What is Abstract Algebraic Logic?

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Abstract

Our aim is to answer the question of the title, explaining which kind of problems form the nucleus of the branch of algebraic logic that has come to be known as Abstract Algebraic Logic. We give a concise account of the birth and evolution of the field, that led from the discovery of the connections between certain logical systems and certain algebraic structures to a general theory of algebraization of logics. We consider first the so-called algebraizable logics, which may be regarded as a paradigmatic case in Abstract Algebraic Logic; then we shift our attention to some wider classes of logics and to the more general framework in which they are studied. Finally, we review some other topics in Abstract Algebraic Logic that, though equally interesting and promising for future research, could not be treated here in detail.

1 Introduction

Algebraic logic can be described in very general terms as the study of the relations between algebra and logic. One of the main reasons that motivate such a study is the possibility to treat logical problems with algebraic methods. This means that, in order to solve a logical problem, we may try to translate it into algebraic terms, find the solution using algebraic techniques, and then translate back the result into logic. This strategy proved to be fruitful, since algebra is an old and well–established field of mathematics, that offers powerful tools to treat problems that would possibly be much harder to solve using the logical machinery alone.

A famous result obtained in this way is Chang's [6] completeness theorem for the infinite-valued logic introduced by J. Łukasiewicz: in this case the relationship exploited is the one between the real-valued semantics of Łukasiewicz logic and lattice-ordered groups, which are well-known structures studied in abstract algebra.

Let us note that the bridge between algebra and logic can also be exploited in the opposite direction, using logical methods to solve algebraic problems; this strategy, however, has been applied more rarely and only in recent years, thanks to the latest developments of algebraic logic.

We may say that, in general, one of the fundamental aims of algebraic logic is to discover and describe the connection between different logical systems and the corresponding classes of algebras. This study led to the formulation of some general results that are sometimes called *bridge theorems*, stating that a logic has some property if and only if the associated class of algebras has some other property (or the algebraic version of the same property).

Another aim, at a more abstract level, is to explain the nature of these connections. One may wonder, for instance, why the relation of some logical systems with the corresponding class of algebras is very strong, while for some other systems it is much weaker. Another fundamental problem is to specify in which sense we may affirm, as we have done, that a certain class of algebras "corresponds" to a given logical system.

In the last decades, this kind of more abstract questions led the researchers to focus their attention on the process of algebraization itself, i.e. the process by which we associate a certain class of algebras to a particular logical system. This topic forms now the core of a subfield of algebraic logic known as Abstract Algebraic Logic (that we will abbreviate AAL).

2 Some history

The first investigations into the relationship between algebra and logic can be traced back to the very beginning of modern mathematical logic, with the work of the nineteenth–century algebraists such as G. Boole and A. De Morgan. The main contribution of these authors consisted in developing algebraic theories (for instance that of Boolean algebras or that of relation algebras) which admitted, among other interpretations, a logical one. At their time, however, the concept of a formal semantics to be associated with a logical system was yet not considered. Neither was it present in the work of G. Frege and B. Russell, who identified a logical system with a set of axioms and inference rules to be used in derivations; the semantical interpretation, when it was given, was only of an informal kind.

The same approach characterized the work of C. S. Lewis and of A. Heyting, who gave the first axiomatizations of modal and intuitionistic logics. Within this tradition, the main emphasis was given to the concept of *theoremhood* or *logical validity*, in the sense that logical systems were identified with sets of formulas: the set of all theorems of a given logic.

In the same years, however, other authors were investigating a different concept, that of *logical consequence*, and it was this notion that proved to be more fruitful for the development of a general theory of the algebraization of logics.

The two concepts are of course closely related, but that of logical consequence is more general. In fact, as we shall see, logical validity can be reduced to logical consequence, and under some circumstances – but not always, and this is the key point – the latter can also be reduced to the first.

The first general studies of logics as consequence operations were carried out mainly by Polish logicians, following the tratidion initiated by J. Łukasiewicz, A. Lindenbaum and A. Tarski in the 1920s. It was also in these years that algebras and algebra–related structures started to be taken as models of logical systems, that is, as providing formal semantics for logical languages.

The first attempts to develop a general theory of the algebraization of logics were also born within the Polish tradition, starting in the 1970s from H. Rasiowa's [18] investigations into the class of the so-called *implicative logics*. At the end of the 1980s, drawing inspiration from Rasiowa's work, W. Blok and D. Pigozzi [3], [4] introduced the concepts of *algebraizable* and *protoalgebraic logics*. During these years this branch of algebraic logic began to be known as Abstract Algebraic Logic, and it is still a very active field of research, with plenty of established results but also many open questions.

The most comprehensive book currently available on AAL is J. Czelakowski's [9], even if it deals almost exclusively with the class of protoalgebraic logics. Rasiowa's [18], already a classical text in algebraic logic, focuses on an even smaller class of logics and contains the first description of a general method of algebraization. Rasiowa's approach is further generalized and extended to a wider class of logics in the fundamental monograph by Blok and Pigozzi [4]. R. Wójcicki's [22] contains a systematic exposition of several results on logics conceived as consequence relations and the semantics of matrices. Finally, we have to cite two contributions from which we have largely drawn for the present work, that contain an historical-theoretical introduction to the field and many references: Font *et al.* [14] and Jansana [16].

3 Consequence relations

In this paper we will deal only with propositional logics, not only to simplify our exposition but also because propositional logics are best suited for exemplifying the application of the general theory of algebraization of logic. First-order logic has also been investigated with algebraic tools, starting from the work of Tarski himself on classical logic; but for the first-order case a general theory is still not so well-developed as for the propositional logics, and there is not a unique, standard way of algebraization.

As we have said, in algebraic logic is central the concept of logical consequence rather than validity; more precisely, logics are indeed conceived as consequence relations. Let us give a formal definition of this notion.

Let Fm be the set of all formulas of a given propositional language. We say that a relation $\vdash \subseteq P(Fm) \times Fm$ is a *consequence relation* when, for all $\Gamma, \Delta \subseteq Fm$ and all $\varphi, \psi \in Fm$, the following conditions are satisfied:

Identity: $\varphi \vdash \varphi$.
Monotonicity: if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$.
Transitivity: if $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for all $\psi \in \Gamma$, then $\Delta \vdash \varphi$.

Note that we use the symbol $\vdash$ to denote a generic consequence relation, without any assumption on the way it may be defined (syntactical, semantical, etc.).

The properties listed above are in some way intended to capture the essence of the concept of consequence, and they seem to be minimal requirements for calling the resulting system *a logic*. Note, however, that although very general, our definition leaves out some well–known logical systems such as the so–called *nonmonotonic logics*, in which Monotonicity does not hold, and also all those logics where Γ is not a set but a sequence of formulas, as happens for instance with some of the so–called *substructural logics*. Algebraic methods have been applied also to these families of logics, but we shall not consider them in this paper as they fall outside the standard framework of algebraic logic, which is our main interest here.

It is easy to see that the approach based on the notion of consequence generalizes the one based on logical validity, for in our framework we can define the theorems simply as the consequences of the empty set (which by monotonicity will be consequences of any set): that is, φ is a theorem whenever $\emptyset \vdash \varphi$. As we have anticipated, in some cases it is also possible to define the notion of consequence in terms of theoremhood. This happens for instance in classical logic, for we know that for a finite set $\Gamma = \{\psi_1, \ldots \psi_n\}$ it holds that $\Gamma \vdash \varphi$ is equivalent to $\vdash (\psi_1 \wedge \ldots \wedge \psi_n) \to \varphi$. Note, however, that this is a special property of classical logic (depending on the existence and the behaviour of certain connectives such as conjunction and implication), which need not hold in general for any logic.

In order to identify a consequence relation, defined as above, with a logic in the modern sense, we need to meet the requirement that it be formal. This is achieved through the following definition.

We call a consequence relation $\vdash$ *substitution-invariant* or *structural* if it satisfies the following property: if $\sigma : Fm \to Fm$ is any substitution (in the usual logical sense), then

Structurality: $\Gamma \vdash \varphi$ implies $\sigma(\Gamma) \vdash \sigma(\varphi)$.

In algebraic logic, a particular logic is identified with a structural consequence relation defined on some set Fm of formulas built using the appropriate propositional language.

The first three properties listed above were first considered by Tarski [21], while the fourth was introduced by J. Łoś and R. Suszko [17]. In fact, Tarski's original definition included also the following property: for all $\Gamma \cup \{\varphi\} \subseteq Fm$,

Finitarity: if $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$ for some finite set $\Delta \subseteq \Gamma$.

Finitarity is of course a desirable property for a logic, related to the well-known compactness theorem, but nowadays it is not considered part of the basic definition of consequence relation. For instance, some many-valued logics, such as the Łukasiewicz logic semantically defined from the real interval $[0, 1]$, do not enjoy it. Note, however, that any consequence operation defined by means of the usual syntactical calculi (Hilbert-style, natural deduction, sequent, etc.) is finitary. Hence all logics definable by such calculi (for instance classical logic, intuitionistic logic, normal modal logics, etc.) satisfy all the properties we have listed.

Now, given a logic in the sense defined above, let us consider the problem of how to associate to it in a standard way a class of algebras that not only should provide a semantics for it, but may be reagarded as a kind of algebraic counterpart of that logic. In the next section we recall some algebraic notions that will be needed in order to understand the algebraic logic approach to this question.

4 Algebras

An *algebra* consists of a non-empty set A together with some operations defined on A, each one with its arity. An n-ary operation on A is a function from A^n to A, i.e. a map that assigns an element of A to each sequence $\langle a_1, \ldots, a_n \rangle$ of elements of A. A 0-ary operation is just an element of A. Some common examples of algebras are groups, rings and lattices; some algebras related to logic are Boolean algebras (corresponding to classical logic), Heyting algebras (intuitionistic logic) and MV-algebras (related to the Łukasiewicz many-valued logic).

Algebras are denoted by $\mathbf{A} = \langle A, f_1^m, f_2^n, \ldots \rangle$, where A is the underlying set or *universe* of the algebra and f_1^m, f_2^n are the algebraic operations with arity m, n. Usually we do not indicate the arity of the connectives, as the context suffices to determine it; on the other hand, we shall sometimes write $f^{\mathbf{A}}$ when we want to stress that the operation f is defined on the algebra $\mathbf{A}$.

In order to speak about algebras we use an ordinary first-order language with equality (for which we use the symbol $\approx$) and without any other relation symbol; we use a function symbol of the appropriate arity to represent each operation of the algebra. We call this kind of language an *algebraic language.*

Since we are interested in relating algebras to propositional logics, we consider algebraic languages that are built from the propositional ones in the following way. Given a propositional language L, consisting of a set of variables plus a set of propositional connectives, we take any propositional variable $p \in L$ as a variable of the algebraic language, and any propositional connective of L as function symbol of the appropriate arity of the algebraic language. In this way we have that the propositional formulas of L coincide with the terms of the algebraic language. The formulas of the algebraic language are then built from terms in the usual way, for instance the atomic ones have the form $\varphi \approx \psi$, where φ and ψ are terms, i.e. formulas of the propositional language L. This kind of atomic formula of the algebraic language is called an *equation.* Equations, as we are about to see, have a key role in algebraic logic; since they are pairs of formulas, we shall sometimes write $\langle \varphi, \psi \rangle$ to denote the equation $\varphi \approx \psi$. Another subclass of formulas that has special interest is that of *quasiequations.* By a quasiequation we mean a formula of the following form: $(E_1 \ \& \ \ldots \ \& \ E_n) \ \Rightarrow \ E$, where all E_i are equations and the connectives & and $\Rightarrow$ refer to the ordinary first-order conjunction and implication. Note that any equation is also a quasiequation (an implication with empty antecedent).

Given an algebra $\mathbf{A}$, the notions of valuation, satisfaction, etc. can be defined in the usual way. A *valuation* is a map from the set of variables into the universe A. Valuations are extended compositionally to propositional formulas, i.e. to the terms of the algebraic language. We have, for instance, $v(p \wedge q) = v(p) \wedge v(q)$, where the symbol $\wedge$ represents both the propositional connective (on the left of the equality) and the function symbol of the algebraic language (on the right).

We say that an algebra $\mathbf{A}$ is a *model* of (or *satisfies*) an equation $\varphi \approx \psi$ when $v(\varphi) = v(\psi)$ for any valuation v into A. Note that this is equivalent to saying that the universal closure of $\varphi \approx \psi$ is true in $\mathbf{A}$ according to the standard semantics for first-order logic with equality. The notion of satisfaction for quasiequations is defined in a similar way: an algebra $\mathbf{A}$ is a model of a quasiequation Q when the universal closure of Q is true in $\mathbf{A}$.

As we have said, equations and quasiequations are particularly interesting, especially in connection with the algebraic notions of variety and quasivariety. A *variety* (or equational class) is a class of algebras definable by means of equations, i.e. the class of all algebras satisfying some set of equations. Analogously, a *quasivariety* is a class of algebras definable by means of quasiequations. Since equations are quasiequations, each variety is a quasivariety; the converse is not true in general.

The aforementioned classes of Boolean algebras, Heyting algebras, MV-algebras and lattices are some examples of varieties. Proper quasivarieties are rarely found as algebras of logic: an example is the class of BCK algebras, related to combinatory logic.

The set of formulas of any logic is itself the universe of an algebra, which has a key role in algebraic logic, and is called the *formula algebra*. The operations on this algebra, which we denote by $\mathbf{Fm}$, are defined as follows. For any propositional n-ary connective $*$, the operation $* : Fm^n \to Fm$ is the map that sends each tuple of propositional formulas $\langle \varphi_1, \ldots, \varphi_n \rangle$ to the formula $*(\varphi_1, \ldots, \varphi_n)$. For instance, if $\wedge$ is a binary connective of our logic, its associated algebraic operation sends any pair of formulas $\langle \varphi_1, \varphi_2 \rangle$ to the formula $\varphi_1 \wedge \varphi_2$. Note that this algebra does not satisfy any equation except those of the form $\varphi \approx \varphi$.

A fundamental concept in universal algebra is that of congruence. Let $\mathbf{A}$ be an algebra and $\theta \subseteq A \times A$ an equivalence relation on A, i.e. a reflexive, symmetric and transitive relation. We say that θ is a *congruence* of $\mathbf{A}$ if it satisfies the following compatibility property. For any n-ary operation $*$ of the algebra and for all elments $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, if $\langle a_i, b_i \rangle \in \theta$ for all i such that $1 \leq i \leq n$, then $\langle *(a_1, \ldots, a_n), *(b_1, \ldots, b_n) \rangle \in \theta$. The set of all congruences of any algebra $\mathbf{A}$, ordered by set-theoretic inclusion, has always a minimum element (the identity relation $Id_{\mathbf{A}}$) and a maximum (the total relation $A \times A$).

Given a congruence θ of $\mathbf{A}$, we can obtain a new algebra by identifying the elements of A that are related by θ; the compatibility property allows us to define algebraic operations on this reduced set so that it will have the same algebraic language as $\mathbf{A}$. This algebra is called *quotient algebra* and is denoted by $\mathbf{A}/\theta$; its universe is the set A/θ of equivalence classes $[a]$ of the elements a of A modulo θ, while the operations are defined as follows. For any n-ary operation $*^{\mathbf{A}}$ and any $a_1, \ldots, a_n \in A$, we set $*^{\mathbf{A}/\theta}([a_1], \ldots, [a_n]) = [*^{\mathbf{A}}(a_1, \ldots, a_n)]$.

5 Algebraic semantics and algebraizable logics

At the birth of twentieth-century algebraic logic, the link between logic and algebra was established through algebraic completeness theorems relating a specific logical system to a class of algebras. Among the first results that were obtained is the well-known completeness theorem of classical propositional logic with respect to the class of Boolean algebras.

To state this formally, let us denote by $\vdash_C$ the consequence relation corresponding to classical logic. Let also $\mathbb{BA}$ be the class of Boolean algebras, and recall that any $\mathbf{A} \in \mathbb{BA}$ has an element $1^{\mathbf{A}}$ which is the maximum of its natural order. Then we have that, for any $\Gamma \cup \{\varphi\} \subseteq Fm$, the following conditions are equivalent:

(a) $\Gamma \vdash_C \varphi$

(b) for any $\mathbf{A} \in \mathbb{BA}$ and any valuation v on $\mathbf{A}$, if $v(\psi) = 1^{\mathbf{A}}$ for all $\psi \in \Gamma$, then $v(\varphi) = 1^{\mathbf{A}}$.

The same result holds for many other logical systems, for instance for intuitionistic logic with respect to the class of Heyting algebras, for Łukasiewicz logic with respect to MV-algebras, and so on. What is important, for these kind of completeness theorems, is to have a class of algebras $\mathbb{K}$ and, for any $\mathbf{A} \in \mathbb{K}$, a set $D \subseteq A$ of *designated elements*. This kind of semantics can be seen as a generalization of truth tables: the elements of the algebra represent the space of truth values of the logic, among which the designated ones are those that are in some way treated like *true* in classical logic. In the case of classical (intuitionistic, Łukasiewicz) logic, we have $D = \{1^{\mathbf{A}}\}$ for any $\mathbf{A} \in \mathbb{K}$, but in general there may be more than one designated value.

In algebraic logic, the pair $\langle \mathbf{A}, D \rangle$ is called a *logical matrix* (or simply a *matrix*). This is a very important notion, already defined by Tarski in the 1920s and later extensively studied by several logicians of the Polish school (see, for instance [17]).

Matrices are the standard algebraic models of logics: formally, we say that a matrix $\langle \mathbf{A}, D \rangle$ is a *model* of a logic $\mathcal{L}$ when it satisfies the soundness part of the result just stated for classical logic, i.e. when $\Gamma \vdash_{\mathcal{L}} \varphi$ implies that, for any valuation v on $\mathbf{A}$, if $v(\psi) \in D$ for all $\psi \in \Gamma$, then $v(\varphi) \in D$. In this situation we say that the set D is a *filter of the logic* $\mathcal{L}$ or an *$\mathcal{L}$-filter* for short. We denote by $\mathcal{F}i_{\mathcal{L}}(\mathbf{A})$ the set of all $\mathcal{L}$-filters on an algebra $\mathbf{A}$.

We say that a class of matrices $\mathbb{M}$ is a *matrix semantics* for a logic $\mathcal{L}$ when, for any $\Gamma \cup \{\varphi\} \subseteq Fm$, the following conditions are equivalent:

(a) $\Gamma \vdash_{\mathcal{L}} \varphi$

(b) for any $\langle \mathbf{A}, D \rangle \in \mathbb{M}$ and any valuation v on $\mathbf{A}$, if $v(\psi) \in D$ for all $\psi \in \Gamma$, then $v(\varphi) \in D$.

The algebraic completeness result for classical logic can then be restated saying that the class of matrices $\{\langle \mathbf{A}, \{1^{\mathbf{A}}\} \rangle : \mathbf{A} \in \mathbb{BA}\}$ is a matrix semantics for classical logic.

An important point about the set D of designated elements is whether it is equationally definable, i.e. whether there exists a set of equations $E(p)$ in one variable p

such that, on any algebra, D is the set of elements that satisfy $E(p)$. For instance, in the case of classical logic $E(p)$ can be taken to be the single equation $p \approx p \to p$ or, equivalently, the equation $p \approx \top$, where $\top$ is a constant interpreted, on any Boolean algebra, as the maximum element of the natural order.

If we denote by $E(\mathbf{A})$ the set of elements of A that satisfy the equations $E(p)$, we can formally state that a logic $\mathcal{L}$ has an *algebraic semantics* when there is a class $\mathbb{K}$ of algebras and a set $E(p)$ of equations such that the class of matrices $\mathbb{M} = \{\langle \mathbf{A}, E(\mathbf{A})\rangle : \mathbf{A} \in \mathbb{K}\}$ is a matrix semantics for $\mathcal{L}$. In this situation we say that $\mathbb{K}$ is an $E(p)$-algebraic semantics for $\mathcal{L}$.

The previous definition establishes in a precise way a link between logics and classes of algebras, but let us note that this link is not quite strong. In fact, the same class of algebras can be an algebraic semantics for different logics with different sets of equations: for instance, it follows from Glivenko's theorem that the class of Heyting algebras is both an $E(p)$-algebraic semantics for intuitionistic logic and an $E'(p)$-algebraic semantics for classical logic, where $E(p) = \{p \approx p \to p\}$ and $E'(p) = \{\neg\neg p \approx p \to p\}$. On the other hand, different classes of algebras can be $E(p)$-algebraic semantics for the same logic; finally, not every logic has an algebraic semantics.

These observations suggest that we need some additional criteria in order to be justified in considering a class of algebras as the algebraic counterpart of a particular logic. One way to strengthen the link between logics and algebras is through the notion of *equivalent algebraic semantics*. This concept, together with that of algebraizable logic, was introduced by Blok and Pigozzi [4], whose investigations in the 1980s may be considered the starting point of the unification and growth of the field of Abstract Algebraic Logic.

Before giving the formal definitions, we have to establish some more notational conventions. Given a set of equations $E(p)$ and a formula φ, let $E(\varphi)$ be the set of equations obtained by replacing in all the equations in $E(p)$ the variable p by φ. Similarly, if $\Gamma \subseteq Fm$, then $E(\Gamma)$ denotes the union of all $E(\psi)$ for any $\psi \in \Gamma$. If $\Delta(p,q)$ is a set of formulas in two variables and $\varphi \approx \psi$ an equation, $\Delta(\varphi,\psi)$ denotes the set of formulas obtained by replacing p by φ and q by ψ in all the formulas in Δ. If E is a set of equations, $\Delta(E)$ is the union of all $\Delta(\varphi,\psi)$ for any equation $\varphi \approx \psi \in E$. Observe also that any set of equations $E(p,q)$ in two variables defines, on every algebra $\mathbf{A}$, a binary relation given by the set of pairs $\langle a,b\rangle \in A \times A$ that satisfy in A all the equations of $E(p,q)$.

Now, we say that a logic $\mathcal{L}$ is *algebraizable* with respect to a class of algebras $\mathbb{K}$, with equivalence formulas $\Delta(p,q)$ and defining equations $E(p)$, if the following conditions hold:

(i) $\mathbb{K}$ is an $E(p)$-algebraic semantics for $\mathcal{L}$

(ii) for any $\mathbf{A} \in \mathbb{K}$, the relation defined by the set of equations $E(\Delta(p,q))$ is the identity on $\mathbf{A}$.

In this situation, we say that $\mathbb{K}$ is an *equivalent algebraic semantics* for $\mathcal{L}$.

Let us consider again the relationship between classical logic and Boolean algebras. We already know that (i) holds for $E(p) = \{p \approx \top\}$ and in this case we can take $\Delta(p,q) = \{p \leftrightarrow q\}$, so we have that $E(\Delta(p,q)) = \{p \leftrightarrow q \approx \top\}$. It is easy to check that the relation defined by this set is indeed the identity on any Boolean algebra: hence classical logic is algebraizable, with $\mathbb{BA}$ as an equivalent algebraic semantics.

Looking at the definition, the significance of the condition (ii) may not be clear. In order to explain it, it is useful to introduce another fundamental concept in Abstract Algebraic Logic: the Leibniz congruence.

Let $\langle \mathbf{A}, D\rangle$ be a matrix and θ a congruence of $\mathbf{A}$. We say that θ is *compatible* with D when it does not relate elements in D with elements outside of it, i.e. when, for all $a, b \in A$, we have that $a \in D$ and $\langle a, b\rangle \in \theta$ imply $b \in D$. For any matrix $\langle \mathbf{A}, D\rangle$, the set of congruences compatible with D (also called *matrix congruences*) has always a maximum element: we call it the *Leibniz congruence* of the matrix and denote it by $\boldsymbol{\Omega}\langle \mathbf{A}, D\rangle$. This name was proposed by Blok and Pigozzi, and is motivated by the fact that in propositional contexts the Leibniz congruence represents, in some sense, the relation of indiscernibility or interchangeability *salva veritate*. To see this, consider the formula algebra $\mathbf{Fm}$ of any logic. We have that, for any $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$, the following are equivalent:

(a) $\langle \varphi, \psi\rangle \in \boldsymbol{\Omega}\langle \mathbf{Fm}, \Gamma\rangle$

(b) $\chi(\varphi, q_1, \ldots, q_n) \in \Gamma$ iff $\chi(\psi, q_1, \ldots, q_n) \in \Gamma$, for all $\chi(p, q_1, \ldots, q_n) \in Fm$.

Here $p, q_1, \ldots, q_n$ are propositional variables and $q_1, \ldots, q_n$ act as parameters: then (b) means that φ and ψ are not discernible, with respect to Γ, using any formula χ of the language. Note the analogy with the definition of identity as indiscernibility that can be formulated in second-order logic: $x \approx y \leftrightarrow \forall P(Px \leftrightarrow Py)$.

These observations suggest that the Leibniz congruence may be used to idenitify the elements of an algebra $\mathbf{A}$ that behave in the same way with respect to some subset of A. Since from a logical point of view we are mainly interested in the distinction between designated and non-designated elements, in our case the subset will be some set of designated elements. Identifying elements in this way, we are able to individuate, as algebraic models of a logic, algebras that avoid the redundancy of having various elements that, from a logical point of view, behave in the same way. Of course, in order to impose this restriction in formal terms, we must have means to characterize uniformly the Leibniz congruence on any algebra $\mathbf{A}$. To make this observation precise, let us introduce the following definitions.

Let $\langle \mathbf{A}, D\rangle$ be a matrix and $\Delta(p,q) \subseteq Fm$ formulas with variables p and q. We say that $\Delta(p,q)$ *defines* the Leibniz congruence of $\langle \mathbf{A}, D\rangle$ when, for any $a, b \in A$, we have that $\langle a, b\rangle \in \boldsymbol{\Omega}\langle \mathbf{A}, D\rangle$ if and only if $\Delta(a, b) \subseteq D$. We say that a logic $\mathcal{L}$ has a set of *congruence formulas* when there is $\Delta(p,q) \subseteq Fm$ that defines the Leibniz congruence of any model $\langle \mathbf{A}, D\rangle$ of $\mathcal{L}$.

Let us note, in passing, that logics possessing the previous property form an interesting class, called of *equivalential logics*, which has been extensively studied, mainly by Czelakowski (see for instance [8] and [10]).

Now, if we add to this the property that the subsets of designated elements of any algebra $\mathbf{A} \in \mathbb{K}$ be uniformly definable, we have exactly what (ii) is saying: that in the class $\mathbb{K}$ the designated elements are characterized by the equations $E(p)$, that the Leibniz congruence is defined by the formulas $\Delta(p,q)$ and that it coincides indeed with the identity. In this case, i.e. when $\mathbf{\Omega}\langle \mathbf{A}, D\rangle = Id_{\mathbf{A}}$, we say that the matrix $\langle \mathbf{A}, D\rangle$ is *reduced*.

It is interesting, not only for historical reasons, to note that the process described above was obtained through successive generalizations (mainly due to Rasiowa [18] and then to Blok and Pigozzi [4]) of the strategy used by Lindenbaum and Tarski to prove their algebraic completeness result for classical logic. The essential idea of the proof is the following. Reasoning by contradiction, we assume that $\Gamma \nvdash \varphi$ and look for an algebraic counterexample, i.e. an algebra of the appropriate class $\mathbf{A}$ (in the case of classical logic, a Boolean algebra) together with a set $D \subseteq A$ and a valuation v such that $v(\psi) \in D$ for all $\psi \in \Gamma$ but $v(\varphi) \notin D$. The desired algebra is obtained from the formula algebra $\mathbf{Fm}$ as a quotient given by a congruence which is (in modern terms) exactly the Leibniz congruence $\mathbf{\Omega}\langle \mathbf{Fm}, \Gamma\rangle$: this quotient algebra is called the *Lindenbaum-Tarski algebra* of Γ. It remains only to individuate D and v. The former is given by the equivalence class of Γ (for this we need the compatibility of the congruence with Γ), which in the case of classical logic is indeed Γ itself, the top element of the quotient algebra; while as v we take the canonical projection, i.e. the map that sends each element of $\mathbf{Fm}$ to its equivalence class modulo $\mathbf{\Omega}\langle \mathbf{Fm}, \Gamma\rangle$.

Let us observe that, as happens with algebraic semantics, also an equivalent algebraic semantics need not be unique; for instance, the whole class of Boolean algebras and the single two-element Boolean algebra are both equivalent algebraic semantics for classical logic. However, any algebraizable logic $\mathcal{L}$ has a greatest one, i.e. one that includes all other classes that are equivalent algebraic semantics of $\mathcal{L}$. This class, usually denoted $\mathbf{Alg}^*\mathcal{L}$, can be defined as follows:

$$\mathbf{Alg}^*\mathcal{L} = \{\mathbf{A} : \text{the matrix } \langle \mathbf{A}, D\rangle \text{ is a reduced model of } \mathcal{L}\}.$$

Due to its unicity, some authors call this class *the* equivalent algebraic semantics of $\mathcal{L}$; one sees that this definition is in keeping with the idea to take as algebraic models of a logic algebras that do not have redundant elements in the sense explained above.

It is generally agreed that, for a wide range of logics (including, but not limited to, the algebraizable), the class $\mathbf{Alg}^*\mathcal{L}$ can legitimately be regarded as the algebraic counterpart of a logic $\mathcal{L}$. For many logics this class is a variety, but this need not be true in general; for algebraizable logics, under some finitarity assumptions, one can prove that it is always a quasivariety.

The strength of the link between an algebraizable logic and its greatest equivalent algebraic semantics allows to obtain some general results that relate properties of the logic with properties of its algebraic counterpart. For instance, an algebraizable logic $\mathcal{L}$ is finitely axiomatizable if and only if its class $\mathbf{Alg}^*\mathcal{L}$ has a finite presentation; similarly, it is decidable if and only if its algebraic counterpart is. Moreover, it can be proved (again, under some finitarity assumptions) that an algebraizable logic $\mathcal{L}$ has the so-called Deduction-Detachment Theorem (a generalized version of the classical deduction theorem) if and only if the principal relative congruences of the algebras in $\mathbf{Alg}^*\mathcal{L}$ are equationally definable; another well-known theorem relates the (logical) property of the Craig interpolation with the (algebraic) property of amalgamation. This kind of results, as we said in the introduction, allows to use algebraic tools to solve logical problems; they are especially useful when trying to prove negative results, e.g. that some logic does not have the deduction theorem (for any possible connective).

6 Non–algebraizable logics

The category of algebraizable logics comprises many well-know logical systems, such as classical, intuitionistic, many modal and fuzzy logics; but there are also several logical systems that fall outside of it: for example the local consequence of some modal logics ($K, S4, S5$) and relevance logics (E, T). In fact, one of the central issues in Abstract Algebraic Logic is how to develop general methods for the algebraization of logical systems that may be applied to an increasingly wide range of logical systems. For the class of algebraizable logics, the problem has been essentially solved by the work of Blok and Pigozzi, who provided several criteria to determine if a logic is algebraizable and, if it is, to characterize its equivalent algebraic semantics. In this context one of the main tools, as we have seen, is the study of the Leibniz congruence of logical matrices. This study proved to be fruitful also when applied to the wider class of protoalgebraic logics, even if in this case it is the class of matrices (not just algebras) that reflects many interesting properties of the corresponding logic. Let us give the formal definitions.

A logic $\mathcal{L}$ is *protoalgebraic* if there is a set $\Delta(p, q) \subseteq Fm$ in two variables such that, for all $\varphi, \psi \in Fm$:

(i) $\vdash_{\mathcal{L}} \Delta(\varphi, \varphi)$

(ii) $\varphi, \Delta(\varphi, \psi) \vdash_{\mathcal{L}} \psi$.

These conditions say that the set $\Delta(p, q)$ behaves in some way like a generalized implication connective: (i) corresponds to the identity law ($\varphi \to \varphi$), while (ii) is a Modus Ponens of sorts.

The previous remark can be extended to other classes of logics studied in AAL: most of them can in fact be characterized in terms of possessing some set of connectives satisfying certain properties (for instance, behaving like an implication or a like a biconditional); these are known as syntactic or intrinsic characterizations. An interesting consequence is that some of these properties are preserved by expansions of the language: for instance, if a logic is protoalgebraic, then any expansion of it obtained by adding new connectives to the language will also be protoalgebraic.

The previous definition is very useful in order to prove that a logical system is protoalgebraic; however, it does not convey a clear idea of what is the meaning or the motivation behind it. It may then be useful to note that the property of being protoalgebraic, like that of being equivalential, is strictly related to the issue of the definability of the Leibniz congruence, as illustrated by the following result.

Let $\langle \mathbf{A}, D \rangle$ be a matrix and $\Delta(p, q, \bar{r})$ a set of formulas, where p, q are variables and $\bar{r}$ is a sequence (not necessarily finite) of variables. We say that $\Delta(p, q, \bar{r})$ defines the Leibniz congruence of $\langle \mathbf{A}, D \rangle$ when, for any $a, b \in A$, the following are equivalent:

(a) $\langle a, b \rangle \in \mathbf{\Omega}\langle \mathbf{A}, D \rangle$

(b) $\Delta(a, b, \bar{c}) \subseteq D$ for all $\bar{c} \in A$.

Clearly, this is a generalization of the definability property stated in the previous section for equivalential logics, the only difference being that now we have the variables $\bar{r}$ acting as parameters. Accordingly, we say that $\Delta(p, q, \bar{r})$ is a set of *parametrized congruence formulas* for a logic $\mathcal{L}$ when $\Delta(p, q, \bar{r})$ defines the Leibniz congruence of any model $\langle \mathbf{A}, D \rangle$ of $\mathcal{L}$. The interesting characterization is then the following: a logic is protoalgebraic if and only if it has a set of parametrized congruence formulas.

The modal and relevance systems mentioned above are examples of protoalgebraic logics; by the above characterization, it is clear that the classes of algebraizable and equivalential logics are (properly) included in the protoalgebraic. The latter, as we have anticipated, is indeed the widest class of logical systems for which the theory of logical matrices gives completely satisfactory results, in the sense that many logical properties have counterparts in the class of reduced matrix models of these logics.

An interesting, alternative characterization of protoalgebraic logics can be given in term of the *Leibniz operator*, which is a function determined by the behaviour of the Leibniz congruence. On any algebra $\mathbf{A}$, we can define a map from the power set of A into the set of all congruences of $\mathbf{A}$, given by $D \longmapsto \mathbf{\Omega}\langle \mathbf{A}, D \rangle$ for any $D \subseteq A$. We indicate this function $\mathbf{\Omega}_{\mathbf{A}} : P(A) \to \mathrm{Con}(\mathbf{A})$ with the same symbol as the Leibniz congruence, and we call it *Leibniz operator*.

We have then that a logic $\mathcal{L}$ is protoalgebraic if and only if, on any algebra $\mathbf{A}$, the Leibniz operator is monotone on the filters of the logic, i.e. if D and D' are $\mathcal{L}$-filters and $D \subseteq D'$, then $\mathbf{\Omega}_{\mathbf{A}}(D) \subseteq \mathbf{\Omega}_{\mathbf{A}}(D')$.

In fact, the study of the properties of the Leibniz operator on different logics constitutes the basis of one of the fundamental classifications of logical systems in AAL, called the *Leibniz hierarchy*. Without entering into the details of this classification, let us note that the widest class is that of protoalgebraic logics, to which all the other classes in the hierarchy belong, while the smallest comprises logics that are algebraizable with finite sets of defining equations and equivalence formulas.

Another fundamental classification of logics in AAL, called the *Frege hierarchy*, is based on certain replacement properties that a consequence relation may satisfy. These properties are also important in order to determine the algebraic behaviour of a logic because they are related to the issue of definability of congruences on the formula algebra. The two hierarchies are to a large extent independent of each other; however, one of the current trends of research in AAL is the study of the interplay between the two perspectives and of some classes of logics that arise when crossing the two classifications.

Note that, by item (i) of the definition, any protoalgebraic logic must have theorems; there is one exception, called the *almost inconsistent* logic, where we can take

$\Delta = \emptyset$, but it is not very interesting. It follows that any other logic that lacks theorems is not protoalgebraic, while on the other hand having theorems is not a sufficient condition for being protoalgebraic. These considerations suggest that there are many interesting logical systems that fall outside of the Leibniz hierarchy: some examples include the conjunction-disjunction fragment of classical logic, positive modal logic, some subintuitionistic logics and the Belnap-Dunn four-valued logic.

As we have anticipated, if $\mathcal{L}$ is non-protoalgebraic, then $\mathbf{Alg}^*\mathcal{L}$ is not always a good algebraic counterpart of $\mathcal{L}$. This is so because, on the one hand, we are no longer able to prove interesting results relating logical properties of $\mathcal{L}$ to algebraic properties of $\mathbf{Alg}^*\mathcal{L}$; on the other hand, because sometimes $\mathbf{Alg}^*\mathcal{L}$ is a "odd" class, that does not coincide with some other class of algebras that seems to be more naturally associated with the logic.

A paradigmatic example of this situation is the conjunction-disjunction fragment of classical logic, which seems to be (and has usually been) naturally associated with the variety $\mathbb{D}$ of distributive lattices. Now, denoting by $C_{\wedge\vee}$ this logic, we have that $\mathbf{Alg}^*C_{\wedge\vee} \neq \mathbb{D}$; one may also observe that $\mathbf{Alg}^*C_{\wedge\vee}$ is not even a quasivariety (see [12], where this class is characterized).

There are at least two ways of overcoming the difficulties described above. One, that here we just mention, is to shift the attention from logics conceived as deductive systems (defined semantically or through Hilbert-style calculi) to logics conceived as Gentzen systems. This study, developed in [19] and [20], led to define a notion of algebraizability for Gentzen systems, parallel to the standard algebraizability for sentential logics, that also allows to prove interesting results relating properties of the Gentzen system (for instance, satisfying some kind of deduction theorem) to properties of its algebraic semantics. It turns out that in some interesting cases, in particular those of implicationless logics, a logic that is not algebraizable (or not even protoalgebraic) may have an associated Gentzen system that is algebraizable in this new sense.

Another possible solution is to consider as algebraic models of logics not logical matrices, as we have done so far, but what have been called *generalized matrices*. This strategy leads to a different way of associating a class of algebras to a given logic; we shall see that this new definition can be regarded as a generalization of the previous one and that, in some cases, it allows to obtain better results.

Formally, a *generalized matrix* or *g-matrix* is a pair $\langle \mathbf{A}, \mathcal{C} \rangle$, where $\mathbf{A}$ is an algebra and $\mathcal{C}$ is a closure system on A.

Recall that a closure system is a family of sets $\mathcal{C} \subseteq P(A)$ such that $A \in \mathcal{C}$ and, for any non-empty $\mathcal{B} \subseteq \mathcal{C}$, we have $\bigcap \mathcal{B} \in \mathcal{C}$. Under this perspective, any logic $\mathcal{L}$ can be seen as a particular case of g-matrix of the form $\langle \mathbf{Fm}, \mathcal{T}h\mathcal{L} \rangle$, where $\mathcal{T}h\mathcal{L}$ denotes the set of all theories of $\mathcal{L}$ (i.e. sets of formulas closed under the relation $\vdash_{\mathcal{L}}$ of logical consequence).

A semantics of g-matrices may be developed as a natural generalization of the semantics of matrices sketched before: we say then that a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is a *g-model* of a logic $\mathcal{L}$ when $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{L}}(\mathbf{A})$.

The role of the Leibniz congruence is played in this context by the *Tarski congruence* of a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$, usually denoted by $\widetilde{\boldsymbol{\Omega}}_{\mathbf{A}}\mathcal{C}$, and defined as the greatest congruence compatible with all $D \in \mathcal{C}$. The name after Tarski is motivated by the fact that this notion was introduced when trying to generalize the procedure usually followed in the literature, and particularly by Tarski, to construct the Lindenbaum-Tarski algebra of a sentential logic.

The Tarski congruence can be characterized in terms of the Leibniz congruence, as follows:

$$\widetilde{\mathbf{\Omega}}_{\mathbf{A}}\mathcal{C} = \bigcap_{D \in \mathcal{C}} \mathbf{\Omega}\langle \mathbf{A}, D\rangle.$$

We say that a g-matrix is *reduced* when its Tarski congruence is the identity. As we have done when defining the class $\mathbf{Alg}^*\mathcal{L}$, we may associate to a logic $\mathcal{L}$ another class of algebras, which we denote by $\mathbf{Alg}\mathcal{L}$, defined as the class of algebraic reducts of all reduced g-matrices of $\mathcal{L}$:

$$\mathbf{Alg}\mathcal{L} = \{\mathbf{A} : \text{the g-matrix } \langle \mathbf{A}, \mathcal{C}\rangle \text{ is a reduced g-model of } \mathcal{L}\}.$$

The theory of g-matrices allows to obtain results that can be legitimately considered generalizations of those relative to matrices. As was to be expected, it holds that $\mathbf{Alg}^*\mathcal{L} \subseteq \mathbf{Alg}\mathcal{L}$ for any logic $\mathcal{L}$; moreover, if $\mathcal{L}$ is protoalgebraic, then $\mathbf{Alg}^*\mathcal{L} = \mathbf{Alg}\mathcal{L}$.

For most logics (even for some among the non-protoalgebraic) the two classes indeed coincide, but it is interesting to note that, in the known cases where they do not, it is the class $\mathbf{Alg}\mathcal{L}$ that seems to be the more naturally associated with $\mathcal{L}$. Some examples include the aforementioned conjunction-disjunction fragment of classical propositional logic $C_{\wedge\vee}$, for which we have that $\mathbf{Alg}C_{\wedge\vee}$ is the variety of distributive lattices, and the Belnap-Dunn logic (see [11]).

The new theory does not allow to prove bridge theorems between $\mathcal{L}$ and $\mathbf{Alg}\mathcal{L}$ that may be compared to those obtained for algebraizable logics; however, there are interesting results relating properties of the consequence relation of a logic to properties of a subclass of its g-models, e.g. the *full models* (see [13]).

7 Further topics

The topics we have mentioned in the previous sections constitute the core of the field which has come to be known as Abstract Algebraic Logic, but of course they do not exhaust it. Among the lines of research that we have left out, let us cite some that appear to be particularly interesting and promising.

As we have anticipated in the last section, shifting the attention from sentential logics to Gentzen systems led in some contexts to a significant advance; besides the aforementioned theory of algebraizability of Gentzen systems, let us cite the study of abstract logics (a notion essentially equivalent to that of g-matrices) as models of Gentzen systems, that yielded some interesting correspondence results between the metalogical properties of a Gentzen system and of its models. Algebraic logic tools have been successfully applied to study various possible generalizations of sequent systems, considering for instance m-sided sequents, i.e. sequences of m finite sequences of formulas (see

[15]), of which ordinary sequents are the particular case where $m = 2$, or hypersequents, i.e. finite sequences of arbitrary length of ordinary sequents (see [1]).

In the same spirit, but in the context of ordinary sentential logic, k-dimensional logics have been considered (see for instance [5]). The main idea is drawn from equational logic viewed as a 2-dimensional logic, i.e. as a consequence relation defined on pairs of formulas instead than on single formulas. One can then define suitable notions of translations between different systems, that allow to determine when two consequence relations are deductively equivalent; under this perspective, the algebraizability of a logic $\mathcal{L}$ with respect to a class $\mathbb{K}$ of algebras amounts to saying that the 1-dimensional consequence of $\mathcal{L}$ and the 2-dimensional equational consequence of $\mathbb{K}$ are equivalent.

At an even more abstract level, Blok and Jónsson [2] recently sketched a theory of equivalence between consequence operators based on the structure of the associated closure systems, which is likely to have interesting future developments.

To conclude, let us mention two "traditional" topics that are still object of ongoing research.

On the one hand there is the task of classifying logics, that led to the construction of the hierarchies we have mentioned: its main aim is to isolate classes of logics according to criteria that allow a uniform algebraic treatment, in order to obtain general results that apply to all logics satisfying certain properties. Recent work has been done with the aim to expand or refine the Leibniz hierarchy (see [7]) and further investigations are needed on the logics that fall outside this hierarchy, which are, as we have said, the most difficult to treat algebraically.

On the other hand there is the issue, which has old roots in algebraic logic, of the nature of the algebras of logic, i.e. the problem of how to identify the classes of algebras that are naturally associated with some logical system. The theory of algebraizability of logics provides some more insight into the problem, for it allows to identify those classes that are algebras of logics in the sense of being the equivalent algebraic semantics of an algebraizable logic, as well as to prove that a certain class cannot be the equivalent algebraic semantics of any algebraizable logic.

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