $(\supset \neg)$, we have that $\neg(\varphi \supset \varphi) \dashv\vdash \{\varphi, \neg\varphi\}$. By monotonicity of $\vdash$ and the DDT, we further have that $\{\varphi, \neg\varphi\} \vdash \{(\varphi \supset \psi) \supset \varphi, \neg\varphi\}$. Also, applying MP to $(\supset 3)$, we have that $\{(\varphi \supset \psi) \supset \varphi, \neg\varphi\} \vdash \{\varphi, \neg\varphi\}$. Finally, using again $(\supset \neg)$ and $(\neg \supset)$, we have that $\{(\varphi \supset \psi) \supset \varphi, \neg\varphi\} \dashv\vdash \neg(((\varphi \supset \psi) \supset \varphi) \supset \varphi)$, so the result follows by transitivity of the inter-derivability relation. □

Lemma 5.6. *For every axiom φ of $\mathcal{L}$, the equation $\varphi = \varphi \supset \varphi$ is valid in the variety of $\mathcal{I}$-algebras.*

Proof. It can be easily proved using the twist-structure representation, or we can directly check that all equations are satisfied in $\mathbf{A_4}$ which is the generator of the variety. □

We are now able to prove the announced result.

Theorem 5.7. *The logic $\mathcal{L}$ is algebraizable with respect to the variety of $\mathcal{I}$-algebras, with equivalence formulas $\{\varphi \supset \psi, \psi \supset \varphi, \neg\varphi \supset \neg\psi, \neg\psi \supset \neg\varphi\}$ and defining equation $\varphi = \varphi \supset \varphi$.*

Proof. We will prove that the equivalent algebraic semantics $\mathbf{Alg}^*\mathcal{L}$ of our logic $\mathcal{L}$ is precisely the class of $\mathcal{I}$-algebras. By [5, Theorem 2.17], we know that the class $\mathbf{Alg}^*\mathcal{L}$ is axiomatized by the following equations and quasiequations (recall that $E(x)$ is a shorthand for the equation $x = x \supset x$):

(a) $E(\varphi)$ for all axioms φ of $\mathcal{L}$,
(b) $E(x)$ & $E(x \supset y) \ \Rightarrow\ E(y)$,
(c) $E(x \supset y)$ & $E(y \supset x)$ & $E(\neg x \supset \neg y)$ & $E(\neg y \supset \neg x) \ \Rightarrow\ x = y$.

In order to prove that $\mathcal{I}\text{-}\mathsf{Alg} \subseteq \mathbf{Alg}^*\mathcal{L}$, it is then sufficient to prove that any $\mathcal{I}$-algebra $\mathbf{A}$ satisfies (a) to (c). The first item is Lemma 5.6. As to (b), applying (I1), we have that, for all $a, b \in A$, the assumption $E(a)$ implies that $(a \supset a) \supset b = a \supset b = b$. But $E(a \supset b)$ means that $b = a \supset b = (a \supset b) \supset (a \supset b) = b \supset b$, so we are done. Finally, we have already established (c) in the proof of Theorem 3.8.

In order to show that $\mathbf{Alg}^*\mathcal{L} \subseteq \mathcal{I}\text{-}\mathsf{Alg}$, we have to check that any $\mathbf{A} \in \mathbf{Alg}^*\mathcal{L}$ satisfies the equations valid in the variety of $\mathcal{I}$-algebras. To see this, using (a) and (c), it will be enough to prove that, for any equation $\varphi = \psi$ of Definition 3.1, it holds that $\vdash \{\varphi \supset \psi,\ \psi \supset \varphi,\ \neg\varphi \supset \neg\psi,\ \neg\psi \supset \neg\varphi\}$. Some cases are straighforward. The non-trivial ones have been proved in Proposition 5.5. □

A well-known result on algebraizable logics (see for instance [5, Corollary 4.9]) tells us that the finitary extensions of $\mathcal{L}$ are also algebraizable and that the corresponding equivalent quasivariety semantics are the subquasivarieties of $\mathcal{I}$-algebras. In particular, we know that the axiomatic extensions of $\mathcal{L}$ are in one-to-one correspondence with subvarieties of $\mathcal{I}$-algebras. It follows then from Proposition 4.10 that $\mathcal{L}$ has only four consistent axiomatic proper extensions, which can be axiomatized by adding the following axioms to the calculus given in Defintion 5.1:

(I) $\neg(p \supset p) \supset ((q \supset \neg q) \supset \neg q)$,
(II) $(\neg p \supset p) \supset p$,
(III) $\neg(p \supset p) \supset \neg(q \supset q)$,
(IV) $(p \supset q) \supset (\neg q \supset \neg p)$.

Using the twist-structure representation of $\mathcal{I}$-algebras, it is easy to check that (I) allows us to axiomatize the logic corresponding to $\mathsf{V}(\{\mathbf{A_3^+}, \mathbf{A_3^-}\})$ and likewise (II), (III) and (IV) characterize the logics corresponding, respectively, to $\mathsf{V}(\mathbf{A_3^+})$, $\mathsf{V}(\mathbf{A_3^-})$ and $\mathsf{V}(\mathbf{A_2})$.

In order to characterize the remaining finitary (non-axiomatic) extensions of $\mathcal{L}$ we would need a description of the lattice of subquasivarieties of $\mathcal{I}$-algebras and this, as mentioned in the previous section, is still an open question. For now, we can observe that it is easy to check that the rule corresponding to the quasiequation 4.2 is the following:

$$\frac{p \supset \neg p \qquad \neg p \supset p}{q}$$

By the considerations made in Section 4, we know then that, if we add this rule to $\mathcal{L}$ or to any of its axiomatic extensions, we obtain in each case a different logic.

6. Further work

We hope that the previous sections have shown that the theory of implicative twist-structures has an independent interest, both from a logical and an algebraic point of view, which does not depend solely on its connections with N4-lattices and bilattices. We would like to conclude the paper by mentioning some topics that, in our opinion, deserve further investigation.

We have already discussed the issue of classifying the subquasivarieties of $\mathcal{I}$-algebras, which corresponds on the logical side to the study of finitary extensions of the logic.

Another problem that is particularly interesting from a logical point of view is how to characterize the $\{\to\}$-fragment of our logic $\mathcal{L}$ and the corresponding algebraic semantics. It seems that the methods used in the paper cannot be straightforwardly applied here, as no twist-structure representation is yet available for algebras in the language with the $\to$ implication as the only operation. However, as observed in Section 5, the $\to$ implication alone is sufficient to define translations that guarantee algebraizability of the corresponding fragment of the logic. This seems to indicate that even such a reduced language might be expressive enough for our purposes, and also that the problem of axiomatizing the logic can be reduced to that of finding a (quasi)equational presentation for the class of $\{\to\}$-subreducts of $\mathcal{I}$-algebras.

A natural question to ask is whether the construction introduced in Section 2 could be easily generalized to define implicative twist-structures over, for example, generalized Heyting algebras (the 0-free subreducts of Heyting algebras). This is certainly possible and will be the object of future investigation, although it seems that weakening certain algebraic properties of classical implicative lattices will imply modifying in a non-trivial way the proof strategy that we used to obtain the representation result of Theorem 3.8.

Another interesting possibility to be investigated is, in our opinion, the development of a topological representation for our twist-structures, perhaps along the same lines of topological studies [13, 14] of other implication-based algebras such as Tarski and Hilbert algebras (i.e., the purely implicational subreducts of, respectively, Boolean algebras and Heyting algebras). The fact that our structures do not have a lattice reduct but belong to a finitely generated variety seems to indicate that the right approach to this problem might be found within the framework of natural duality theory [15] rather than Priestley-style duality.

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