Algebraic modal correspondence: Sahlqvist and beyond

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Abstract

This paper provides a bridge in the gap between the model-theoretic and the algebraic side of modal correspondence theory. We give a new, algebraic proof of the classical Sahlqvist correspondence theorem, as well as a new, algebraic proof of the analogous result for the *atomic inductive formulas*, which form a proper extension of the Sahlqvist class.

Contents

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Introduction

Modal logics are perhaps the best known logics other than classical logic. In their modern form they were introduced in the 1930s, as enriched formal languages in which one can express and reason about *modes of truth*, e.g., the *possible*, *necessary*, *usual* or *past* truth of propositions. Syntactically, the language of modal logic is an expansion of classical propositional logic with new connectives, so as to have formulas such as $\Box\varphi$ or $\Diamond\varphi$, the intended meaning of which respectively is 'φ is necessary/obligatory/always true in the past…' and 'φ is possible/permitted/sometimes true in the past…'. Modal logics are widely applied in fields as diverse as program verification in theoretical computer science [21], natural language semantics in formal philosophy [43], multi-agent systems in AI [14], foundations of arithmetics [1], game theory in economics [27], categorization theory in social and management science [28].

The key to this success is the peculiar but natural way modal logic is interpreted in relational structures, paired with the ubiquity of these structures in science and philosophy. A *relational structure*, also called a *Kripke frame* in honor of Saul Kripke who introduced this interpretation [24], consists of a non-empty set (the domain) and a collection of relations (possibly of different arity). The basic version of this notion is given by structures (W, R) with nonempty domain W and one binary relation R. When regarded as interpretations for modal formulas, relational structures typically represent an environment or system which is subject to *variation* in some respects. The domain W should be thought of as the collection of all the alternative states of affairs allowed by the system. Elements in W can then stand for, e.g. the stages of a process evolving through time, parallel realities, conceivable alternatives over the same scenario due to insufficient knowledge, mis-perception, etc. The relation R should be thought of as specifying, e.g., the allowed transitions in the evolution of the system, the number of alternatives over a situation which are still entertained by an agent, etc. The aspects in which the system or environment under consideration varies is further specified by a *valuation function* V, assigning subsets of the domain to propositional variables, effectively telling us which pieces of atomic information hold at which states in the system. A relational structure with a valuation, or *Kripke model*, (W, R, V) can obviously be seen as a relational structure $(W, R, \{V(p) \mid p \in \mathsf{Prop}\})$ with $V(p)$ being a unary predicate for each atomic proposition p. Thus both relational structures and Kripke models can serve as models for classical first and second-order logic.

It is important to emphasize that the perspective offered on relational structures by modal logic differs from that of classical languages in certain important respects: Firstly, modal logic is intrinsically *local*. A modal formula is interpreted *at a given point* in a structure or Kripke model, and its truth value is unaffected by what happens in the model further than a certain number of relational steps away from this point. Secondly, modal semantics differentiates between *absolute* and *contingent* information. In a Kripke model, the accessibility relations represent the invariant "structure of possibility", while the propositional valuation represents the truth value of pieces of contingent information across these possibilities. Thirdly, modal logic operates on at least two different levels: on *models*, where only one particular distribution of contingent information is taken into account, and on *Kripke frames* (relational structures) where implicitly all contingencies are universally quantified.

Correspondence theory arises as the subfield of the model theory of modal logic aimed at answering the question of how precisely these languages — modal, first-order, second-order, and possibly others — interact and overlap in their shared semantic environment. Modal logic inhabits a sphere somewhere between first and second-order logic. Over models, modal languages constitute well behaved fragments of first-order languages. On the other hand, a modal formula is valid on a frame if it is true under all possible valuations. Validity thus involves quantification over all subsets of the domain — clearly a second order notion — making modal languages expressive fragments of universal monadic second-order logic when interpreted over relational structures. However, many of these second-order conditions are in fact equivalent to simple first-order ones. Further natural questions thus arise: which properties, as expressed in a given classical logic, are expressible by means of modal formulas? Which modal formulas express elementary properties of relational structures? Modal correspondence theory seeks answers to these questions.

Being able to compare modal logic to classical logic brings several benefits to both logics: fragments of first-order logic inherit the good computational behaviour of modal logic, and conversely the theory of first-order logic (e.g. compactness and Skolem-Löwenheim theorems) and many existing automated proof tools may be made applicable to modal languages. Correspondence theory, broadly understood, has become a standard modus operandi in such fields as program verification, where different logical languages are interpreted over the same structures, so that e.g. decidability of one logic can be bounded by the decidability of the other.

The best-known results in correspondence theory were achieved in the 1970s independently by Sahlqvist and van Benthem. These results involve the syntactic characterization of a class of formulas in the basic modal language (the so-called *Sahlqvist formulas*) which are guaranteed to express elementary conditions which are effectively computable from the given modal formula. These results constitute what is commonly referred to as Sahlqvist correspondence theory.

Correspondence theory is a presently active field of research, and has significantly broadened its scope in recent years, being now able to uniformly produce Sahlvist-type results for an array of nonclassical logics which includes, but is not limited to, intuitionistic and lattice-based (modal) logics [11], substructural logics (such as linear, relevance logic, Lambeck calculus and its expansions)[32], and mu-calculus [7].

Key to this extension is the recasting of the original Sahlqvist correspondence problem into the setting of the complex algebras associated with relational models. Here, the effective computation of the elementary property corresponding to any given Sahlqvist formula can be analyzed from an *algebraic* and *order-theoretic* viewpoint.

This analysis makes it possible to explain the syntactic characterization of Sahlqvist formulas in terms of the order-theoretic properties of their associated term functions. In its turn, this order-theoretic explanation provides the guideline to extend the definition of Sahlqvist formulas to the previously mentioned array of logics.

In the present paper, an *ab ovo*, self-contained exposition of Sahlvist correspondence theory is given in the original language of basic modal logic treated by Sahlqvist and van Benthem. The present exposition is closely related to but also very different from the standard textbook treatment (cf. e.g. [2, 4]), and aims at laying the conceptual and technical foundations of the recent developments indicated above. We believe that the present treatment can be useful in making the new developments in correspondence theory accessible to a wider community of logicians.

1 Modal correspondence: a quick introduction and survey

The present section begins by collecting the formal preliminaries to correspondence theory for basic modal logic, up to the standard translation. From there, we start surveying landmark results, including van Benthem and Chagrova's theorems. We then give a bird's-eye view of the most important syntactic classes and algorithmic strategies in the literature. Then we survey the existing literature on algebraic Sahlqvist theory, up to the most recent developments.

1.1 Modal logic

Syntax and semantics. The *basic modal language*, denoted ML, is defined using a set Prop of propositional variables, also called atomic propositions, and a unary modal operator $\Diamond$ ('diamond'). The well-formed *formulas* of this language are given by the rule

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \psi \vee \varphi \mid \Diamond\varphi,$$

where $p \in \mathsf{Prop}$. The connective $\Box$, the dual of $\Diamond$, is defined as $\Box\varphi := \neg\Diamond\neg\varphi$. The boolean connectives $\wedge$, $\rightarrow$, and $\leftrightarrow$ and also the constant $\top$ are defined as usual. We interpret this language on Kripke frames and models: A *Kripke frame* is a structure $\mathcal{F} = (W, R)$ with W a non empty set and R a binary relation on W. Augmenting $\mathcal{F}$ with a valuation function $V : \mathsf{Prop} \rightarrow \mathcal{P}(W)$ we obtain a *Kripke model* $\mathcal{M} = (W, R, V)$.

The *complex algebra* of a frame $\mathcal{F} = (W, R)$ is the boolean algebra with operator (BAO)

$$\mathcal{F}^+ = (\mathcal{P}(W), \cap, \cup, -_W, \varnothing, W, m_R)$$

where $-_W$ denotes set complementation relative to W, and $m_R : \mathcal{P}(W) \rightarrow \mathcal{P}(W)$ is given by

$$m_R(X) := \{w \in W \mid \text{there exists } v \in X \text{ such that } Rwv\}.$$

We also let $l_R(X) := \{w \in W \mid \text{for all } v \text{ if } Rwv \text{ then } v \in X\}$, or equivalently, $l_R(X) := -_W m_R(-_W X)$.

The perspective we develop in Section 3 is based on $(\mathcal{P}(W), \subseteq)$ being a partial order and the operations of $\mathcal{F}^+$ enjoying certain properties w.r.t. this order.

We now define the semantics of ML on models and frames via the following *meaning function*. This formulation will be convenient later on in Section 3.1, when we will develop the discussion on the reduction strategies.

Meaning Function For a formula $\varphi \in \mathrm{ML}$ we write $\varphi = \varphi(p_1, \ldots, p_n)$ to indicate that *at most* the atomic propositions $p_1, \ldots p_n$ occur in φ. Every such φ induces an n-ary operation on $\mathcal{P}(W)$,

$$[\![\varphi]\!] : \mathcal{P}(W)^n \longrightarrow \mathcal{P}(W),$$

inductively given by:

$$\begin{array}{rl} [\![\bot]\!] & \text{is the constant function } \varnothing \\ [\![p]\!] & \text{is the identity map } Id_{\mathcal{P}(W)} \\ [\![\neg\varphi]\!] & \text{is the complementation } W \setminus [\![\varphi]\!] \\ [\![\varphi \vee \psi]\!] & \text{is the union } [\![\varphi]\!] \cup [\![\psi]\!] \\ [\![\Diamond\varphi]\!] & \text{is the semantic diamond } m_R([\![\varphi]\!]). \end{array}$$

It follows that

$$\begin{array}{rl} [\![\top]\!] & \text{is the constant function } W \\ [\![\varphi \wedge \psi]\!] & \text{is the intersection } [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\Box\varphi]\!] & \text{is the semantic box } l_R([\![\varphi]\!]). \end{array}$$

For every formula φ, the n-ary operation $[\![\varphi]\!]$ can be also regarded as a map that takes valuations as arguments and gives subsets of $\mathcal{P}(W)$ as its output. Indeed, for every $\varphi \in \mathrm{ML}$, let

$$[\![\varphi]\!](V) := [\![\varphi]\!](V(p_1), \ldots, V(p_n)). \tag{1}$$

Then $[\![\varphi]\!](V)$ is the *extension* of φ under the valuation V, i.e. the set of the states of $(\mathcal{F}, V)$ at which φ is true. Since this happens for all valuations, we can think of $[\![\varphi]\!]$ as the *meaning* function of φ. We can now define the notion of *truth* of a formula φ at a point w in a model $\mathcal{M} = (W, R, V)$, denoted $\mathcal{M}, w \Vdash \varphi$, by

$$\mathcal{M}, w \Vdash \varphi \quad \text{iff} \quad w \in [\![\varphi]\!](V).$$

Similarly, *validity* at a point in a frame is given by

$$\mathcal{F}, w \Vdash \varphi \quad \text{iff} \quad w \in [\![\varphi]\!](V) \text{ for every valuation } V \text{ on } \mathcal{F}.$$

The *global versions* of truth and validity is obtained by quantifying universally over w in the above clauses. Thus we have $\mathcal{M} \Vdash \varphi$ iff $[\![\varphi]\!](V) = W$, and $\mathcal{F}, w \Vdash \varphi$ iff $[\![\varphi]\!](V) = W$ for every valuation V.

General frames, admissible valuations, and canonicity. Appealing as they are, Kripke frames are not adequate to provide uniform completeness results for *all* modal logics (the first examples of frame-incomplete modal logic were given by Thomason [35]. This issue has been further clarified by Blok [3]).

For uniform completeness, Kripke frames need to be equipped with extra structure. A *general frame* is a triple $\mathcal{G} = (W, R, \mathcal{A})$, such that $\mathcal{G}^\sharp = (W, R)$ is a Kripke frame, and $\mathcal{A}$ is a sub-BAO of the complex algebra $(\mathcal{G}^\sharp)^+$ (cf. page 5). An *admissible valuation* on $\mathcal{G}$ is a map $v : \mathsf{Prop} \longrightarrow \mathcal{A}$. Satisfaction and validity of modal formulas w.r.t. general frames are defined as in the case of Kripke frames, but by restricting to admissible valuations.

In fact, the desired uniform completeness can be given in terms of the following proper subclass of general frames: a general frame $\mathcal{G}$ is *descriptive* if $\mathcal{A}$ forms a base for a Stone topology[1] on W, in which $R[w]$ is a closed set for each $w \in W$.

In the light of the uniform completeness w.r.t. descriptive general frames, to prove that a given modal logic is frame-complete, it is sufficient to show that its axioms are valid on a given descriptive general frame $\mathcal{G}$ if, and only if, they are valid on its underlying Kripke frame $\mathcal{G}^\sharp$. Formulas the validity of which is preserved in this way are called *canonical*.

The standard translation. When interpreted on models, modal logic is essentially a fragment of first-order logic, into which we can effectively and straightforwardly translate it using the so called *standard translation*. In order to introduce it, we need some preliminary definitions.

Let L_0 be the first-order language with $=$ and a binary relation symbol R, over a set of denumerably many individual variables $\mathsf{VAR} = \{x_0, x_1, \ldots\}$. Also, let L_1 be the extension of L_0 with a set of unary predicate symbols $P, Q, P_0, P_1 \ldots$, corresponding to the propositional variables $p, q, p_0, p_1 \ldots$ of Prop. The language L_2 is the extension of L_1 with universal second-order quantification over the unary predicates $P, Q, P_0, P_1 \ldots$

Clearly, Kripke frames are structures for *both* L_0 and L_2. Moreover, (modal) models $\mathcal{M} = (W, R, V)$ can be regarded as structures for L_1, by interpreting the predicate symbols P associated with any given atomic proposition $p \in \mathsf{Prop}$ as the subset $V(p) \subseteq W$.

ML-formulas are translated into L_1 by means of the following *standard translation* ST_x from [2]. Given a first-order variable x and a modal formula

[1] A topology τ on a set W is a *Stone* topology if (W, τ) is compact, and every two distinct points can be separated by some clopen set.

φ, this translation yields a first-order formula $\mathrm{ST}_x(\varphi)$ in which x is the only free variable. $\mathrm{ST}_x(\varphi)$ is given inductively by

$$
\begin{aligned}
\mathrm{ST}_x(p) &= Px, \\
\mathrm{ST}_x(\bot) &= x \neq x, \\
\mathrm{ST}_x(\neg\varphi) &= \neg(ST_x(\varphi)), \\
\mathrm{ST}_x(\varphi \vee \psi) &= ST_x(\varphi) \vee ST_x(\psi), \\
\mathrm{ST}_x(\Diamond\varphi) &= \exists y(xRy \wedge ST_y(\varphi)), \text{ where } y \text{ is any fresh variable.}
\end{aligned}
$$

The *standard second-order translation* of a modal formula φ is the L_2-formula obtained by universal second-order quantification over all predicates corresponding to proposition letters occurring in φ, that is, the formula $\forall P_1 \ldots \forall P_n \mathrm{ST}_x(\varphi)$.

As is well known and easy to check, for every model $(\mathcal{F}, V)$ and state w in it, it holds that $(\mathcal{F}, V), w \Vdash \varphi$ iff $(\mathcal{F}, V) \models \mathrm{ST}_x(\varphi)[x := w]$. Moreover, $\mathcal{F}, w \Vdash \varphi$ iff $\mathcal{F} \models \forall P_1 \ldots \forall P_n \mathrm{ST}_x(\varphi)[x := w]$. The analogous global versions of these results are, respectively, $(\mathcal{F}, V) \Vdash \varphi$ iff $(\mathcal{F}, V) \models \forall x \mathrm{ST}_x$ and $\mathcal{F} \Vdash \varphi$ iff $\mathcal{F} \models \forall P_1 \ldots \forall P_n \forall x \mathrm{ST}_x(\varphi)$.

1.2 Correspondence

As seen in the previous subsection, the correspondence between modal languages and predicate logic depends on where one focusses in the multi-layered hierarchy of relational semantics notions. At the bottom of this hierarchy lies the model. At this level, the question of correspondence, at least when approached from the modal side, is trivial: all modal formulas define first-order conditions on these structures. This can be made more precise: we have the following elegant theorem by van Benthem:

Theorem 1.1 (cf. [40])**.** *Modal logic is exactly the bisimulation invariant fragment of* L_1.

At the top of the hierarchy, the interpretation of modal languages over relational structures via the notion of validity turns them into fragments of monadic second-order logic, and rather expressive fragments at that. Indeed, as Thomason [36] has shown, second-order consequence may be effectively reduced to the modal consequence over relational structures.

On the other hand, as already indicated, some modal formulas actually define first-order conditions on *Kripke frames.* For instance, in the standard second-order translation of any formula φ which contains no propositional variables (called a *constant formula*), the second-order quantifier prefix is empty. Hence, to mention a concrete example, the standard second-order translation $\mathrm{ST}_x(\Box\bot)$ is $\forall y(Rxy \to y \neq y) \equiv \forall y(\neg Rxy)$.

We refer to $\Box\bot$ and $\forall y(\neg Rxy)$ as *local frame correspondents*, since for all Kripke frames $\mathcal{F}$ and states w,

$$\mathcal{F}, w \Vdash \Box\bot \quad \text{iff} \quad \mathcal{F} \models \forall y(\neg Rxy)[x := w].$$

A modal formula φ and a first-order sentence α are *global frame correspondents* if $\mathcal{F} \Vdash \varphi$ iff $\mathcal{F} \models \alpha$ for all Kripke frames $\mathcal{F}$.

But formulas need not be constant to define first-order conditions: indeed, $p \to \Diamond p$ and Rxx are local frame correspondents. A short proof of this fact might be instructive: Let $\mathcal{F} = (W, R)$ and $w \in W$. Suppose Rww, and let V be any valuation such that $(\mathcal{F}, V), w \Vdash p$. Then, since Rww, the state w has a successor satisfying p, and hence $(\mathcal{F}, V), w \Vdash \Diamond p$. Since V was arbitrary, we conclude that $\mathcal{F}, w \Vdash p \to \Diamond p$. Conversely, suppose $\neg Rww$, and let V be *some* valuation such that $V(p) = \{w\}$. Then $(\mathcal{F}, V), w \Vdash p$ but $(\mathcal{F}, V), w \nVdash \Diamond p$, hence $\mathcal{F}, w \nVdash p \to \Diamond p$.
The latter direction is an instance of the so called *minimal valuation argument*, which pivots on the fact that some "first-order definable" minimal element exists in the class of valuations which make the antecedent of the formula true at w. We will take full stock of this observation in Section 3.1.

According to the picture emerging from the facts collected so far, it is the correspondence of modal logic and first-order logic on frames that is open and most interesting. It is here where our efforts are needed in order to try and rescue as much of modal logic as we can from the computational disadvantages of second-order logic. Indeed, there is much that can be salvaged.

In [40], van Benthem provides an elegant model-theoretic characterization of the modal formulas which have global first-order correspondents. The constructions involved in this characterization (viz. ultrapowers) are infinitary. So it would be useful to couple this result with a theorem providing an effective way to check whether a formula is elementary. This would, of course, be much too good to be true, and indeed, our skepticism is confirmed by Chagrova's theorem:

Theorem 1.2 (cf. [5]). *It is algorithmically undecidable whether a given modal formula is elementary.*

An effective characterization is therefore impossible, but if we are willing to be satisfied with approximations, all is not lost. Various large and interesting, syntactically defined classes of (locally) elementary formulas are known.

1.3 Syntactic classes

A large part of the study of correspondence between modal and first-order logic has traditionally consisted of the identification of syntactically specified classes of modal formulas which have local frame correspondents.

Formulas without nesting. These are the modal formulas in which no nesting of modal operators occur. Their elementarity was proved by van Benthem [39].

Sahlqvist formulas. This is the archetypal class of elementary modal formulas, due to Sahlqvist [29]. The definition, which is rather involved, will be given in full in section 2. Over the years, many extensions, variations and analogues of this result have appeared, including alternative proofs (e.g. [30]), generalizations to arbitrary modal signatures (e.g. [13]), variations of the correspondence language (e.g. [25] and [41]), Sahlqvist results for hybrid logics (e.g. [34]).

Apart from being elementary, the Sahlqvist formulas have the added virtue of being *canonical* (i.e., of being valid in the canonical, or Henkin, models of the logics axiomatized by them), and hence, of axiomatizing complete normal modal logics. In other words, logics axiomatized using Sahlqvist formulas are sound and complete with respect to elementary classes of Kripke frames.

Inductive formulas. The inductive formulas are a generalization of the Sahlqvist class, introduced by Goranko and Vakarelov [20]. These will be discussed in section 2.

Modal reduction principles. A *modal reduction principle* is an ML-formula of the form $Q_1Q_2\dots Q_np \to Q_{n+1}Q_{n+2}\dots Q_{n+m}p$ where $0 \leq n, m$ and $Q_i \in \{\Box, \Diamond\}$ for $1 \leq i \leq n+m$. Many well known modal axioms take this form, e.g. $\Box p \to p$ (reflexivity), $\Box p \to \Box\Box p$ (transitivity), $p \to \Box\Diamond p$ (symmetry), $\Diamond\Box p \to \Box\Diamond p$ (the Geach axiom), and $\Box\Diamond p \to \Diamond\Box p$ (the McKinsey axiom). In [38], van Benthem provides a complete classification of the modal reduction principles that define first-order properties on frames. For example, as we have already seen, $\Diamond\Box p \to \Box\Diamond p$ defines such a property and $\Box\Diamond p \to \Diamond\Box p$ does not.

In [38], it is also shown that, when interpreted over transitive frames, *all* modal reduction principles define first-order properties.

Complex formulas. This class was introduced by Vakarelov [37]. Complex formulas can be seen as substitution instances of Sahlqvist formulas obtained through the substitution of certain elementary disjunctions for propositional variables. The resulting formulas may violate the Sahlqvist definition.

1.4 Algorithmic strategies

The standard proof of the elementarity of the Sahlqvist formulas takes the form of an algorithm, known as the Sahlqvist-van Benthem algorithm, which computes first-order correspondents for the members of this class. However, the syntactic definition of the Sahlqvist formulas is taken as primary. Approaches which take the algorithm as primary have gained momentum in recent years.

One approach is to apply algorithms for second-order quantifier elimination to the second-order translations of modal formulas. Two notable examples are the algorithms SCAN [15] and DLS [33]. SCAN is a resolution based algorithm and can successfully compute the first-order frame correspondent of every Sahlqvist formula [19]. The same is true of DLS [6], which, in contrast, is based on Ackermann's lemma.

The algorithm SQEMA [9] was designed specifically for modal logic. It computes first-order frame correspondents for all inductive (and hence Sahlqvist) formulas, among others. It has been extended and developed in a series of papers by the same authors. It also guarantees the canonicity of all formulas on which it succeeds (more details about canonicity will be discussed in the next paragraph).

The algorithm ALBA [11] is the analogue of SQEMA in the setting of distributive modal logic (DML) [17]. ALBA has recently been generalized to non-distributive logics [10] and to mu-calculus on an intuitionistic propositional base [7].

1.5 Sahlqvist theory: the algebraic approach

As mentioned earlier, Sahlqvist formulas have the added benefit of being canonical (cf. page 7), and hence complete w.r.t. the elementary class of frames defined by their correspondents. In the literature, there exist two prominent algebraic approaches to Sahlqvist theory, which are motivated by the study of modal canonicity. They can be respectively traced back to Sambin and Vaccaro [30], and to Jónsson [22] and Ghilardi and Meloni [18]. Recall that proving that a given modal formula is canonical involves proving that its validity transfers from any descriptive general frame to its underlying Kripke frame.

Sambin and Vaccaro [30] gave a simplified proof of the Sahlqvist canonicity theorem using order-topological methods. Their strategy, which is sometimes referred to as *canonicity-via-correspondence*, makes use of the existence of a first-order correspondent for any given Sahlqvist formula *in the first-order language of the underlying frame.*

$$\begin{array}{ccc} \mathcal{G} \Vdash \varphi & & \mathcal{G}^\sharp \Vdash \varphi \\ \Updownarrow & & \Updownarrow \\ \mathcal{G} \models \mathrm{FO}(\varphi) & \Leftrightarrow & \mathcal{G}^\sharp \models \mathrm{FO}(\varphi). \end{array}$$

The crucial observation is that the truth of a first-order sentence in a Kripke frame, seen as a relational model, is independent of the assignments being admissible or arbitrary, as illustrated by the horizontal bi-implication in the diagram above. From this, and the correspondence (represented by the two vertical bi-implications in the diagram) the canonicity result follows.

Goranko and Vakarelov [20] give a proof of canonicity for inductive formulas using a similar strategy to that of Sambin and Vaccaro. Conradie and Palmigiano [11] use duality theory to extend the results in [20] to distributive modal logic (DML); in particular, they prove the canonicity of inductive DML-formulas via correspondence.

The second algebraic approach to Sahlqvist theory heavily relies on the theory of canonical extensions [23], and focuses on canonicity independently of correspondence. In his seminal work, Jónsson [22] proved the canonicity of Sahlqvist identities using the fact that the operations interpreting the logical connectives can be extended from a given BAO to its canonical extension, and then using the order-theoretic properties of these extensions and of their resulting compositions. Independently, Ghilardi and Meloni [18] gave a constructive proof of canonicity in intuitionistic logic using filter and ideal completions.

Following [22], Gehrke, Nagahashi and Venema [17] proved the canonicity of Sahlqvist inequalities in distributive modal logic. Suzuki [31] extends this method and proves canonicity for Sahlqvist formulas in substructural logics.

Recent work [45, 26] extends the techniques in [22, 17] to the inductive formulas in DML. The techniques in [22, 17] do not trivially generalize to the inductive formulas, and the proof makes use of ALBA in a novel way which, interestingly, does not involve correspondence. Besides bringing together the two algebraic approaches to Sahlqvist theory, this work promises new insights into rather advanced open questions, concerning e.g. canonicity via pseudo-correspondence [44], or canonicity in the presence of additional axioms.

2 Sahlqvist formulas and atomic inductive formulas

The aim of the present section is to give precise definitions of the syntactic classes that we are going to treat in Section 3. For the reader's convenience, we will do this in a hierarchical form which is conceptually akin to the treatment in [2], although it is slightly different from it, to better fit our account in the following section. In particular, we also introduce the class of *atomic inductive formulas*, which goes beyond the Sahlqvist class, and is a restriction of the definition of the inductive formulas [20]. To further set the stage for the treatment in Section 3, we will also show that the formulas in each of these classes can be equivalently rewritten in certain normal forms. We confine our presentation to the basic modal language.

2.1 Sahlqvist implications and atomic inductive implications

Closed and Uniform formulas ([2]). The *closed* modal formulas are those that contain no proposition letter. An occurrence of a proposition letter p in a formula φ is a *positive* (*negative*) if it is under the scope of an even (odd) number of negation signs. (To apply this definition correctly one of course has to bear in mind the negation signs introduced by the defined connectives $\rightarrow$ and $\leftrightarrow$.) A formula φ is *positive* in p (*negative* in p) if all occurrences of p in φ are positive (negative).

A proposition letter occurs *uniformly* in a formula if it occurs only positively or only negatively. A modal formula is *uniform* if all the propositional letters it contains occur uniformly. Let UF be the class of uniform formulas.

Very simple Sahlqvist implications. A *very simple Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, negative formulas and proposition letters, using $\vee$, $\wedge$ and $\Diamond$. A *very simple Sahlqvist implication* is an implication $\varphi \rightarrow \psi$ in which ψ is positive and φ is a very simple Sahlqvist antecedent. Let VSSI be the class of very simple Sahlqvist implications.

Sahlqvist implications ([2]). A *boxed atom* is a propositional variable preceded by a (possibly empty) string of boxes, i.e. a formula of the form $\Box^n p$ where $n \in \mathbb{N}$ and $p \in \mathsf{Prop}$. A *Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, negative formulas and boxed atoms, using $\vee$, $\wedge$ and $\Diamond$. A *Sahlqvist implication* is an implication $\varphi \to \psi$ in which ψ is positive and φ is a Sahlqvist antecedent. Let SI be the class of Sahlqvist implications.

Atomic inductive implications. Let $\sharp$ be a symbol not belonging to ML. An *atomic box-form of* $\sharp$ in ML is defined recursively as follows:

1. for every $k \in \mathbb{N}$, $\Box^k \sharp$ is an atomic box-form of $\sharp$;
2. If $B(\sharp)$ is an atomic box-form of $\sharp$, then for any proposition letter p, $\Box(p \to B(\sharp))$ is an atomic box-form of $\sharp$.

Thus, atomic box-forms of $\sharp$ are of the type

$$\Box(p_0 \to \Box(p_1 \to \ldots \Box(p_n \to \Box^k \sharp) \ldots)),$$

where the p's are are not necessarily different.

By substituting a propositional variable $p \in \mathsf{Prop}$ for $\sharp$ in an atomic box-form $B(\sharp)$ we obtain an *atomic box-formula of* p, namely $B(p)$. The last occurrence of the variable p is the *head* of $B(p)$ and every other occurrence of a variable in $B(p)$ is *inessential* there. An *atomic regular antecedent* is a formula built up from $\top$, $\bot$, negative formulas, and atomic box-formulas, using $\vee$, $\wedge$ and $\Diamond$.
The *dependency digraph* of a set of box-formulas $\mathcal{B} = \{B_1(p_1), \ldots, B_n(p_n)\}$ is the directed graph $G_{\mathcal{B}} = \langle V, E \rangle$ where $V = \{p_1, \ldots, p_n\}$ is the set of heads of members of $\mathcal{B}$, and E is a binary relation on V such that $p_i E p_j$ iff p_i occurs as an inessential variable in a box formula in $\mathcal{B}$ with head p_j. A digraph is *acyclic* if it contains no directed cycles or loops. Note that the transitive closure of the edge relation E of an acyclic digraph is a strict partial order, i.e. it is irreflexive and transitive, and consequently also antisymmetric. The dependency digraph of a formula φ is the dependency digraph of the set of box-formulas that occur as subformulas of φ.

An *atomic inductive antecedent* is an atomic regular antecedent with an acyclic dependency digraph. An *atomic regular (resp. inductive) implication* is an implication $\varphi \to \psi$ in which ψ is positive and φ is an atomic regular (resp. inductive) antecedent. Let AII be the class of atomic inductive implications.

Example 2.1. Consider the following formulas:

$$\begin{aligned}\varphi_1 &:= p \wedge \Box(p \to \Box q) \to \Diamond\Box\Box q,\\ \varphi_2 &:= \Diamond\Box p \wedge \Diamond(\Box(p \to q) \vee \Box(p \to \Box\Box r)) \to \Diamond\Box(q \vee \Diamond r)\\ \varphi_3 &:= \Diamond(\Box(p \to \Box\Box q) \vee \Box(q \to \Box p)) \to \Diamond\Box p.\end{aligned}$$

φ_1 is an atomic inductive implication, which is not a Sahlqvist implication. The antecedent is the conjunction of the atomic box-formulas p and $\Box(p \to \Box q)$. The dependency digraph over the set of heads $\{p, q\}$ has only one edge, from p to q, and thus linearly orders the variables.

φ_2 is an atomic inductive implication. Its dependency digraph has three vertices p, q, and r, and arcs from p to q and from p to r.

φ_3 is an atomic regular but not inductive implication. Its dependency digraph contains a 2-cycle on vertices p and q.

2.2 Sahlqvist and atomic inductive formulas

Sahlqvist formulas ([2]). A *Sahlqvist formula* is a formula that is built up from Sahlqvist implications by freely applying boxes, conjunctions and disjunctions[2]. Let SF be the class of Sahlqvist formulas.

Atomic inductive formulas. An *atomic inductive formula* is a formula that is built up from atomic inductive implications by freely applying boxes, conjunctions, and disjunctions. Let AIF be the class of atomic inductive formulas.

The correspondence results for Sahlqvist and atomic inductive formulas can be respectively reduced to the correspondence results for Sahlqvist and atomic inductive *implications*: this is an immediate consequence of the following proposition:

Proposition 2.2. *Let $\Phi \in \{\mathsf{SF}, \mathsf{AIF}\}$. Every $\varphi \in \Phi$ is semantically equivalent to a negated Φ-antecedent, and hence to a Φ-implication*[3].

Proof. Fix a formula $\varphi \in \Phi$, and let φ' be the formula obtained from $\neg\varphi$ by importing the negation over all connectives. Since $\varphi \equiv \varphi' \to \bot$, it is

[2] Actually, in the definition in [2] the application of disjunction is restricted, and is only allowed between formulas that do not share any propositional variables. Proposition 2.2 shows that this restriction is unnecessary in the Boolean case. More about this in Remark 2.6.

[3] By a Φ-*antecedent* we mean a Sahlqvist antecedent if $\Phi = \mathsf{SF}$ or an atomic inductive antecedent if $\Phi = \mathsf{AIF}$; by a Φ-*implication* we mean a Sahlqvist implication if $\Phi = \mathsf{SF}$ or an atomic inductive implication if $\Phi = \mathsf{AIF}$.

enough to show that φ' is a Φ-antecedent, in order to prove the statement. This is done by induction on the construction of φ from Φ-implications. If φ is a Φ-implication $\alpha \to \mathsf{Pos}$, negating and rewriting it as $\alpha \wedge \neg\mathsf{Pos}$ already turns it into a Φ-antecedent. If $\varphi = \Box\psi$, where ψ satisfies the claim, then $\neg\varphi \equiv \Diamond\neg\psi$ hence the claim follows for φ, because Φ-antecedents are closed under diamonds. Likewise, if $\varphi = \psi_1 \wedge \psi_2$, where ψ_1 and ψ_2 satisfy the claim, then $\neg\varphi \equiv \neg\psi_1 \vee \neg\psi_2$ hence the claim follows for φ, because Φ-antecedents are closed under disjunctions. The case of $\varphi = \psi_1 \vee \psi_2$ is completely analogous. □

2.3 Definite implications

In the previous subsection, we saw how the correspondence results for formulas in SF and in AIF can be respectively reduced to the correspondence results for formulas in SI and in AII. In their turn, the latter correspondence results can be respectively reduced to the correspondence results for the subclasses of their *definite implications*. These are defined by forbidding the use of disjunction, except within negative formulas, in the building of antecedents. To be precise:

Definition 2.3.

- A *definite very simple Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, negative formulas and propositional letters, using only $\wedge$ and $\Diamond$.
- A *definite Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, negative formulas and boxed atoms, using only $\wedge$ and $\Diamond$.
- A *definite atomic regular antecedent* is a formula built up from $\top$, $\bot$, negative formulas, and atomic box formulas, using only $\wedge$ and $\Diamond$. A *definite atomic inductive antecedent* is a definite atomic regular antecedent with an acyclic dependency digraph.

Let $\Phi \in \{\mathsf{VSSI}, \mathsf{SI}, \mathsf{AII}\}$. Then $\varphi \to \psi \in \Phi$ is a *definite Φ-implication* if φ is a definite Φ-antecedent.

In the next section, we will be able to confine our attention w.l.o.g. to the definite implications in each class, thanks to Fact 2.4 and to Proposition 2.5 below.

Fact 2.4. *If $\varphi_i \in \mathrm{ML}$ locally corresponds to $\alpha_i(x) \in L_0$ for $1 \le i \le n$, then $\bigwedge_{i=1}^n \varphi_i$ locally corresponds to $\bigwedge_{i=1}^n \alpha_i(x)$.*

Proposition 2.5. *Let $\Phi \in \{\mathsf{VSSI}, \mathsf{SI}, \mathsf{AII}\}$. Every $\varphi \in \Phi$ is equivalent to a conjunction of definite implications in Φ.*

Proof. Note that φ can be equivalently rewritten as a conjunction of definite implications in Φ by exhaustively distributing $\Diamond$ and $\wedge$ over $\vee$ in the antecedent, and then applying the equivalence $A \vee B \to C \equiv (A \to C) \wedge (B \to C)$. $\square$

Remark 2.6. As we mentioned already in Section 1, correspondence theory has been extended to logics with a weaker than classical propositional base. Accordingly, as will be clear from the current account, the correspondence mechanism is independent of the Boolean setting we are in. However, there is one single point in our presentation in which we took advantage of the specific properties of the classical setting, namely Proposition 2.2. Thanks to classical negation, we are able to pack (i.e. prove the semantical equivalence of) any Sahlqvist/atomic inductive formula in (to) *one* Sahlqvist/atomic inductive implication. In settings where classical negation is not available, this cannot be done. Still, Sahlqvist formulas can be defined as in [2], i.e. allowing the application of disjunction only between formulas that do not share any proposition letters, and the correspondence result for this class can still be reduced to the correspondence result for Sahlqvist implications, thanks to Fact 2.4 and the following facts (cf. [2, Lemma 3.53]):

1. If $\varphi \in \mathrm{ML}$ locally corresponds to $\alpha(x) \in L_0$, then for every $k \in \mathbb{N}$, $\Box^k\varphi$ locally corresponds to $\forall y(xR^k y \to \alpha(y))$.
2. If $\varphi_i \in \mathrm{ML}$ locally corresponds to $\alpha_i(x) \in L_0$ for $1 \leq i \leq n$, and for every $1 \leq i, j \leq n$ if $i \neq j$ then φ_i and φ_j do not have proposition letters in common, then $\bigvee_{i=1}^n \varphi_i$ locally corresponds to $\bigvee_{i=1}^n \alpha(x)_i$.

3 Algebraic correspondence

The present section is the heart of the paper. In it, we will proceed incrementally and give the algebraic correspondence argument for the definite formulas of each class defined in Section 2.1. More details on the methodology are given in the next subsection.

3.1 The general reduction strategy

Our starting point is the well-known fact, already mentioned above, that any modal formula φ *locally* corresponds to its standard second-order translation, i.e.,

$$\mathcal{F}, w \Vdash \varphi \quad \text{iff} \quad \mathcal{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi)[x := w]. \tag{2}$$

We are interested in strategies that produce a semantically equivalent first-order condition out of the default local second-order correspondent of φ on the right-hand side of (2).

A large and natural class of formulas for which, by definition, this is possible is introduced by van Benthem [39]:

Definition 3.1. The class of *van Benthem-formulas*[4] consists of those formulas $\varphi \in \mathrm{ML}$ for which $\forall P_1 \ldots \forall P_n ST_x(\varphi)$ is equivalent to $\forall P'_1 \ldots \forall P'_n ST_x(\varphi)$ where the quantifiers $\forall P'_1 \ldots \forall P'_n$ range, not over all subsets of the domain, but only those that are definable by means of L_0-formulas.

The van Benthem-formulas are the designated targets of the reduction strategy in its most general form. To see this, for every frame $\mathcal{F} = (W, R)$, let

$$\mathsf{Val}_{\mathsf{L_0}}(\mathcal{F}) = \{V' : \mathsf{Prop} \to \mathcal{P}(W) \mid V'(p) \text{ is } L_0\text{-definable for every } p \in \mathsf{Prop}\}.$$

This is the set of the *tame* valuations on $\mathcal{F}$. Using the notation introduced in (1), if $\varphi \in \mathrm{ML}$ is a van Benthem formula, then the following chain of equivalences holds for every $\mathcal{F}$ and every w:

$$\begin{array}{rll} \mathcal{F}, w \Vdash \varphi & \text{iff} & w \in [\![\varphi]\!](V) \text{ for every } V \text{ on } \mathcal{F} \\ & \text{iff} & w \in [\![\varphi]\!](V') \text{ for every } V' \in \mathsf{Val}_{\mathsf{L_0}}(\mathcal{F}). \end{array} \tag{3}$$

Theorem 3.2. *Every van Benthem-formula has a local first-order frame correspondent.*

Proof. Let φ be a van Benthem-formula and let Σ be the set of all L_0 substitution instances of $\mathrm{ST}_x(\varphi)$, i.e. the set of all formulas obtained by substituting L_0-formulas $\alpha(y)$ for occurrences $P(y)$ of predicate symbols in $\mathrm{ST}_x(\varphi)$. Clearly, $\forall\overline{P}\mathrm{ST}_x(\varphi) \models \Sigma[x := w]$, where $\overline{P}$ is the vector of all predicate symbols occurring in $\mathrm{ST}_x(\varphi)$. Also, since φ is a van Benthem-formula, $\Sigma \models \forall\overline{P}\mathrm{ST}_x(\varphi)[x := w]$. Then $\Sigma \models \mathrm{ST}_x(\varphi)[x := w]$, and since this is a first-order consequence, we may appeal to the compactness theorem to find some finite subset $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models \mathrm{ST}_x(\varphi)[x := w]$.

We claim that $\Sigma' \models \forall\overline{P}\mathrm{ST}_x(\varphi)[x := w]$. Indeed, let $\mathcal{M}$ be any L_1-model such that $\mathcal{M} \models \Sigma'[x := w]$. Since the predicate symbols in $\overline{P}$ do not occur in Σ', every $\overline{P}$-variant of $\mathcal{M}$ also models Σ', and hence also $\mathrm{ST}_x(\varphi)$. It follows that $\mathcal{M} \models \forall\overline{P}\mathrm{ST}_x(\varphi)[x := w]$. Thus we may take $\bigwedge\Sigma'$ as a local first-order frame correspondent for φ. □

[4]this name first appears in [8].

However, relying on compactness, as it does, Theorem 3.2 is of little use if we want to explicitly calculate the first-order correspondent for a given $\varphi \in$ ML, or devise an algorithm which produces first-order frame correspondents for each member of a given *class* of modal formulas; therefore a more refined strategy is in order, the development of which is the core of correspondence theory.

Each class of modal formulas of Subsection 2.1 is defined so as to guarantee that the second 'iff' (i.e. the non trivial one) of (3) can be proved not just for V' ranging arbitrarily over $\mathsf{Val}_{\mathsf{L}_0}(\mathcal{F})$ but rather ranging over a much more restricted and nicely defined subset of it. Moreover, each of these subsets of tame valuations is defined in such a way as to enable the algorithmic generation of the first-order correspondents of the members of the class of formulas it is paired with. More specifically, as we will see next, the following pairings hold between classes of formulas and subsets of tame valuations:

$$\begin{array}{r|l} \mathsf{UF} & V' : \mathsf{Prop} \to \{\varnothing, W\} \\ \mathsf{VSSI} & V' : \mathsf{Prop} \to \mathcal{P}_{fin}(W) \\ \mathsf{SI} & V' : \mathsf{Prop} \to \{R^n[X] \mid n \in \mathbb{N}, X \in \mathcal{P}_{fin}(W)\}. \end{array}$$

So far, our account has been faithful to the textbook exposition, albeit with slightly different notation. The algebraic treatment which we are about to introduce crucially provides an intermediate step which clarifies the textbook account: each class of modal formulas of Section 2.1 is defined so as to guarantee that, for every formula φ in the given class, the meaning function $[\![\varphi]\!]$ enjoys certain purely order-theoretic properties that make sure that the second crucial 'iff' can be proved for V' ranging in the corresponding subclass of tame valuations (the definition of which, as we already mentioned, underlies the algorithmic generation of the first-order correspondent of φ). We start to see how this works in the next subsection.

3.2 Uniform and Closed formulas

The reduction strategy. Among all the first-order definable valuations V on $\mathcal{F}$, the simplest ones are those which assign W or $\varnothing$ to each propositional variable. Indeed, let V_0 be such a valuation and suppose that the following were equivalent for the modal formula φ:

$$\begin{array}{lll} \mathcal{F}, w \Vdash \varphi & \text{iff} & w \in [\![\varphi]\!](V) \text{ for all } V \text{ on } \mathcal{F} \\ & \text{iff} & w \in [\![\varphi]\!](V_0) \end{array}$$

This would in turn mean that

$$\begin{array}{ll} & \mathcal{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi)[x := w] \\ \text{iff} & \mathcal{F} \models ST_x(\varphi)[x := w, P_1 := V_0(p_1), \ldots, P_n := V_0(p_n)] \end{array}$$

Therefore, we could equivalently transform the formula above into a first-order formula by replacing each occurrennce $P_i z$ with either $z \neq z$ if $V_0(p_i) = \varnothing$, or with $z = z$ if $V_0(p_i) = W$. This is enough to effectively generate the first-order correspondent of φ.

Order theoretic conditions. For which formulas φ would it be possible to implement the reduction strategy outlined above? The answer to this question can be given in purely order-theoretic terms:

Proposition 3.3. *Let $\langle X_i, \leq \rangle$, $i = 1, \ldots, n$, and $\langle Y, \leq \rangle$ be posets. Let each X_i have a maximum, $\top_i$, and a minimum, $\bot_i$. Let $f : X_1 \times \cdots \times X_n \longrightarrow Y$. If f is either order preserving or order reversing in each coordinate, then the minimum of f exists and is $f(c_1, \ldots c_n)$, where, for every i, $c_i = \bot_i$ if f is order preserving in the i-th coordinate, and $c_i = \top_i$ if f is order reversing in the i-th coordinate.*

Corollary 3.4. For every $\varphi \in \mathrm{ML}$, if $[\![\varphi]\!] : \mathcal{P}(W)^n \to \mathcal{P}(W)$ is order preserving or order reversing in each coordinate, then the following are equivalent:

1. $\forall V[w \in [\![\varphi]\!](V)]$.
2. $w \in [\![\varphi]\!](V_0)$, where $V_0(p_i) = W$ if φ is order reversing in p_i and $V_0(p_i) = \varnothing$ if φ is order preserving in p_i.

Proof. ($1 \Rightarrow 2$) Clear. ($2 \Rightarrow 1$) It follows from Proposition 3.3 that $[\![\varphi]\!](V_0)$ is the minimum of $[\![\varphi]\!]$ and hence $[\![\varphi]\!](V_0) \subseteq [\![\varphi]\!](V)$ for every valuation V. $\square$

Syntactic conditions. Now that we have the reduction strategy and sufficient order-theoretic conditions for the strategy to apply, it only remains to verify that these conditions are met by the uniform formulas. And indeed, the following proposition can be easily shown by induction on φ:

Proposition 3.5. *If $\varphi \in \mathrm{ML}$ is a uniform formula, then $[\![\varphi]\!]$ is order preserving (reversing) in those coordinates corresponding to propositional variables in which φ is positive (negative).*

Example 3.6. Let us consider the uniform formula $\Box\Diamond p$. The minimal valuation for p is $V_0(p) = \varnothing$, since the formula is positive and hence order-preserving in p. The standard translation of this formula gives

$$\mathcal{F} \models \forall P\, \forall y(Rxy \to \exists z(Ryz \wedge Pz)[x := w]$$
$$\text{iff} \quad \mathcal{F} \models \forall y(Rxy \to \exists z(Ryz \wedge P^0 z)[x := w]$$

where the predicate $P^0 z$ can be replaced with $z \neq z$ giving a first-order equivalent formula $\forall y(Rxy \to \exists z(Ryz \wedge z \neq z)$ which simplifies to $\forall y(\neg xRy)$.

To sum up: although the uniform formulas and their accompanying valuations are extremely simple, the key features of our account are already present: first, the subclass of tame valuations is identified, using which the desired first-order correspondent can be effectively computed; second, the order theoretic properties are highlighted, which guarantee the crucial preservation of equivalence; third, the syntactic specification of the formulas φ of the given class guarantees that their associated meaning functions $[\![\varphi]\!]$ meet the required order theoretic properties.

Non-uniform formulas and 'minimal valuation' argument. The discussion above also shows that every uniform formula is locally equivalent on frames to some closed formula (which is obtained by replacing every positive variable with $\bot$ and every negative variable with $\top$). This elimination of variables can in fact be applied not only to uniform formulas but also to formulas that are uniform *with respect to some variables*, so as to eliminate those 'uniform' variables separately. Therefore, modulo this elimination, in the following subsections we are going to assume w.l.o.g. that the formulas we consider are non-uniform in each of their variables. Modulo equivalent rewriting, we can assume w.l.o.g. that every such formula is of the form $\varphi \to \psi$, where ψ is positive, and all the variables occurring in ψ also occur in φ. For such formulas, we have:

$$\begin{array}{lll} \mathcal{F}, w \Vdash \varphi \to \psi & \text{iff} & w \in [\![\varphi \to \psi]\!](V) \text{ for all } V \text{ on } \mathcal{F} \\ & \text{iff} & \text{for all } V \text{ on } \mathcal{F}, \text{ if } w \in [\![\varphi]\!](V) \text{ then } w \in [\![\psi]\!](V). \end{array}$$

The textbook heuristics for producing the correspondent of formulas of this form is the 'minimal valuation' method (see [42] subsection 9.4): find the (class of) minimal valuation(s) V^* on $\mathcal{F}$ such that $w \in [\![\varphi]\!](V^*)$ (and plug their description in the standard translation of the consequent). The success of this heuristics rests on two conceptually different requirements: first, that 'minimal valuations' exist; second, provided they exists, that they are tame. The account we will present in the next sections, as described in the discussion at the end of Subsection 3.1, will deal with these two requirements separately and in the reverse order. Namely, first we will single out subclasses of L_0-definable valuations (our target class of 'minimal valuations'), restricting the universal quantification to which guarantees that the first-order correspondents can be effectively computed (essentially by way of the 'plug-in' method alluded to above); second, we will show that certain order theoretic properties of the extension maps $[\![\varphi]\!]$, seen as operations on $\mathcal{F}^+$, guarantee that restricting the universal quantification to the target class preserves equivalence (essentially by guaranteeing the existence of a suitable 'minimal valuation' taken in the target class). Third, we will show that the syntactic conditions on φ guarantee that $[\![\varphi]\!]$ satisfies the required order-theoretic properties.

3.3 Very simple Sahlqvist implications

The reduction strategy. Consider the subclass of the tame valuations which assign finite subsets of some bounded size $m \in \mathbb{N}$ to propositional variables, i.e. valuations $V_1 : \mathsf{Prop} \longrightarrow \mathcal{P}_m(W)$, where

$$\mathcal{P}_m(W) := \{S \subseteq W \mid |S| \leq m\},$$

and suppose the following were equivalent:

1. $\forall V(w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V))$
2. $\forall V_1(w \in [\![\varphi]\!](V_1) \Rightarrow w \in [\![\psi]\!](V_1))$.

This would mean that

$$\begin{array}{ll} & \mathcal{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi \to \psi)[x := w] \\ \text{iff} & \mathcal{F} \models \forall P_1^1 \ldots \forall P_n^1 ST_x(\varphi \to \psi)[x := w], \end{array}$$

where the variables P_i^1 would not range over arbitrary subsets of W, but only over those of size at most m. Provided the equivalence between 1 and 2 above holds, we would effectively obtain the local first-order correspondent of $\varphi \to \psi$ by replacing each $\forall P_i^1$ in the prefix with $\forall z_i^1 \forall z_i^2 \ldots \forall z_i^m$ and each atomic formula of the form $P_i^1 y$ with $y = z_i^1 \vee y = z_i^2 \vee \cdots \vee y = z_i^m$, where all the z's are fresh variables.

Order theoretic conditions. An operation $f : \mathcal{P}(W)^n \to \mathcal{P}(W)$ is a *complete operator* if f preserves arbitrary joins in each coordinate, i.e., for every $i = 1 \ldots n$, every $\mathcal{X} \subseteq \mathcal{P}(W)$, and all $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n \in \mathcal{P}(W)$,

$$\begin{array}{ll} & f(X_1, \ldots, X_{i-1}, \bigcup \mathcal{X}, X_{i+1}, \ldots, X_n) \\ = & \bigcup_{Y \in \mathcal{X}} f(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_n). \end{array} \tag{4}$$

This implies in particular that for all $X_1, \ldots, X_n \in \mathcal{P}(W)$,

$$\text{if } X_i = \varnothing \text{ for some } i \in \{1, \ldots, n\}, \text{ then } f(X_1, \ldots, X_n) = \varnothing. \quad (5)$$

Recall that any complete operator is order preserving in each coordinate.

Composition of complete operators will be important for our account: in order to describe their order-theoretic properties, the following definition will be useful.

Definition 3.7. Let $g : \mathcal{P}(W)^n \to \mathcal{P}(W)$ be a composition of complete operators.

1. The *degree of g in the ith coordinate*, notation δ^i_g, is defined by induction on g:
   (a) If g is itself a complete operator, then $\delta^i_g = 1$ for every coordinate $1 \leq i \leq n$ whose corresponding variable occurs in g, and $\delta^i_g = 0$ otherwise;
   (b) If $g = f(h_1, h_2, \ldots, h_m)$ for some complete operator f and compositions of complete operators, $h_1, \ldots, h_m$, then $\delta^i_g = \delta^i_{h_1} + \cdots + \delta^i_{h_m}$.
2. The *degree of g*, notation δ_g, is $max\{\delta^i_g \mid 1 \leq i \leq n\}$.

Lemma 3.8. *If $g : \mathcal{P}(W)^n \to \mathcal{P}(W)$ is a composition of complete operators, then*

1. *g is order preserving in each coordinate, and*
2. *for all $X_1, \ldots X_n \in \mathcal{P}(W)$, if $X_i = \varnothing$ for some $1 \leq i \leq n$ whose corresponding variable occurs in g, then $g(X_1, \ldots, X_n) = \varnothing$.*

Proof. 1. Every complete operator is order preserving and the composition of order preserving maps is order preserving.
2. By induction on δ_g. □

The composition of *unary* complete operators yields complete operators, but that this is not generally the case for non-unary complete operators:

Example 3.9. Consider the extension map $[\![\varphi]\!]$ for the very simple Sahlqvist antecedent $\varphi(p) = \Diamond p \wedge \Diamond\Diamond p$, defined on the complex algebra of the frame $\mathcal{F} = (W, R)$ with $W = \{w, v, u\}$ and $R = \{(w, v), (v, u)\}$. Then,

$$\begin{aligned}
\varphi(\{v\} \cup \{u\}) &= m_R(\{v,u\}) \cap m_R(m_R(\{v,u\})) \\
&= \{w,v\} \cap \{w\} \\
&= \{w\}. \\
\varphi(\{v\}) \cup \varphi(\{u\}) &= (m_R(\{v\}) \cap m_R(m_R(\{v\}))) \cup m_R(\{u\}) \cap m_R(m_R(\{u\})) \\
&= (\{w\} \cap \varnothing) \cup (\{v\} \cap \{w\}) \\
&= \varnothing \cup \varnothing = \varnothing.
\end{aligned}$$

However, compositions of complete operators do retain a certain semblance of the join-preservation of the operators from which they are built, as the next lemma shows. We will write $Y \subseteq_k X$ or $Y \in \mathcal{P}_k(X)$ to indicate that $Y \subseteq X$ and $|Y| \leq k$, for $k \in \mathbb{N}$.

Lemma 3.10. *If $g : \mathcal{P}(W)^n \to \mathcal{P}(W)$ is a composition of complete operators, and $X_1, \ldots, X_n \in \mathcal{P}(W)$, then*

$$g(X_1, \ldots, X_n) = \bigcup \{g(S_1, \ldots, S_n) \mid S_i \subseteq_{\delta_g^i} X_i,\ 1 \leq i \leq n\}.$$

Proof. By induction on the degree of g. If $\delta_g = 1$, then g is a complete operator f and hence

$$\begin{aligned}
f(X_1, \ldots, X_n) &= f(\bigcup_{x_1 \in X_1} \{x_1\}, \ldots, \bigcup_{x_n \in X_n} \{x_n\}) \\
&= \bigcup \{f(\{x_1\}, \ldots, \{x_n\}) \mid \{x_i\} \subseteq_1 X_i,\ 1 \leq i \leq n\}.
\end{aligned}$$

If $\delta_g > 1$, then g is of the form $f(h_1, \ldots, h_m)$ where f is a complete operator and each h_i is a composition of complete operators. Then

$$\begin{aligned}
g(X_1, \ldots, X_n) &= f(h_1(X_1, \ldots, X_n), \ldots, h_m(X_1, \ldots, X_n)) \\
&= f(\bigcup \{h_i(S_1^i, \ldots, S_n^i) \mid S_j^i \subseteq_{\delta_{h_i}^j} X_j,\ 1 \leq j \leq n\})_{i=1}^m \\
&= \bigcup \{f(h_i(S_1^i, \ldots, S_n^i))_{i=1}^m \mid S_j^i \subseteq_{\delta_{h_i}^j} X_j,\ 1 \leq j \leq n\} \\
&\subseteq \bigcup \{f(h_i(S_1, \ldots, S_n))_{i=1}^m \mid S_j \subseteq_{\delta_{h_1}^j + \cdots + \delta_{h_m}^j} X_j,\ 1 \leq j \leq n\} \\
&= \bigcup \{g(S_1, \ldots, S_n) \mid S_j \subseteq_{\delta_g^j} X_j,\ 1 \leq j \leq n\}.
\end{aligned}$$

Here the second equality holds by the inductive hypothesis, and the third since f is a complete operator. The inclusion holds since the set of which the union is taken in the third line is a subset of the corresponding set in the fourth line. The last equality holds by the assumptions on g and by definition of δ_g.
The converse inclusion follows from g being order preserving (Lemma 3.8).

□

Consider the following conditions on $[\![\varphi]\!]$:

(a) $[\![\varphi(p_1, \ldots, p_n)]\!] = [\![\varphi']\!](p_1, \ldots, p_n, [\![\gamma_1]\!], \ldots, [\![\gamma_\ell]\!])$, where

(b) $[\![\varphi'(p_1, \ldots, p_n, s_1, \ldots, s_l)]\!]$ is a composition of complete operators on $\mathcal{P}(W)$,

(c) $[\![\gamma_1]\!]$ to $[\![\gamma_\ell]\!]$ are order reversing in each coordinate.

For every frame $\mathcal{F}$ and every $m \in \mathbb{N}$, let $\mathsf{Val}_1(\mathcal{F})$ be the set of valuations on $\mathcal{F}$ of type $V_1 : \mathsf{Prop} \to \mathcal{P}_m(W)$.

Proposition 3.11. *Let $\varphi \to \psi \in$ ML be such that $[\![\varphi]\!]$ verifies the conditions (a)-(c) above and $[\![\psi]\!]$ is order-preserving. Let $m = max_{i=1}^n m_i$ where m_i is the degree of $[\![\varphi]\!]$ relative to its ith coordinate. Then the following are equivalent for every frame $\mathcal{F}$:*

1. $(\forall V \in \mathsf{Val}(\mathcal{F}))[w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V)]$

2. $(\forall V_1 \in \mathsf{Val}_1(\mathcal{F}))[w \in [\![\varphi]\!](V_1) \Rightarrow w \in [\![\psi]\!](V_1)]$.

Proof. ($1 \Rightarrow 2$) Clear. ($2 \Rightarrow 1$) Let m_i be the degree of $[\![\varphi]\!]$ in the i-th coordinate, for $1 \leq i \leq n$. Fix V and let $w \in [\![\varphi]\!](V)$. Hence,

$$\varnothing \neq [\![\varphi]\!](V) = [\![\varphi']\!](V(p_1), \ldots V(p_n), [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)).$$

By Lemma 3.8(2), this implies that $V(p_i) \neq \varnothing$ for every $i = 1, \ldots, n$. By Lemma 3.10,

$$[\![\varphi]\!](V) = \bigcup\{[\![\varphi']\!](S_1, \ldots, S_n, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid S_i \subseteq_{m_i} V(p_i),\ 1 \leq i \leq n\}.$$

Hence, $w \in [\![\varphi]\!](V)$ implies that $w \in [\![\varphi']\!](T_1, \ldots, T_n, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V))$ for some $T_i \subseteq_{m_i} V(p_i)$, $1 \leq i \leq n$. Let V_1 be the valuation that maps any $q \in \mathsf{Prop} \setminus \{p_i \mid 1 \leq i \leq n\}$ to $\varnothing$ and such that $V_1(p_i) = T_i$: clearly, $V_1 \in \mathsf{Val}_1(\mathcal{F})$; moreover $w \in [\![\varphi]\!](V_1)$: indeed,

$$\begin{array}{rcl} w & \in & [\![\varphi']\!](T_1, \ldots, T_n, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \\ & \subseteq & [\![\varphi']\!](T_1, \ldots, T_n, [\![\gamma_1]\!](V_1), \ldots, [\![\gamma_\ell]\!](V_1)) \\ & = & [\![\varphi']\!](V_1(p_1), \ldots, V_1(p_n), [\![\gamma_1]\!](V_1), \ldots, [\![\gamma_\ell]\!](V_1)) \\ & = & [\![\varphi]\!](V_1); \end{array}$$

the inclusion in the chain above follows since $V_1(p) \subseteq V(p)$ for every $p \in \mathsf{Prop}$ and the extensions of the γ's are reversing. Hence, by assumption (2), $w \in [\![\psi]\!](V_1)$. Since $[\![\psi]\!]$ is order preserving in every coordinate, and again $V_1(p) \subseteq V(p)$ for every $p \in \mathsf{Prop}$, we get $w \in [\![\psi]\!](V_1) \subseteq [\![\psi]\!](V)$, which concludes the proof. $\square$

Syntactic conditions. It remains to verify that the very simple Sahlqvist implications verify the assumptions of Proposition 3.11. The assumptions on ψ are verified because of Proposition 3.5. As to the assumptions on φ:

Proposition 3.12. *If $\varphi = \varphi(p_1, \ldots, p_n)$ is a very simple Sahlqvist antecedent then it verifies the assumptions (a)-(c) of Proposition 3.11. In particular, the maps $[\![\gamma]\!]$'s are exactly the ones induced by the negative formulas occurring in the construction of φ, the map $[\![\varphi']\!]$ is induced by the compound occurrences of $\wedge$ and $\Diamond$, and for every $1 \leq i \leq n$, the degree of $[\![\varphi']\!]$ in the ith coordinate is the number of positive occurrences of p_i in φ.*

Proof. For the first part, note that the identity map $id : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$, the intersection $\cap : \mathcal{P}(W) \times \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ ($\langle X, Y\rangle \mapsto X \cap Y$) and semantic diamond operations $m_R : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ ($X \mapsto m_R(X)$) are complete operators. The second part is proven by induction on φ. $\square$

Example 3.13. Let us consider the very simple Sahlqvist formula

$$p \wedge \Diamond p \rightarrow \Box p,$$

which locally corresponds to the property of having at most one successor. The variable p occurs twice positively in the antecedent, giving $[\![p \wedge \Diamond p]\!]$ degree 2 in the corresponding coordinate. Hence, according to our reduction strategy, the monadic second order quantification in the second-order translation

$$\forall P[P(x) \wedge \exists y(xRy \wedge P(y)) \rightarrow \forall u(xRu \rightarrow P(u))]$$

can be equivalently restricted to subsets of size at most 2. Doing this yields the equivalent L_0-formula

$$\forall z_1 \forall z_2[(x = z_1 \vee x = z_2) \wedge \exists y(xRy \wedge (y = z_1 \vee y = z_2)) \rightarrow \forall u(xRu \rightarrow (u = z_1 \vee u = z_2))].$$

This can be simplified to

$$\forall z_1 \forall z_2[(x = z_1 \vee x = z_2) \wedge (xRz_1 \vee xRz_2)) \rightarrow \forall u(xRu \rightarrow (u = z_1 \vee u = z_2))],$$

and reasoning a bit further this can be seen to be equivalent to

$$\forall z \forall u(xRu \wedge xRz \rightarrow u = z).$$

As seen above, the reduction strategy does not immediately yield the simplest first-order equivalent possible. Some further simplification will usually be possible, as will also be seen in examples 3.20 and 3.25. More optimal equivalents could be produced at the cost of complicating the reduction strategy. This will be further discussed in the conclusion.

3.4 Multiple occurrences of variables

Before moving on to the more general classes of formulas in our hierarchy, let us present some observations that will allow us to significantly simplify the presentation in the following sections.

Definition 3.14. A non-uniform implication $\varphi \to \psi$ is a *1-implication* if every variable $p \in \mathsf{Prop}$ occurs positively in φ at most once.

All the best known examples of Sahlqvist implications in the literature are 1-implications, and from an order theoretic point of view, these formulas are much better behaved: for instance, the following is an easy consequence of Proposition 3.12:

Proposition 3.15. *Let $\varphi \to \psi \in \mathrm{ML}$ be a very simple Sahlqvist implication s.t. φ is a positive formula. If $\varphi \to \psi$ is a 1-implication then $[\![\varphi]\!]$ is a complete operator.*

Moreover, as an immediate consequence of Proposition 3.11, the tame valuations corresponding to the 1-very simple Sahlqvist implications map atomic propositions to singleton subsets. This section is aimed at showing that the correspondence result of any class of implications $\Phi \subseteq \mathrm{ML}$ can be obtained as a consequence of the correspondence result for the 1-implications in Φ.

Given a frame $\mathcal{F} = (W, R)$, let V be the class of all valuations on $\mathcal{F}$, and let V' be a subclass of V. Let $m, k \in \mathbb{N}$, $\varphi = \varphi(r_1, \ldots, r_m, s_1, \ldots, s_k) \in \mathrm{ML}$ be positive in the r-variables, and $\psi = \psi(s_1, \ldots, s_k) \in \mathrm{ML}$. Finally, let $\mathsf{P} \cup \{p\} = \{p_1, \ldots, p_m\} \cup \{p\} \subseteq \mathsf{Prop}$ a subset of fresh variables, i.e. not occurring in φ or ψ, and let $\mathsf{V}'^{p}_{\mathsf{P}}$ be defined as follows: $V'' \in \mathsf{V}'^{p}_{\mathsf{P}}$ iff there exists some $V' \in \mathsf{V}'$ s.t. $V'' \sim_p V'$ and $V''(p) = V'(p_1) \cup \cdots \cup V'(p_m)$.

Proposition 3.16. *Suppose that the following are equivalent:*

$$\forall V \in \mathsf{V} \quad [w \in [\![\varphi(p_1/r_1, \ldots, p_m/r_m, \textstyle\bigvee_{i=1}^m p_i/s_1, \ldots, \bigvee_{i=1}^m p_i/s_k)]\!](V) \Rightarrow w \in [\![\psi(\textstyle\bigvee_{i=1}^m p_i/s_1, \ldots, \bigvee_{i=1}^m p_i/s_k)]\!](V)] \tag{6}$$

$$\forall V' \in \mathsf{V}' \quad [w \in [\![\varphi(p_1/r_1, \ldots, p_m/r_m, \textstyle\bigvee_{i=1}^m p_i/s_1, \ldots, \bigvee_{i=1}^m p_i/s_k)]\!](V') \Rightarrow w \in [\![\psi(\textstyle\bigvee_{i=1}^m p_i/s_1, \ldots, \bigvee_{i=1}^m p_i/s_k)]\!](V')] \tag{7}$$

Then the following statements are equivalent:

$$\forall V \in \mathsf{V} \quad [w \in [\![\varphi(p/r_1, \ldots, p/r_m, p/s_1, \ldots, p/s_k)]\!](V) \Rightarrow w \in [\![\psi(p/s_1, \ldots, p/s_k)]\!](V)] \tag{8}$$

$$\forall V' \in \mathsf{V}'^{p}_{\mathsf{P}} \quad [w \in [\![\varphi(p/r_1,\ldots,p/r_m,p/s_1,\ldots,p/s_k)]\!](V') \\ \Rightarrow w \in [\![\psi(p/s_1,\ldots,p/s_k)]\!](V')]. \tag{9}$$

Proof. Clearly, (8) implies (9). To prove that (9) implies (8), we will show that (9) ⇒ (7) ⇒ (6) ⇒ (8).

(7) ⇒ (6) we have by assumption. Condition (6) implies (8): indeed, since the variables $p_1,\ldots,p_m$ are fresh, then (8) is equivalent to the special case of (6) obtained by imposing the restriction that $V(p) = V(p_1) = \cdots = V(p_m)$ (the details of this proof are left to the reader).

For the sake of the implication from (9) to (7), assume (9) and that

$$w \in [\![\varphi(p_1/r_1,\ldots,p_m/r_m,\bigvee_{i=1}^{m} p_i/s_1,\ldots,\bigvee_{i=1}^{m} p_i/s_k)]\!](V') \tag{10}$$

for some $V' \in \mathsf{V}'$. Let V'' be s.t. $V'' \sim_p V'$ and $V''(p) = \bigcup_{i=1}^{m} V'(p_i)$. Clearly, $V'' \in \mathsf{V}'^{p}_{\mathsf{P}}$. Since φ is positive in the r-variables, the following chain holds:

$$\begin{array}{rcl} w & \in & [\![\varphi(p_1/r_1,\ldots,p_m/r_m,\bigvee_{i=1}^{m} p_i/s_1,\ldots,\bigvee_{i=1}^{m} p_i/s_k)]\!](V') \\ & \subseteq & [\![\varphi(p/r_1,\ldots,p/r_m,p/s_1,\ldots,p/s_k)]\!](V''). \end{array}$$

Therefore, by assumption (9), $w \in [\![\psi(p/s_1,\ldots,p/s_k)]\!](V'')$. But again, $V'(\bigvee_{i=1}^{m} p_i) = V''(p)$ implies that

$$[\![\psi(p/s_1,\ldots,p/s_k)]\!](V'') = [\![\psi(\bigvee_{i=1}^{m} p_i/s_1,\ldots,\bigvee_{i=1}^{m} p_i/s_k)]\!](V'),$$

which finishes the proof. □

The statement of the Proposition above is more general than we will need: when it is applied to our case of interest, that of the 1-implications, the r-variables and the s-variables in φ are thought of as the place holders for the positive and negative occurrences of the variable p. This statement is also less general than we need, treating just multiple occurrences of *one* variable. However, it is straightforward, modulo introducing another set of indexes, to extend it so as to treat multiple occurrences of n variables.

As we mentioned early on, the proposition above provides us with a uniform way of deriving the correspondence result for any class of implications Φ from the correspondence result for the class 1-Φ of the 1-implications in Φ: indeed, notice that if the class V' is L_0-definable, then so is $\mathsf{V}'^{p}_{\mathsf{P}}$; therefore if a reduction strategy is available for 1-Φ w.r.t. the class V' of tame valuations, then the Proposition above guarantees that a reduction strategy for any formula $\varphi \to \psi \in \Phi$ is available w.r.t. a class V'_φ of tame valuations that only depends on the multiplicity of occurrences of each variable in φ. Moreover, the reduction algorithm for Φ can be effectively derived from the reduction algorithm for 1-Φ in the following way (here again we just consider multiple occurrences of just *one* variable, leaving the multi-variable case to the reader): if $p \in \mathsf{AtProp}$ occurs positively in φ m times, then consider the *1-formula transform* of $\varphi \to \psi$, i.e. the formula $\varphi^* \to \psi^*$, where φ^* is obtained by replacing each positive occurrence of p in φ by a fresh variable in $\mathsf{P} \subseteq \mathsf{AtProp}$ as above, and each negative occurrence of p in ψ by $\bigvee_{i=1}^m p_i$, and $\psi^* = \psi(\bigvee_{i=1}^m p_i/p)$; consider the standard translation of $\varphi^* \to \psi^*$; by assumption we can eliminate the second order variables P_i from this standard translation by replacing the quantification $\forall P_i$ with $\forall z_i$, where z_i is a fresh variable, and the single occurrences of $P_i y$ with its first-order description $\beta_i(z_i, y)$ derived from the tame valuations. Then the first order correspondent of $\varphi \to \psi$ is effectively obtained by replacing the quantification $\forall P$ with $\forall z_1 \cdots \forall z_m$, where the z_i's are fresh variables, and each occurrence of Py with $\bigvee_{i=1}^m \beta_i(z_i, y)$.

In the remainder of the paper, we will restrict our treatment to the 1-formulas in each class, and will provide more details on the general account only when needed.

3.5 Sahlqvist implications

The reduction strategy. Another promising subclass of tame valuations is formed by those $V_2 \in \mathsf{Val}(\mathcal{F})$ such that for every $p \in \mathsf{Prop}$, $V_2(p) = R[z]$ for some $z \in W$. Indeed, suppose that the following were equivalent:

1. $\forall V(w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V))$
2. $\forall V_2(w \in [\![\varphi]\!](V_2) \Rightarrow w \in [\![\psi]\!](V_2))$.

This would mean that

$$\mathcal{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi \to \psi)[\![w]\!] \quad \text{iff} \quad \mathcal{F} \models \forall P_1^2 \ldots \forall P_n^2 ST_x(\varphi \to \psi)[\![w]\!]$$

where the variables P_i^2 would not range on $\mathcal{P}(W)$, but only on $\{R[z] \mid z \in W\}$. Therefore the formula above on the right-hand side can be transformed into a first-order formula by replacing each $\forall P_i^2$ in the prefix with $\forall z_i$ and each atomic formula of the form $P_i^2 y$ with $z_i R y$ $\bigvee_{j=1}^m z_i^j R y$, where all the z's are fresh variables.

Actually, this argument can be refined and extended to valuations V_2 such that for every $p \in \mathsf{Prop}$, $V_2(p) = R^k[z]$ for some $z \in W$, and some

$k \in \mathbb{N}$ relation on W (Notice that the valuations V_1 ranging over singletons are the special case of V_2 where $k = 0$). In this case, the formula above on the right-hand side can be equivalently transformed into a first-order formula by replacing each $\forall P_i^2$ in the prefix with $\forall z_i$, and each formula of the form $P_i^2 y$ with an L_0-formula which says 'there exists an R-path from z_i to y in k_i steps', such as:

$$\exists v_0, \ldots v_{k_i}[(z_i = v_0 \wedge \bigwedge_{j=0}^{k_i - 1} v_j R v_{j+1} \wedge v_{k_i} = y]$$

This time we are after some conditions on φ and ψ that guarantee that the universal quantification $\forall V$ can be equivalently replaced with the universal quantification $\forall V_2$.

Order theoretic conditions. The maps $f, g : \mathcal{P}(W) \to \mathcal{P}(W)$ form an *adjoint pair* (notation: $f \dashv g$) iff for every $X, Y \in \mathcal{P}(W)$,

$$f(X) \subseteq Y \quad \text{iff} \quad X \subseteq g(Y).$$

Whenever $f \dashv g$, f is the *left adjoint* of g and g is *the right adjoint* of f. One important property of adjoint pairs of maps is that if a map admits a left (resp. right) adjoint, the adjoint is unique and can be computed pointwise from the map itself and the order (which in our case is the inclusion). This means that admitting a left (resp. right) adjoint is an *intrinsically* order-theoretic property of maps.

Proposition 3.17. *1. Right adjoints between complete lattices are exactly the completely meet-preserving maps, i.e., in the concrete case of powerset algebras $\mathcal{P}(W)$ they are exactly those maps g such that $g(\bigcap S) = \bigcap\{g(X) \mid X \in S\}$ for all $S \subseteq \mathcal{P}(W)$;*

2. *Right adjoints on a powerset algebra $\mathcal{P}(W)$ are exactly maps of the form $l_{\mathcal{S}}$ for some binary relation $\mathcal{S}$ on W.*

3. *For any binary relation $\mathcal{S}$ on W, the left adjoint of $l_{\mathcal{S}}$ is the map $m_{\mathcal{S}^{-1}}$, defined by the assignment $X \mapsto \mathcal{S}[X]$.*

Proof. 1. See [12, Proposition 7.34]. 2. We leave to the reader to verify that every map of form $l_{\mathcal{S}}$ is completely meet preserving, hence it is a right adjoint. Conversely, let $g : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ be a right adjoint. Then by item 1 above, g is completely meet preserving. Define $\mathcal{S} \subseteq W \times W$ as follows: for every $x, z \in W$,

$$x\mathcal{S}z \quad \text{iff} \quad x \notin g(W \setminus \{z\}).$$

Hence,

$$x \in l_{\mathcal{S}}(W \setminus \{z\}) \;\text{ iff }\; \mathcal{S}[x] \subseteq (W \setminus \{z\}) \;\text{ iff }\; z \notin \mathcal{S}[x] \;\text{ iff }\; x \in g(W \setminus \{z\}),$$

which shows our claim for all the special subsets of W of type $W \setminus \{z\}$. In order to show it in general, fix $X \in \mathcal{P}(W)$ and notice that $X = \bigcap_{z \notin X}(W \setminus \{z\})$. Using the fact that g is completely meet preserving and the special case shown above, we get:

$$\begin{array}{rcl} g(X) & = & g(\bigcap\{(W \setminus \{z\}) \mid z \notin X\}) \\ & = & \bigcap\{g(W \setminus \{z\}) \mid z \notin X\} \\ & = & \bigcap\{l_{\mathcal{S}}(W \setminus \{z\}) \mid z \notin X\} \\ & = & l_{\mathcal{S}}(\bigcap\{(W \setminus \{z\}) \mid z \notin X\} \quad (*) \\ & = & l_{\mathcal{S}}(X). \end{array}$$

The marked equality can be verified directly, but also follows from the more general fact that $l_{\mathcal{S}}$ is completely meet preserving for every $\mathcal{S}$. 3. Left to the reader. □

Consider the following conditions on $[\![\varphi]\!]$:

(a) $\varphi(p_1, \ldots, p_n) = \varphi'(\chi_1(p_1), \ldots, \chi_n(p_n), \gamma_1, \ldots, \gamma_\ell)$, moreover

(b) $[\![\varphi']\!]$ is a complete operator;

(c) for $1 \leq i \leq n$, $[\![\chi_i]\!] : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ is a right adjoint, i.e. there exists some $f_i : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ such that for every $X, Y \in \mathcal{P}(W)$, $f_i(X) \subseteq Y$ iff $X \subseteq [\![\chi_i]\!](Y)$;

(d) for every $1 \leq i \leq n$, f_i is defined by $X \mapsto R^{k_i}[X]$

(e) $[\![\gamma_1]\!]$ to $[\![\gamma_\ell]\!]$ are order reversing in each coordinate.

Notice that, by Proposition 3.17, condition (c) already guarantees that for every i, f_i is defined by $X \mapsto \mathcal{S}_i[X]$ for some *arbitrary* binary relation $\mathcal{S}_i$ on W; however, since $\mathcal{S}_i$ is arbitrary, this is not yet enough to guarantee that valuations defined by $p_i \mapsto f_i(\{z_i\})$ be L_0-definable. Condition (d) above guarantees this last point.

For every frame $\mathcal{F}$, let $\mathsf{Val}_2(\mathcal{F})$ be the set of valuations on $\mathcal{F}$ such that, for every $p \in \mathsf{Prop}$, $V_2(p) = R^k[x]$ for some $x \in W$ and some $k \in \mathbb{N}$.

Proposition 3.18. *Let $\varphi \to \psi \in \mathrm{ML}$ be such that $[\![\varphi]\!]$ verifies the conditions (a)-(d) above and $[\![\psi]\!]$ is order preserving in each coordinate. Then the following are equivalent:*

1. $(\forall V \in \mathsf{Val}(\mathcal{F}))[w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V)]$
2. $(\forall V_2 \in \mathsf{Val}_2(\mathcal{F}))[w \in [\![\varphi]\!](V_2) \Rightarrow w \in [\![\psi]\!](V_2)]$.

Proof. $(1 \Rightarrow 2)$ Clear. $(2 \Rightarrow 1)$ Fix V and let $w \in [\![\varphi]\!](V)$. Hence,

$$\varnothing \neq [\![\varphi]\!](V) = [\![\varphi']\!]([\![\chi_1]\!](V(p_1)), \ldots, [\![\chi_n]\!](V(p_n)), [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)),$$

and since by assumption (b) $[\![\varphi']\!]$ is a complete operator, $[\![\chi_i]\!](V(p_i)) \neq \varnothing$ for every $1 \leq i \leq n$. Moreover, because every set is the union of the singletons of its elements and complete operators preserve arbitrary unions in each coordinate, the following chain of equalities holds:

$$\begin{array}{ll} & w \in [\![\varphi]\!](V) \\ = & [\![\varphi']\!]([\![\chi_1]\!](V(p_1)), \ldots, [\![\chi_n]\!](V(p_n)), [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \\ = & \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid \{x_i\} \subseteq [\![\chi_i]\!](V(p_i))\}_{i=1}^n \\ = & \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid f_i(\{x_i\}) \subseteq V(p_i)\}_{i=1}^n \end{array}$$

where the last equality is a consequence of assumption (c). Then $w \in [\![\varphi']\!](\{z_1\}, \ldots, \{z_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)))$ for some $z_i \in W$, $1 \leq i \leq n$, such that $f_i(\{z_i\}) \subseteq V(p_i)$. Let V_2 be the valuation that maps any $q \in \mathsf{Prop} \setminus \{p_i \mid 1 \leq i \leq n\}$ to $\varnothing$ and such that $V_2(p_i) = f_i(\{z_i\})$. By assumption (d), $V_2 \in \mathsf{Val}_2(\mathcal{F})$. Let us show that $w \in [\![\varphi]\!](V_2)$: indeed,

$$\begin{array}{rl} w \in & [\![\varphi']\!](\{z_1\}, \ldots, \{z_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \\ \subseteq & \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid f_i(\{x_i\}) \subseteq V_2(p_i)\}_{i=1}^n \\ \subseteq & \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V_2), \ldots, [\![\gamma_\ell]\!](V_2)) \mid f_i(\{x_i\}) \subseteq V_2(p_i)\}_{i=1}^n \\ = & \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V_2), \ldots, [\![\gamma_\ell]\!](V_2)) \mid \{x_i\} \subseteq [\![\chi_i]\!](V_2(p_i))\}_{i=1}^n \\ = & [\![\varphi']\!]([\![\chi_1]\!](V_2(p_1)), \ldots, [\![\chi_n]\!](V_2(p_n)), [\![\gamma_1]\!](V_2), \ldots, [\![\gamma_\ell]\!](V_2)) \\ = & [\![\varphi]\!](V_2). \end{array}$$

The second inclusion holds since $V_2(p) \subseteq V(p)$ for every $p \in \mathsf{Prop}$ and the $[\![\gamma]\!]'s$ are order reversing. By assumption (2), we can conclude that $w \in [\![\psi]\!](V_2)$. Now, since $[\![\psi]\!]$ is order preserving in every coordinate, and $V_2(p_i) = f_i(\{z_i\}) \subseteq V(p_i)$ for every $1 \leq i \leq n$, we get $w \in [\![\psi]\!](V_2) \subseteq [\![\psi]\!](V)$, which concludes the proof. □

Syntactic conditions.

Proposition 3.19. *If* $\varphi \to \psi \in \mathrm{ML}$ *is a Sahlqvist 1-implication, then* $\varphi \to \psi$ *verifies the hypotheses of Proposition 3.18. In particular, the maps* $[\![\chi_i]\!]$ *are exactly the ones induced by the boxed atoms.*

Proof. It follows from Propositions 3.12 and 3.17, and the additional fact that, for every $R_1, R_2 \subseteq W \times W$, $l_{R_2} \circ l_{R_1} = l_{R_1 \circ R_2}$. $\square$

Example 3.20. Consider the (definite) 1-Sahlqvist implication $\Diamond\Box p \wedge \Box q \to \Box\Diamond(p \wedge q)$. This has standard second-order frame equivalent

$$\forall P \forall Q[(\exists y(xRy \wedge \forall u(yRu \to P(u))) \wedge \forall v(xRv \to Q(v))) \to \forall w(xRw \to \exists s(Rws \wedge P(s) \wedge Q(s)))].$$

The reduction strategy prescribes that in the above we replace $\forall P \forall Q$ with $\forall z_1 \forall z_2$, and substitute $P(y)$ and $Q(y)$ with $\exists v_0 \exists v_1(v_0 = z_1 \wedge v_0 R v_1 \wedge v_1 = y)$ and $\exists v_0 \exists v_1(v_0 = z_2 \wedge v_0 R v_1 \wedge v_1 = y)$, respectively, which simplify to $z_1 R y$ and $z_2 R y$, respectively. Doing this we obtain the first-order frame equivalent

$$\forall z_1 \forall z_2[(\exists y(xRy \wedge \forall u(yRu \to z_1 Ru)) \wedge \forall v(xRv \to z_2 Rv)) \to \forall w(xRw \to \exists s(Rws \wedge z_1 Rs \wedge z_2 Rs))],$$

Using the well-known fact that for any first-order formula $\beta(x, y)$ it holds that $\forall x \forall y \beta(x, y) \models \forall x \forall x \beta(x, x)$, we see (by pulling out quantifiers and setting $z_1 = y$ and $z_2 = x$) that the above has as consequence

$$\forall y \forall w(xRy \wedge xRw \to \exists s(xRs \wedge yRs \wedge wRs)).$$

An easy semantic argument shows that the converse also holds, and hence that the last formula is actually a local first-order frame correspondent for $\Diamond\Box p \wedge \Box q \to \Box\Diamond(p \wedge q)$.

3.6 Atomic inductive formulas

The reduction strategy. Let us introduce the following piece of notation: For every $i+1$-ary relation $\mathcal{S}_i$ on W and all $X_1, \ldots, X_i \subseteq W$, let

$$\begin{array}{ll} & \mathcal{S}_i[X_1, \ldots, X_i] \\ := & \{y \in W \mid \exists x_1 \cdots \exists x_i[\bigwedge_{h=1}^{i} x_h \in X_h \ \wedge \ \mathcal{S}_i(x_1, \ldots, x_i, y)]\}. \end{array}$$

The last subclass of tame valuations we are going to consider in this paper can be described as follows: Given an arbitrary finite set $\{p_1, \ldots, p_n\} \subseteq \mathsf{Prop}$ strictly and linearly ordered by $p_i < p_j$ iff $i < j$, assume that there exist some $k_1, \ldots, k_n \in \mathbb{N}$ and some $m_i \in \mathbb{N}$ for $1 < i \leq n$, such that V_3 can be inductively defined as follows:

1. $V_3(p_1) = \mathcal{S}_1[w_1]$ for some $w_1 \in W$, where each $\mathcal{S}_1$ is the composition R^{k_1} of R with itself k_1 times;

2. for every $1 < i \leq n$, $V_3(p_i) = \mathcal{S}_i[\{w_i\}, V_3(p_{i_1}), \ldots, V_3(p_{i_{m_i}})]$ for some $w_i \in W$, $p_{i_1}, \ldots, p_{i_{m_i}} \in \{p_0, p_1, \ldots, p_{i-1}\}$, and where $\mathcal{S}_i$ is such that for all $x_0, \ldots, x_{m_i}, y \in W$,

$$\mathcal{S}_i(x_0, \ldots, x_{m_i}, y) \text{ iff } (\bigwedge_{0 \leq h < m_i} x_h R x_{h+1}) \wedge x_{m_i} R^{k_i} y, \tag{11}$$

where, as before, R^{k_i} is the composition of R with itself k_i times. (Notice that when $n = 1$, the valuations V_3 reduce to special valuations V_2 of the previous section on simple Sahlqvist formulas).

Suppose that the following were equivalent:

1. $\forall V (w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V))$

2. $\forall V_3 (w \in [\![\varphi]\!](V_3) \Rightarrow w \in [\![\psi]\!](V_3))$.

This would mean that

$$\mathcal{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi \to \psi)[\![w]\!] \quad \text{iff} \quad \mathcal{F} \models \forall P_1^3 \ldots \forall P_n^3 ST_x(\varphi \to \psi)[\![w]\!], \tag{12}$$

where the variables P_i^3 would not range on $\mathcal{P}(W)$, but only on sets as described in the first enumeration above. In this case, the right-hand side of (12) can be equivalently transformed into a first-order formula by the following procedure, that we define inductively:

1. Replace $\forall P_1^3$ in the prefix with $\forall z_1$, and substitute each subformula of the form $P_1^3 y$ with $z_1 R^{k_1} y$, where the latter is an abbreviation for $\exists u_1 \ldots \exists u_{k_1+1}(z_1 = u_1 \wedge y = u_{k_1+1} \wedge \bigwedge_{j=1}^{k_1} u_j R u_{j+1})$.

2. Suppose that, for each $1 \leq h < i$, $\forall P_h^3$ in the prefix has been replaced by first-order quantifiers and in the matrix each subformula of the form $P_h^3 y$ has been substituted with an L_0-formula $\alpha_h(y)$. Then, replace $\forall P_i^3$ in the prefix by $\forall z_i$ and substitute each subformula of the form $P_i^3 y$ with

$$\exists v_0, \ldots \exists v_{m_i} [z_i = v_0 \wedge (\bigwedge_{j=0}^{m_i - 1} v_j R v_{j+1}) \wedge (\bigwedge_{h=1}^{m_i} \alpha_h(v_h)) \wedge v_{m_i} R^{k_i} y],$$

where $v_{m_i} R^{k_i} y$ is defined similarly to $z_1 R^{k_1} y$ in clause 1 above.

This time we are after some conditions on φ and ψ that guarantee that the universal quantification $\forall V$ can be replaced with the universal quantification $\forall V_3$.

Order theoretic conditions. The notion of adjunction of monotone maps can be generalized to j-ary maps in a component-wise fashion: a j-ary map $f : \mathcal{P}(W)^j \to \mathcal{P}(W)$ is *residuated* if there exists a collection of maps

$$\{g_h : \mathcal{P}(W)^j \to \mathcal{P}(W) \mid 1 \leq h \leq j\}$$

s.t. for every $1 \leq h \leq j$ and for all $X_1, \ldots, X_j, Y \in \mathcal{P}(W)$,

$$f(X_1, \ldots, X_j) \subseteq Y \quad \text{iff} \quad X_h \subseteq g_h(X_1, \ldots, X_{h-1}, Y, X_{h+1}, \ldots, X_j).$$

The map g_h is the *h-th residual* of f. The facts stated in the following example and proposition are well known in the literature in their binary instance (cf. [16, Subsection 3.1.3]):

Example 3.21. For every $j+1$-ary relation $\mathcal{S}$ on W and every $(X_1, \ldots, X_j) \in \mathcal{P}(W)^j$, let

$$\begin{array}{rl} & \mathcal{S}[X_1, \ldots, X_j] \\ := & \{y \in W \mid \exists x_1 \cdots \exists x_j [\bigwedge_{h=1}^{j} x_h \in X_h \ \wedge \ \mathcal{S}(x_1, \ldots, x_j, y)]\}. \end{array}$$

The j-ary operation on $\mathcal{P}(W)$ defined by the assignment

$$(X_1, \ldots, X_j) \mapsto \mathcal{S}[X_1, \ldots, X_j] \tag{13}$$

is residuated and its h-th residual is the map

$$g_h : \mathcal{P}(W)^j \to \mathcal{P}(W)$$

which maps every j-tuple $(X_1, \ldots, X_{h-1}, Y, X_{h+1}, \ldots, X_j)$ to the set

$$\{w \in W \mid \alpha_{\mathcal{S}}^h(w)\},$$

where $\alpha_{\mathcal{S}}^h(w)$ is the following first-order formula:

$$\forall x_1 \cdots \forall y \cdots \forall x_j [(\bigwedge_{k \in \mathbf{j}_h} x_k \in X_k \ \& \ \mathcal{S}(x_1, \ldots, w, \ldots, x_j, y)) \Rightarrow y \in Y],$$

and for every $j \in \mathbb{N}$ and $1 \leq h \leq j$, $\mathbf{j}_h = \{1, \ldots, j\} \setminus \{h\}$.

Proposition 3.22. *If $f : \mathcal{P}(W)^j \to \mathcal{P}(W)$ is residuated and $\{g_h : \mathcal{P}(W)^j \to \mathcal{P}(W) \mid 1 \leq h \leq j\}$ is the collection of its residuals, then:*

1. *f is order preserving in each coordinate and for $1 \leq h \leq j$, g_h is order preserving in its h-th coordinate;*
2. *f preserves arbitrary joins in each coordinate;*

3. *f coincides with the map defined by the assignment (13), for some $j+1$-ary relation $\mathcal{S}$ on W.*

Proof. 1. Fix $1 \leq h \leq j$, let $X_1, \ldots, X_j, Y, Z \in \mathcal{P}(W)$, and assume that $Y \subseteq Z$. By residuation, the "tautological" inclusion

$$f(X_1, \ldots, Z, \ldots, X_j) \subseteq f(X_1, \ldots, Z, \ldots, X_j)$$

is equivalent to the second inclusion in the following chain:

$$Y \subseteq Z \subseteq g_h(X_1, \ldots, f(X_1, \ldots, Z, \ldots, X_j), \ldots, X_j),$$

which yields, again by residuation,

$$f(X_1, \ldots, Y, \ldots, X_j) \subseteq f(X_1, \ldots, Z, \ldots, X_j).$$

The proof that g_h is monotone in the h-th coordinate goes likewise.

2. Fix $\mathcal{Y} \subseteq \mathcal{P}(W)$, $X_1, \ldots, X_{h-1}, X_{h+1}, \ldots X_j \in \mathcal{P}(W)$ and let us show that

$$f(X_1, \ldots, \bigcup \mathcal{Y}, \ldots, X_j) = \bigcup \{f(X_1, \ldots, Y, \ldots, X_j) \mid Y \in \mathcal{Y}\}.$$

The right-to-left inclusion follows by f being order preserving in each coordinate. By residuation, the converse inclusion is equivalent to

$$\bigcup \mathcal{Y} \subseteq g_h(X_1, \ldots, \bigcup \{f(X_1, \ldots, Y, \ldots, X_j) \mid Y \in \mathcal{Y}\}, \ldots, X_j),$$

to prove which, the following chain suffices:

$$\begin{array}{lll} \bigcup \mathcal{Y} & \subseteq & \bigcup \{g_h(X_1, \ldots, f(X_1, \ldots, Y, \ldots, X_j), \ldots, X_j) \mid Y \in \mathcal{Y}\} \\ & \subseteq & g_h(X_1, \ldots, \bigcup \{f(X_1, \ldots, Y, \ldots, X_j) \mid Y \in \mathcal{Y}\}, \ldots, X_j). \end{array}$$

The first inclusion readily follows by applying residuation to the "tautological" inclusions

$$f(X_1, \ldots, Y, \ldots, X_j) \subseteq f(X_1, \ldots, Y, \ldots, X_j)$$

for every $Y \in \mathcal{Y}$. The second one follows by the monotonicity of g_h in its h-th coordinate.

3. Let $\mathcal{S} \subseteq W^{j+1}$ be defined as follows: for all $y, x_1, \ldots, x_j \in W$,

$$(x_1, \ldots, x_j, y) \in \mathcal{S} \text{ iff } \{y\} \subseteq f(\{x_1\}, \ldots \{x_j\}).$$

Then $f(\{x_1\}, \ldots \{x_j\}) = \mathcal{S}[\{x_1\}, \ldots \{x_j\}]$. Using this observation and the fact that, by item 2 above, every residuated map is completely join preserving in each argument, then it is easy to show that for all $X_1, \ldots, X_j \in \mathcal{P}(W)$,

$$f(X_1, \ldots, X_j) = \mathcal{S}[X_1, \ldots, X_j].$$

□

Consider the following conditions on $[\![\varphi]\!]$: There exist $k_1, \ldots, k_n, m_1, \ldots, m_n \in \mathbb{N}$, such that

(a) $[\![\varphi]\!](p_1, \ldots, p_n)$
$= [\![\varphi']\!]([\![\chi_1]\!](p_1), \ldots, [\![\chi_n]\!](p_{n_1}, \ldots, p_{n_{m_n}}, p_n), [\![\gamma_1]\!], \ldots, [\![\gamma_\ell]\!])$,
where

(b) $[\![\varphi']\!]$ is a complete operator on $\mathcal{P}(W)$,

(c) for $1 \leq i \leq n$, $[\![\chi_i]\!](p_{i_1}, \ldots, p_{i_{m_i}}, p_i)$ is the $(m_i + 1)$-th residual of some (m_i+1)-ary residuated operation f_i, and $p_{i_1}, \ldots, p_{i_{m_i}} \in \{p_1, \ldots, p_{i-1}\}$;

(d) the binary relations $\mathcal{S}_1$ corresponding to f_1 is of the form R^{k_1};

(e) for every $1 < i \leq n$, the $(m_i + 2)$-ary relations $\mathcal{S}_i$ on W corresponding to f_i is given, for all $x_0, \ldots, x_{m_i}, y \in W$, by

$$\mathcal{S}_i(x_0, \ldots, x_{m_i}, y) \text{ iff } (\bigwedge_{0 \leq h < m_i} x_h R x_{h+1}) \wedge x_{m_i} R^{k_i} y.$$

(f) γ_1 to γ_ℓ are negative formulas.

For every frame $\mathcal{F}$, every formula $\varphi(p_1, \ldots, p_n)$ satisfying condition (a) to (f), let Val_3 be the set of valuations V_3 on $\mathcal{F}$ which map any $q \in \mathsf{Prop} \setminus \{p_1, \ldots, p_n\}$ to $\varnothing$ and are defined inductively on $\{p_1, \ldots, p_n\}$ as follows:

- $V_3(p_1) = \mathcal{S}_1[\{w_1\}]$ for some $w_1 \in W$, and where $\mathcal{S}_1$ is as specified in (d), above.
- For every $1 < i \leq n$, $V_3(p_i) = \mathcal{S}_i[\{w_i\}, V_3(pi_1), \ldots, V_3(p_{i_{m_i}})]$, and where $\mathcal{S}_i$ is as specified in (e), above.

Proposition 3.23. *Let $\varphi \to \psi \in \mathrm{ML}$ be such that $[\![\varphi]\!]$ verifies the conditions (a)-(f) above and $[\![\psi]\!]$ is order preserving in each coordinate. Then the following are equivalent:*

1. $(\forall V \in \mathsf{Val}(\mathcal{F}))[w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V)]$

2. $(\forall V_3 \in \mathsf{Val}_3(\mathcal{F}))[w \in [\![\varphi]\!](V_3) \Rightarrow w \in [\![\psi]\!](V_3)]$.

Proof. ($1 \Rightarrow 2$) Clear. ($2 \Rightarrow 1$) Fix $V \in \mathsf{Val}(\mathcal{F})$ and let $w \in [\![\varphi]\!](V)$. Hence $\varnothing \neq [\![\varphi']\!]([\![\chi_1]\!](V), \ldots, [\![\chi_n]\!](V), \ldots, [\![\chi_\ell]\!](V))$, and since by assumption (b) $[\![\varphi']\!]$ is a complete operator, we get, $[\![\chi_i]\!](V) \neq \varnothing$ for every $1 \leq i \leq n$. Then

$$\begin{array}{rcl} w & \in & [\![\varphi']\!](V(\chi_1), \ldots, V(\chi_n), V(\gamma_1), \ldots, V(\gamma_\ell)) \\ & = & \bigcup\{[\![\varphi']\!](z_1, \ldots, z_n, V(\gamma_1), \ldots, V(\gamma_\ell)) \mid z_i \in V(\chi_i), 1 \leq i \leq n\} \\ & = & \bigcup\{[\![\varphi']\!](z_1, \ldots, z_n, V(\gamma_1), \ldots, V(\gamma_\ell)) \mid f_i(V(p_{i_1}), \ldots, V(p_{i_{m_i}}), \{z_i\}) \subseteq V(p_i), 1 \leq i \leq n\} \end{array}$$

where the first equality follows from the fact that $[\![\varphi']\!]$ is a complete operator and the second from assumption (c).

Then $w \in [\![\varphi']\!](w_1, \ldots, w_n, V(\gamma_1), \ldots, V(\gamma_\ell))$ for some $w_1, \ldots, w_n \in W$ such that, for all $1 \leq i \leq n$,

$$f_i(V(p_{i_1}), \ldots, V(p_{i_{m_i}}), \{w_i\}) \subseteq V(p_i). \tag{14}$$

Let V_3 be the valuation that maps any $q \in \mathsf{Prop} \setminus \{p_i \mid 1 \leq i \leq n\}$ to $\varnothing$ and is defined inductively on $\{p_1, \ldots, p_n\}$ as follows:

- $V_3(p_1) = f_1(\{w_1\})$, and
- for $1 < i \leq n$, $V_3(p_i) = f_i(V_3(p_{i_1}), \ldots, V_3(p_{i_{m_i}}), \{w_i\})$.

By assumptions (d) and (e), $V_3 \in \mathsf{Val}_3$. Moreover, $V_3(p_i) \subseteq V(p_i)$, for all $1 \leq i \leq p$. Indeed, the latter can be shown by induction on i. For $i = 1$, since w_1 satisfies (14), we get $V_3(p_1) = f_1(\{w_1\}) \subseteq V(p_1)$. Let $i > 1$ and suppose that the inclusion holds for the first $i - 1$ variables. Since w_i satisfies(14) and, by Proposition 3.22, f_i is order preserving in each coordinate, we get

$$\begin{array}{rcl} V_3(p_i) & = & f_i(V_3(p_{i_1}), \ldots, V_3(p_{i_{m_i}}), \{w_i\}) \\ & \subseteq & f_i(V(p_{i_1}), \ldots, V(p_{i_{m_i}}), \{w_i\}) \subseteq V(p_i). \end{array}$$

Let us now show that $w \in [\![\varphi]\!](V_3)$:

$$\begin{array}{rl} w \in & [\![\varphi']\!](w_1, \ldots, w_n, V(\gamma_1), \ldots, V(\gamma_\ell)) \\ \subseteq & [\![\varphi']\!](w_1, \ldots, w_n, V_3(\gamma_1), \ldots, V_3(\gamma_\ell)) \\ \subseteq & \bigcup\{[\![\varphi']\!](z_1, \ldots, z_n, V_3(\gamma_1), \ldots, V_3(\gamma_\ell)) \mid f_i(V_3(p_{i_1}), \ldots, V_3(p_{i_{m_i}}), \{z_i\}) \subseteq V_3(p_i), 1 \leq i \leq n\} \\ = & \bigcup\{[\![\varphi']\!](z_1, \ldots, z_n, V_3(\gamma_1), \ldots, V_3(\gamma_\ell)) \mid z_i \in [\![\chi_i]\!](V_3(p_{i_1}), \ldots, V_3(p_{i_{m_i}}), V_3(p_i)), 1 \leq i \leq n\} \\ = & [\![\varphi']\!](V_3(\chi_1), \ldots, V_3(\chi_n), V_3(\gamma_1), \ldots, V_3(\gamma_\ell)) \\ = & [\![\varphi]\!](V_3). \end{array}$$

Here the first inclusion holds since the $[\![\gamma]\!]$'s are order reversing by assumption (f), $[\![\varphi']\!]$ is order-preserving, and as mentioned above $V_3(p_i) \subseteq V(p_i)$ for all $1 \leq i \leq n$. The second inclusion holds by the definition of V_3. By assumption 2., from $w \in [\![\varphi]\!](V_3)$ we have that $w \in [\![\psi]\!](V_3)$. Now, from the facts that $[\![\psi]\!]$ is order preserving in each coordinate and that $V_3(p_i) \subseteq V(p_i)$ for all $1 \leq i \leq n$, we conclude that that $w \in [\![\psi]\!](V)$. □

Syntactic conditions.

Proposition 3.24. *If φ is definite 1-atomic inductive antecedent, then $[\![\varphi]\!]$ verifies the hypotheses (a) to (f) of Proposition 3.23. In particular, the maps $[\![\chi_i]\!]$ are exactly the ones induced by the atomic box-formulas.*

Proof. (Sketch) As far as conditions (a), (b), and (f) are concerned, we note the following. If φ is a definite 1-atomic inductive antecedent, then the atomic box formulas and negative formulas used in its construction correspond to $\chi_1, \ldots \chi_n$ and $\gamma_1, \ldots, \gamma_\ell$, respectively, in

$$[\![\varphi']\!]([\![\chi_1]\!](p_1), \ldots, [\![\chi_n]\!](p_{n_1}, \ldots, p_{n_{m_n}}, p_n), [\![\gamma_1]\!], \ldots, [\![\gamma_\ell]\!]).$$

The 'skeleton' consisting of the composition of $\Diamond$'s and $\wedge$'s used in the construction corresponds to the complete operator $[\![\varphi']\!]$.

As for conditions (c), (d) and (e), we note that since the dependency digraph of φ is acyclic, its transitive closure is a strict partial order. We can therefore assume, without loss of generality, that the variables are ordered $p_1 < p_2 < \cdots < p_n$ by some linear extension of this partial order. Hence we will have that $p_{i_1}, \ldots, p_{i_{m_i}} \in \{p_1, \ldots, p_{i-1}\}$ in each $[\![\chi_i]\!](p_{i_1}, \ldots, p_{i_{m_i}}, p_i)$. By Propositions 3.12, 3.17, and 3.19, in order to complete the proof, it is enough to show that for every atomic box-formula

$$\chi_i(p_{i_1}, \ldots, p_{i_{m_i}}, p_i) = \Box(p_{i_1} \to \Box(p_{i_2} \to \ldots \Box(p_{i_{m_i}} \to \Box^k p_i) \ldots)),$$

$[\![\chi_i]\!]$ is the $(m_i + 1)$-th residual of the map $f_i : \mathcal{P}(W)^{m_i+1} \to \mathcal{P}(W)$ defined as

$$f(X_1, \ldots, X_{m_i}, Y) = R^{k_i}[X_{m_i} \cap R[X_{m_i-1} \cap \cdots R[X_2 \cap R[X_1 \cap Y] \cdots]]].$$

The details of the proof are left to the reader. □

Example 3.25. Let us consider the 1-atomic inductive formula

$$\varphi := p \wedge \Box(p \to q) \to \Diamond q,$$

which locally corresponds to the property of being a reflexive state. The dependency digraph induces the order $p < q$ on the variables. We have $k_1 = 0$, $m_1 = 0$, $k_2 = 0$ and $m_2 = 1$. The standard local second-order translation is

$$\forall P \forall Q[P(x) \wedge \forall y(xRy \rightarrow (P(y) \rightarrow Q(y))) \rightarrow \exists u(xRu \wedge Q(u))].$$

The reduction strategy prescribes that we replace $\forall P \forall Q$ in the prefix with $\forall z_1 \forall z_2$ and that we substitute occurrences of the form $P(y)$ with $\alpha_1(y) := \exists u_1(z_1 = u_1 \wedge y = u_1)$ which is equivalent to $y = z_1$. It further prescribes that occurrences of the form $Q(y)$ should be substituted with $\alpha_2(y) := \exists v_0 \exists v_1(z_2 = v_0 \wedge v_0 R v_1 \wedge v_1 = z_1 \wedge v_1 = y)$, where we have already used the simplified version of α_1. Now $\alpha_2(y)$ can be further simplified to $z_2 R z_1 \wedge z_1 = y$. Doing the substitution we obtain

$$\forall z_1 \forall z_2[x = z_1 \wedge \forall y(xRy \rightarrow (y = z_1 \rightarrow z_2 R z_1 \wedge y = z_1)) \rightarrow \exists u(xRu \wedge z_2 R z_1 \wedge u = z_1)].$$

This is equivalent to

$$\begin{aligned} & \forall z_1 \forall z_2[\forall y(xRy \rightarrow (y = x \rightarrow z_2 Rx)) \rightarrow \exists u(xRu \wedge z_2 Rx \wedge u = x)] \\ \equiv\ & \forall z_1 \forall z_2[(xRx \rightarrow z_2 Rx) \rightarrow \exists u(xRx \wedge z_2 Rx)] \\ \equiv\ & xRx. \end{aligned}$$

4 Conclusions

In this paper, the Sahlqvist-style syntactic identification of classes of modal formulas that are endowed with local first order correspondents has been explained in terms of certain order-theoretic properties of the extension maps corresponding to the formulas of these classes. These properties and the resulting methodology hold beyond the Sahlqvist class, as the example of atomic inductive formulas shows. Further features which we would like to emphasize are:

Generalizing the signatures. Our treatment is modular: in particular, we neatly divided the correspondence proof for each class of formulas in three stages. Although, for simplicity, we confined our treatment to the basic modal signature, the most important stage, i.e. the one referred to as 'order theoretic conditions' is intrinsically independent from any algebraic signature. Therefore, it can be applied to any one, and in particular to any modal signature.

Modifying the description of tame valuations. We have showed that at the core of the correspondence mechanism there are special classes of (tame) valuations, the members of which can be described uniformly in the language L_0. Of course the classes $\mathsf{Val}_1(\mathcal{F})$, $\mathsf{Val}_2(\mathcal{F})$ and $\mathsf{Val}_3(\mathcal{F})$, on which we settled as a compromise between simplicity of presentation and generality, are just three instances. A plethora of variations on them is possible. On the one hand, by complicating the definitions of these classes, using more parameters and taking into account the positions where boxed atoms / box formulas occur within the antecedent, the shape of the first-order correspondent obtained can be improved. On the other hand, the class $\mathsf{Val}_3(\mathcal{F})$, for example, is designed to be targeted by *atomic* inductive implications, and as such is too small for the whole class of inductive implications. The essential difference lies in the definition of the (general, not necessarily atomic) box formulas, which yields shapes such as the following:

$$\Box(A_0 \to \Box(A_1 \to \dots \Box(A_n \to \Box^k p) \dots)),$$

where the A_i's are arbitrary positive formulas. The correspondence result for the whole class of inductive formulas can be obtained by the same methodology we proposed in Section 3.6, applied to a suitably larger class of special tame valuations.

From the Boolean to the distributive setting. The statements and proofs about the order-theoretic conditions only use the following two features of powerset algebras: that they are complete distributive lattices and that they are completely join-generated by their completely join prime elements[5] (the singleton subsets). Therefore, these proofs go through virtually unchanged in the more general setting of distributive lattices enjoying these two properties. In fact, it is well known that the modal expansion of each of these special lattices is isomorphic to the complex algebra of some frame for distributive modal logic. This observation motivates e.g. the development of the algorithmic correspondence theory for distributive modal logic in [11] in the purely algebraic setting of lattices of such kind.

Duality and correspondence. Although it was not strictly needed for our exposition, in this paper correspondence is regarded as a byproduct of the duality between Kripke frames and complete and atomic BAO's. For instance, results such as Proposition 3.17.2 and .3 and Proposition 3.22.3

[5]An element $c \neq \bot$ of a complete lattice L is *completely join prime* if, for every $S \subseteq L$, $c \leq s$ for some $s \in S$ whenever $c \leq \bigvee S$.

are essentially characterizations of objects across a duality. More in general, the relational interpretation of modal logic can be obtained by dualizing its canonical algebraic interpretation on BAO's. This modus operandi is not confined to modal logic: any duality involving the class of algebras canonically associated with a given propositional logic provides the appropriate setting for correspondence results.

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