Neutrosophic Logics: Prospects and Problems

Umberto Rivieccio

University of Genoa
Department of Philosophy
Via Balbi 4 - Genoa, Italy
email riv@tele2.it

April 13, 2007

Abstract

Neutrosophy has been introduced some years ago by Florentin Smarandache as a new branch of philosophy dealing with "the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra". A variety of new theories has been developed on the basic principles of neutrosophy: among them is neutrosophic logics, a family of many-valued systems that can be regarded as a generalization of fuzzy logics. In this paper we present a critical introduction to neutrosophic logics, focusing on the problem of defining suitable neutrosophic propositional connectives and discussing the relationship between neutrosophic logics and other well-known frameworks for reasoning with uncertainty and vagueness, such as (intuitionistic and interval-valued) fuzzy systems and Belnap's logic.

Keywords: neutrosophic logics, neutrosophy, many-valued logics, fuzzy logics, intuitionistic fuzzy logic, Belnap logic.

1 Introduction

Neutrosophy has been proposed by Smarandache [16] as a new branch of philosophy, with ancient roots, dealing with "the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra". The fundamental thesis of neutrosophy is that every idea has not only a certain degree of truth, as is generally assumed in many-valued logic contexts, but also a falsity degree and an indeterminacy degree that have to be considered independently from each other. Smarandache seems to understand such "indeterminacy" both in a subjective and an objective sense, i.e. as uncertainty as well as imprecision, vagueness, error, doubtfulness etc.

Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts, such as neutrosophic set theory, neutrosophic probability, neutrosophic statistics and neutrosophic logic. The neutrosophic framework has already found practical applications in a variety of different fields, such as relational database systems, semantic web services [17], financial data set detection [11] and new economies growth and decline analysis [12].

It is clear that all of these proposals, promising as they are, still need to be refined from a formal point of view. In this paper we will introduce and discuss some basic features of neutrosophic logics, which is a family of many-valued systems that can be regarded as a generalization of fuzzy logics; we will try to point out its many appealing aspects as well as the most controversial ones.

2 From fuzzy to neutrosophic values

Since the introduction of fuzzy logic many systems have been developed in order to deal with approximate and uncertain reasoning: among the latest and most general proposals is neutrosophic logic, introduced by Smarandache [16] as a generalization of fuzzy logic and several related systems. We shall now briefly review some of these systems in order to gradually introduce the basic notions of neutrosophic logic.

In fuzzy logics the two-point set of classical truth values $\{0, 1\}$ is replaced by the real unit interval $[0, 1]$: each real value in $[0, 1]$ is intended to represent a different degree of truth, ranging from 0, corresponding to *false* in classical logic, to 1, corresponding to *true*. The standard logical connectives are defined as functions on $[0, 1]$, such as $x \wedge y = min\,(x, y)$, $x \vee y = max\,(x, y)$ and so on.

Given a sentence p whose truth degree is $v\,(p) = t \in [0, 1]$, in fuzzy logic it is implicitly assumed that it also has a falsity degree given by $1 - t$. This need not hold in general in *intuitionistic fuzzy logic*, a generalization of fuzzy logic introduced by Atanassov [2]. The falsity degree of each sentence is now explicitly represented by a second real value $f \in [0, 1]$ so that the intuitionistic fuzzy value of a sentence p is an ordered pair $v\,(p) = (t, f)$ with $t + f \leq 1$. The main novelty of Atanassov's approach is that since one may have $t + f < 1$, a certain amount indeterminacy or incomplete information is allowed. The main novelty of neutrosophic logic, as we shall see, is that we do not even assume that the incompleteness or "indeterminacy degree" is always given by $1 - (t + f)$.

One can also consider the possibility that $t+f>1$: so inconsistent beliefs are also allowed, that is a sentence may be regarded as *both* true and false at the same time. In this way we obtain a family of paraconsistent logics which have been investigated by Priest (see for instance [13], [14]) and by Ginsberg [9], Fitting [7] and others within the framework of algebraic structures known as *bilattices*. The best known among these systems is Belnap's four-valued logic [4], which is based on the following set of truth values: $F=\{0,1\}\times\{0,1\}=\{(0,0),(0,1),(1,0),(1,1)\}$. We can interpret Belnap's values in terms of the classical ones as follows: $(1,0)$ corresponds to *true*, $(0,1)$ to *false*, $(0,0)$ to *unknown* (i. e. not known to be either true or false) and $(1,1)$ to *contradictory* (i. e. known to be both true and false). Kleene's three-valued logic is also related to these systems: in fact it can be regarded as a special case of Belnap's logic where the set of truth values is $K=\{(0,0),(0,1),(1,0)\}\subset F$, with $(0,0)$ meaning *undefined* as in the theory of partial functions.

Another important issue when reasoning with uncertainty is whether to employ "crisp" truth values or intervals. In fact, many authors contend that in most cases we may be unable to determine the exact truth value of a sentence, perhaps knowing only that it belongs to some interval $[t_1,t_2]$ with $0\le t_1\le t_2\le 1$. This is the main idea behind the so called *interval-valued fuzzy sets* (see for instance [3]) and may be regarded as a further stage in the process of fuzzification.

Combining interval-values with paraconsistency, we may consider not just one interval but an ordered pair of intervals for each sentence, representing respectively its truth and falsity degree: in this way we obtain a simplified form of *interval neutrosophic logic* [17].

We are now ready to give Smarandache's original definition of the neutrosophic set of truth values.

Let N be a set defined as follows: $N=\{(T,I,F):T,I,F\subseteq[0,1]\}$. A *neutrosophic valuation* is a mapping from the set of propositional formulas to N, that is for each sentence p we have $v(p)=(T,I,F)$. So to each sentence is assigned an ordered triple representing its truth degree, indeterminacy degree and falsity degree. Intuitively, the set $I\subseteq[0,1]$ may represent not only indeterminacy but also vagueness, uncertainty, imprecision, error etc. Note also that T,I,F, called the *neutrosophic components*, are subsets of $[0,1]$ and not necessarily intervals, so that we may be able to handle information coming from different, possibly conflicting sources.

For instance, suppose we consult two experts, the first saying that the truth degree of a sentence p is $0\le t\le 0.3$, the second saying it is $0.7\le t\le 0.9$: we may represent this as $v(p)=(\{[0,0.3]\cup[0.7,0.9]\},I,F)$.

As another example, consider a voting process where 5 voters out of 10 say "yes" to some proposal p, 3 say "no" and 2 are undecided. We may represent this as $v(p)=(0.5,0.2,0.3)$. Or maybe we are even unable to determine the exact number of the votes, knowing only that the "yes" are between 5 and 7, the "no" between 1 and 3 and the undecided between 0 and 4, thus having $v(p) = ([0.5, 0.7], [0, 0.4], [0.1, 0.3])$.

We mention a further generalization of the neutrosophic set of truth values that may be used to deal with modal contexts, though it has never been employed in applications so far. We need to extend N in order to represent absolute or necessary truth (as well as absolute falsity and indeterminacy), that is truth (falsity, indeterminacy) in every possible world as distinguished from truth in at least one world. To do this we let T, I, F be subsets of the non-standard unit interval $]^{-}0, 1^{+}[$ as defined in non-standard analysis [15]. Recall that $1^{+} = 1 + \epsilon$ and ${}^{-}0 = 0 - \epsilon$, where $\epsilon > 0$ is an infinitesimal, that is $|\epsilon| < \frac{1}{n}$ for all positive integers n. In this way we may represent necessary truth, that is truth in all possible worlds, with the neutrosophic truth value $(1^{+}, {}^{-}0, {}^{-}0)$, while $(1, 0, 0)$ may stand for possible truth, that is truth in at least one world.

It is easy to see that the neutrosophic set of truth values can be regarded as a generalization of all the previous ones: for instance if we set $v(p) = (t, 0, 1-t)$ for every sentence p we obtain the set of truth values corresponding to fuzzy logic, if we set $v(p) = (t, 0, f)$ with $t + f \leq 1$ we get intuitionistic fuzzy logic and so on. We may then define propositional operators that generalize the standard connectives of fuzzy logic, intuitionistic fuzzy logic etc.

3 Neutrosophic connectives

We shall now consider some possible definitions for the basic propositional connectives of neutrosophic logic; we will concentrate on the simplest case, that is when the neutrosophic components are real values instead of intervals or subsets of the unit interval.

Negation. Three kinds of negation have been proposed for neutrosophic logic so far. Given a sentence p and a neutrosophic valuation v such that $v(p) = (t, i, f) \in N$, the truth value of $\neg p$ may be defined as:

(N1) $v(\neg p) = (1-t, 1-i, 1-f)$ (Smarandache [16])

(N2) $v(\neg p) = (f, i, t)$ (Ashbacher [1])

(N3) $v(\neg p) = (f, 1-i, t)$ (Smarandache *et al.* [17])

(N1), introduced by Smarandache [16], is a rather straightforward generalization of the most widely used negation in fuzzy logic. However, from the standpoint of neutrosophic logic, it has the serious drawback that the truth and falsity degree are not related to each other via the negation operator. Consider for example a proposition p whose value is $v(p) = (0,1,0)$: intuitively we may say that we do not have any evidence for the truth or falsity of p, and we have a high degree of indeterminacy. If we adopt (N1) we get $v(\neg p) = (1,0,1)$, that is a sentence known with the greatest precision to be both true and false. Therefore the negation of a sentence whose value is completely unknown becomes a paradoxical statement: this result seems quite unintuitive.

We may avoid this difficulty adopting (N2) or (N3) instead, for then we get respectively $\neg(0,1,0) = (0,1,0)$ and $\neg(0,1,0) = (0,0,0)$. It is not difficult to see that (N3) also leads to some unintuitive result: in fact, if $v(p) = (1,0,0)$ means that we know with the greatest precision that p is true, then we should also know with the greatest precision that $\neg p$ is false. But we have $\neg(1,0,0) = (0,1,1)$, that is we correctly conclude that $\neg p$ is false but we also get an unjustified amount of indeterminacy.

We may also consider our voting example again. We have 10 voters, 5 of them saying "yes" to some proposal p, 3 saying "no" and 2 being undecided: so we write $v(p) = (0.5, 0.2, 0.3)$. What about $\neg p$? Intuitively we may argue that those who said "yes" to p would say "no" to $\neg p$, and conversely those who said "no" to p would now say "yes to $\neg p$, while those who were undecided about p would still be undecided about $\neg p$. Therefore we should have $v(\neg p) = (0.3, 0.2, 0.5)$: but if we apply (N3) to determine the value of $\neg p$ we get $v(\neg p) = (0.3, 0.8, 0.5)$, as if we had now 16 voters instead of our original 10.

We may conclude that, from an intuitive point of view, (N2) seems to be the best definition for a negation operator in neutrosophic logic. As Ashbacher [1] points out, the intuition behind this choice is simply that the amount of indeterminacy associated with a sentence should remain unchanged when we apply the negation operator.

Conjunction and disjunction. Given two sentences p_1, p_2 and a neutrosophic valuation v such that $v(p_1) = (t_1, i_1, f_1) \in N$, and $v(p_2) = (t_2, i_2, f_2) \in N$, the truth value of the conjunction $p_1 \wedge p_2$ may be defined as:

(C1) $v(p_1 \wedge p_2) = (t_1 \cdot t_2, i_1 \cdot i_2, f_1 \cdot f_2)$ [16]

(C2) $v(p_1 \wedge p_2) = (min(t_1, t_2), min(i_1, i_2), max(f_1, f_2))$ [1]

(C3) $v(p_1 \wedge p_2) = (min(t_1, t_2), max(i_1, i_2), max(f_1, f_2))$ [1]

Each one of these functions enjoys some basic properties of classical conjunction, i.e. is associative, commutative and admits a unit element, that is for all $x, y, z \in N$:

1. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (associativity)

2. $x \wedge y = y \wedge x$ (commutativity)

3. there is $e \in N$ such that $e \wedge x = x \wedge e = x$ (identity)

(C1) is a generalization of the conjunction used in *product logic* [10], where $x \cdot y$ denotes the usual product between real numbers, but it apparently needs some refinement before being applied to the neutrosophic framework. For instance, consider two sentences p and q whose truth values are $v(p) = (0, 0, 1)$ and $v(q) = (0, 0, 0)$. We may interpret this saying that p is *false* while q is *unknown*. Now adopting (C1) we have $v(p \wedge q) = (0 \cdot 0 = 0, 0 \cdot 0 = 0, 1 \cdot 0 = 0)$, so that the conjunction $p \wedge q$ has a strictly lower falsity degree (that is, a higher truth degree) than p. This is quite unusual and also in contrast with the behaviour of conjunction in classical logic.

(C2) and (C3) are both generalizations of the conjunction used in *Gödel logic* [6], and unlike (C1) are also idempotent, that is $x \wedge x = x$ for all $x \in N$. We may note that if we adopt (C2) then we have, for instance, $(0, 1, 0) \wedge (1, 0, 0) = (0, 0, 0)$: that is $(1, 0, 0)$ is not an identity. If we interpret $(1, 0, 0)$ as *true* this may be an unwanted result because in fuzzy logic it is usually required that 1 (corresponding to *true* in classical logic) be the unit element of conjunction.

As in classical logic, $\neg$ and $\wedge$ may be used as a basis to define disjunction through De Morgan's laws. If we combine negations (N2) and (N3) with conjunctions (C2) and (C3) we obtain the following disjunction connectives:

(D1) $v(p_1 \vee p_2) = (max(t_1, t_2), max(i_1, i_2), min(f_1, f_2))$

(D2) $v(p_1 \vee p_2) = (max(t_1, t_2), min(i_1, i_2), min(f_1, f_2))$

Both (D1) and (D2) are associative, commutative, idempotent and admit unit elements which are respectively $(0, 0, 1)$ and $(0, 1, 1)$. (N2), (C3) and (D1) have been used by Ashbacher [1] as the basic connectives for *paraconsistent neutrosophic logic*, whose underlying set of truth values is $N = \left\{(t, i, f) \in [0, 1]^3\right\}$. (N2), (C2) and (D2) have been employed by the same author for *intuitionistic neutrosophic logic*, where $N = \left\{(t, i, f) \in [0, 1]^3\right\}$ with $t + i + f \leq 1$.

If we adopt (N3), (C3) and (D2) we obtain a fragment of the so called *interval neutrosophic logic* [17], but in order to obtain the whole system we need to introduce also an implication connective.

Implication. Given two sentences p_1 and p_2 and a neutrosophic valuation v such that $v(p_1) = (t_1, i_1, f_1) \in N$ and $v(p_2) = (t_2, i_2, f_2) \in N$, we may define $p_1 \to p_2$ in the following ways:

(I1) $v(p_1 \to p_2) = v(\neg p_1 \vee p_2)$

(I2) $v(p_1 \to p_2) = (min(1, 1 - t_1 + t_2), max(0, i_2 - i_1), max(0, f_2 - f_1))$

(I1) is just the standard definition of classical logic, usually called an *S-implication* in the fuzzy logics literature. *S-implications* are not widely employed in fuzzy systems because they are too weak, in the sense that even basic tautologies such as $p \to p$ do not hold.

Consider for instance Ashbacher's intuitionistic neutrosophic logic, whose basic connectives are (N2), (C2), (D2) and (I1). We can easily see that this system has no tautologies, that is there is no sentence p such that $v(p) = (1, 0, 0)$ for every neutrosophic valuation v.

(I2) can be regarded as a generalization of the implication connective used in *Łukasiewicz logic* [5], and unlike (I1) admits some tautologies like $p \to p$. If we add (I2) to (N3), (C3) and (D2) we obtain interval neutrosophic logic, which we shall describe in further detail in the following section.

4 Neutrosophic systems

In order to be able to compare the neutrosophic truth values, we need to define an order relation on the elements of $N = \left\{(t, i, f) \in [0, 1]^3\right\}$. This may not be a trivial matter since we have to consider each one of the neutrosophic components.

For instance, suppose we want an order relation $\leq_N$ reflecting the degree of truth associated with each element of N. An obvious choice would be to require that $(t_1, i_1, f_1) \leq_N (t_2, i_2, f_2)$ iff $t_1 \leq t_2$ and $f_1 \geq f_2$, but then we must deal with the indeterminacy component too. So we may consider two possible definitions:

1. $(t_1, i_1, f_1) \leq_{N1} (t_2, i_2, f_2)$ iff $t_1 \leq t_2$, $f_1 \geq f_2$ and $i_1 \leq i_2$
2. $(t_1, i_1, f_1) \leq_{N2} (t_2, i_2, f_2)$ iff $t_1 \leq t_2$, $f_1 \geq f_2$ and $i_1 \geq i_2$

It can be easily verified that $\leq_{N1}$ and $\leq_{N2}$ are well-defined partial order relations on N, but neither of them seems to have a clear intuitive interpretation. Consider for instance $\leq_{N2}$, which has been employed by Smarandache *et al.* [17] for interval neutrosophic logic. The least and greatest elements with respect to $\leq_{N2}$ are $(0, 1, 1)$ and $(1, 0, 0)$, which we may interpret as *false* and *true*. One should expect that $\neg$ *false* = *true*, i.e. $\neg(0, 1, 1) = (1, 0, 0)$, but if we want this we are forced to adopt negation (N1) or (N3), which we have already criticized in the previous section.

Ashbacher [1] defined several alternative orderings on N, but they do not help much in this context because they are even harder to interpret in an intuitive way. Perhaps a possible solution would be to simultaneously apply more than one order relation on N, as has been done introducing the truth and the knowledge ordering on bilattices [9]. Then the next problem would be to define connectives that provide a suitable relation between the two (or more) orderings.

In order to define a consequence relation in neutrosophic logic we have to choose a set $D \subset N$ of truth-like values, which are usually called the *designated elements* of N. Then we have that p implies q ($p \models q$) iff for every neutrosophic valuation v, $v(p) \in D$ implies $v(q) \in D$. A sentence p is a tautology ($\models p$) iff $v(p) \in D$ for every neutrosophic valuation v. If no such sentence exists, we will say that the system is purely *inferential*; from a syntactical point of view, this means that there can be no axioms but only rules of inference.

The simplest choice is to set $D = \{x\}$ for some $x \in N$, e. g. $D = \{(1,0,0)\}$, but if we have defined a suitable order relation $\leq_N$ on N we may also set $D = \{x : y \leq_N x\}$ for some $y \in N$.

We are now going to examine two particularly interesting examples of neutrosophic systems due respectively to Ashbacher [1] and Smarandache *et al.* [17].

Paraconsistent neutrosophic logic (PNL). The underlying set of truth values is $N = \left\{(t,i,f) \in [0,1]^3\right\}$. The basic connectives are negation (N2) and conjunction (C3). Disjunction (D1) is defined via De Morgan's laws as $p \vee q = \neg(\neg p \wedge \neg q)$, while implication (I1) is defined as $p \to q = \neg p \vee q$. The only designated element is $(1,0,0)$, so that we have $p \models q$ iff for every neutrosophic valuation v, $v(p) = (1,0,0)$ implies $v(q) = (1,0,0)$.

It is not difficult to see that PNL is purely inferential, i.e. has no tautologies. For instance, if $v(p) = v(q) = (1,1,1)$ then:

$$v(\neg p) = v(\neg q) = v(p \wedge q) = (1,1,1)$$

So there can be no sentence p such that $v(p) = (1,0,0)$ for every neutrosophic valuation v. Therefore PNL has no axioms. Note also that the structure $\langle N, \wedge, \vee\rangle$ is not a lattice since the absorption laws do not hold.

We have already noted that Belnap's logic [4] is related to the neutrosophic systems: we are now going to show that in fact it can be regarded as a special case of PNL.

The set of Belnap's truth values is $F = \{(0,0),(0,1),(1,0),(1,1)\}$, with $(1,0)$ as the only designated element, and the basic connectives are $\langle \neg, \wedge, \vee\rangle$, defined by the following truth tables:

$\neg$	(0,0)	(0,1)	(1,0)	(1,1)
	(0,0)	(1,0)	(0,1)	(1,1)

$\wedge$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(0,0)	(0,1)
(0,1)	(0,1)	(0,1)	(0,1)	(0,1)
(1,0)	(0,0)	(0,1)	(1,0)	(1,1)
(1,1)	(0,1)	(0,1)	(1,1)	(1,1)

$\vee$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,0)	(1,0)	(1,0)
(0,1)	(0,0)	(0,1)	(1,0)	(1,1)
(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
(1,1)	(1,0)	(1,1)	(1,0)	(1,1)

If we restrict the set N of neutrosophic truth values requiring that $t, f \in \{0, 1\}$ and $i = 0$ for all $(t, i, f) \in N$, we may verify that the connectives of PNL (N2), (C3) and (D1) yield exactly Belnap's truth tables.

It follows that, given two sentences p and q, if p implies q in PNL, then the same also holds in Belnap's logic. Denoting by $\models_{PNL}$ and $\models_B$ respectively the logical consequence relations in PNL and in Belnap's logic, we have that:

$$\text{if} \quad p \models_{PNL} q \quad \text{then} \quad p \models_B q \tag{1}$$

It is possible to show that the converse is also true. In order to prove this result, let us consider the following set of rules:

(R1) $\dfrac{p \wedge q}{p}$ (R2) $\dfrac{p \wedge q}{q}$ (R3) $\dfrac{p \quad q}{p \wedge q}$

(R4) $\dfrac{p}{p \vee q}$ (R5) $\dfrac{p \vee q}{q \vee p}$ (R6) $\dfrac{p \vee p}{p}$

(R7) $\dfrac{p \vee (q \vee r)}{(p \vee q) \vee r}$ (R8) $\dfrac{p \vee (q \wedge r)}{(p \vee q) \wedge (p \vee r)}$ (R9) $\dfrac{(p \vee q) \wedge (p \vee r)}{p \vee (q \wedge r)}$

(R10) $\dfrac{p \vee q}{\neg\neg p \vee q}$ (R11) $\dfrac{\neg (p \vee q) \vee r}{(\neg p \wedge \neg q) \vee r}$ (R12) $\dfrac{\neg (p \wedge q) \vee r}{(\neg p \vee \neg q) \vee r}$

(R13) $\dfrac{\neg\neg p \vee q}{p \vee q}$ (R14) $\dfrac{(\neg p \wedge \neg q) \vee r}{\neg (p \vee q) \vee r}$ (R15) $\dfrac{(\neg p \vee \neg q) \vee r}{\neg (p \wedge q) \vee r}$

We will write $p \vdash q$ iff q can be derived from p using rules (R1)$-$(R15). It is easy to check that this set of rules is sound with respect to the semantics of PNL, that is, for all sentences p and q:

$$\text{if} \quad p \vdash q \quad \text{then} \quad p \models_{PNL} q \tag{2}$$

Moreover, it has been shown (for instance by Font [8]) that rules (R1)−(R15) are sound and complete with respect to the semantics of Belnap's logic, that is, for all sentences p and q:

$$p \vdash q \quad \text{if and only if} \quad p \models_{B} q \tag{3}$$

Combining (1), (2) and (3) we obtain:

$$p \models_{PNL} q \quad \text{if and only if} \quad p \models_{B} q \tag{4}$$

Therefore we see that, from an inferential point of view, PNL turns out to be equivalent to Belnap's logic. Clearly this is an important limitation of PNL, since it means that the neutrosophic set of truth values can be essentially reduced to Belnap's four-point set. In particular, the indeterminacy component turns out to be "useless" within PNL. We may conclude that, if we want to take advantage of all the neutrosophic components, the definitions of the connectives of PNL need to be modified in some way. An obvious suggestion would be to introduce an alternative implication connective such as (I2): in this way we would obtain a system very similar to interval neutrosophic logic, which we are now going describe.

Interval neutrosophic logic (INL). As in PNL, the set of truth values is $N = \left\{(t, i, f) \in [0, 1]^3\right\}$. The basic connectives are negation (N3) and implication (I2). Disjunction (D2) and conjunction (C3) can be defined as:

$$\begin{aligned} p \vee q &= (p \rightarrow q) \rightarrow q \\ p \wedge q &= \neg\left(\neg p \vee \neg q\right) \end{aligned}$$

As in PNL, the only designated element is $(1, 0, 0)$, but unlike PNL, in INL there are some tautologies such as $p \rightarrow p$.

Smarandache *et al.* [17] noted that, from an algebraic point of view, the structure $\langle N, \wedge, \vee\rangle$ is a distributive lattice. We may add that $\langle N, \wedge, \vee\rangle$ is bounded and $\neg\left(x \vee y\right) = \neg x \wedge \neg y$ for all $x, y \in N$, so that the following structure is a De Morgan algebra:

$$\langle N, \wedge, \vee, (1, 0, 0), (0, 1, 1), \neg\rangle$$

The lattice order on N is given by the latter relation we have considered at the beginning of this section, that is we have $(t_1, i_1, f_1) \leq_N (t_2, i_2, f_2)$ iff $t_1 \leq t_2$, $f_1 \geq f_2$ and $i_1 \geq i_2$. We may verify that conjunction (C3) is an *N-norm* in the sense of Smarandache *et al.* [17], that is, for all $x, y, z \in N$:

1. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (associativity)
2. $x \wedge y = y \wedge x$ (commutativity)
3. $(1, 0, 0) \wedge x = x \wedge (1, 0, 0) = x$ (identity)
4. if $x \leq_N y$ then $x \wedge z \leq_N y \wedge z$ (monotonicity)

Conversely, disjunction (D2) is an *N-conorm*, that is, for all $x, y, z \in N$:

1. $(x \vee y) \vee z = x \vee (y \vee z)$ (associativity)
2. $x \vee y = y \vee x$ (commutativity)
3. $(0, 1, 1) \vee x = x \vee (0, 1, 1) = x$ (identity)
4. if $x \leq_N y$ then $x \vee z \leq_N y \vee z$ (monotonicity)

We may list some well known laws of classical logic that do not hold in INL:

1. $p \vee \neg p$ (excluded middle)
2. $\neg (p \wedge \neg p)$ (non contradiction)
3. $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$ (contraposition)
4. $(p \wedge \neg p) \rightarrow q$ (Pseudo Scotus)

We have already noted that (I2) can be regarded as a generalization of the Łukasiewicz implication connective. Indeed, if we restrict the set of neutrosophic truth values requiring that $t + f + i = 1$ and $i = 0$ for all $(t, i, f) \in N$, we may verify that the connectives of INL (N3) and (I2) coincide with the negation and implication of Łukasiewicz infinite-valued logic.

Therefore, denoting by $\models_{INL}$ and $\models_{L}$ respectively the logical consequence relations in INL and in Łukasiewicz logic, we have that for arbitrary sentences p

and q:

$$\text{if} \quad p \models_{INL} q \quad \text{then} \quad p \models_{L} q$$

In this case the converse does not hold. Consider the following axioms, where p, q and r denote arbitrary sentences:

$$
\begin{array}{ll}
\text{(A1)} & p \to (q \to p) \\
\text{(A2)} & (p \to q) \to ((q \to r) \to (p \to r)) \\
\text{(A3)} & ((p \to q) \to q) \to ((q \to p) \to p) \\
\text{(A4)} & (\neg p \to \neg q) \to (q \to p)
\end{array}
$$

It has been shown (see for instance [5]) that (A1)−(A4), together with the *modus ponens* rule, provide a sound and complete axiomatization of Łukasiewicz infinite-valued logic. While (A1)−(A3) are also valid in INL, (A4) is not. However, we can introduce the following rule in INL:

$$\frac{\neg p \to \neg q}{q \to p}$$

Smarandache *et al.* [17] have given a list of axioms and rules for INL assuming $\langle \neg, \wedge, \vee, \to \rangle$ as basic connectives, but no proof of completeness has been published so far.

5 Conclusion and future work

Throughout the previous sections we have tried to show that the neutrosophic formalism is a very general and appealing framework, both needing and deserving further investigation from a logical point of view. First of all, as we have said, it would be very useful to define suitable order relations on the set of neutrosophic truth values; the next step would be to introduce propositional connectives that provide well founded and (if possible) intuitive relations between the two or more different orders.

Another central issue will be to define suitable syntactical consequence relations and to prove completeness with respect to the various neutrosophic semantics that have been considered in the literature, such as the semantics of paraconsistent neutrosophic logic, of interval neutrosophic logic etc.

And finally, the neutrosophic formalism may be further extended in many directions. We have already mentioned the possibility to deal with modal contexts;

temporal neutrosophic logics may also be considered, where the components T, I, F are set-valued vector functions or operators depending on many parameters such as space, time, etc. Another rather straightforward extension would be to let T, I, F be subsets of some partially or linearly ordered lattice L instead of the real unit interval $[0, 1]$, or even to consider different lattices L_1, L_2, L_3 such that $T \subseteq L_1$, $I \subseteq L_2$ and $F \subseteq L_3$.

Atanassov *et al.* [3] said about neutrosophy that "these ideas, once properly formalized, will have a profound impact on our future dealings with imprecision". We share their opinion, and hope that this paper will encourage others to pursue deeper investigations that may lead to such proper formalization.

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