Probabilistic Epistemic Updates on Algebras

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Abstract. The present paper contributes to the development of the mathematical theory of epistemic updates using the tools of duality theory. Here we focus on Probabilistic Dynamic Epistemic Logic (PDEL). We dually characterize the product update construction of PDEL-models as a certain construction transforming the complex algebras associated with the given model into the complex algebra associated with the updated model. Thanks to this construction, an interpretation of the language of PDEL can be defined on algebraic models based on Heyting algebras. This justifies our proposal for the axiomatization of the intuitionistic counterpart of PDEL.
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1 Introduction

The contributions of the present paper pertain to the research program, started in [MPS14,KP13] and continued in [GKP13,FGK$^+$14b,Riv14,BR15,FGK$^+$14a] [FGK$^+$14c,FGKP14], which is aimed at developing the mathematical theory of epistemic updates with the tools of duality theory.
The present paper lays the semantic ground for the introduction of a logical framework generalizing probabilistic dynamic epistemic logic (PDEL) [Koo03], [vBGK09]. The generalization concerns the following respects:
(a) weakening the underlying reasoning machinery from classical propositional logic to nonclassical formalisms (e.g. intuitionistic logic);
(b) generalizing the formal treatment of agents' epistemics by relaxing the requirement of normality for the epistemic modal operators;
(c) considering intuitionistic probability theory as the background framework for probabilistic reasoning.
A major motivation for (c) is the need to account for situations in which the probability of a certain proposition p is interpreted as an agent's propensity to bet on p given some evidence for or against p. If there is little or no evidence for

or against p, it should be reasonable to attribute low probability values to both p and $\neg p$, which is forbidden by classical probability theory (cf. [Wea03]).

A major motivations for (a) is the need to account for situations in which truth emerges as the outcome of a complex procedure (rather than e.g. being ascertained instantaneously). Examples of these situations are ubiquitous in social science. For instance, consider the case of the assessment of the authenticity of works of art. Turner's painting *The Beacon Light* is a case in point: after doubts had been cast on its being a genuine Turner, recent investigations into the materials and painting techniques have established its authenticity[4]. A fully fledged formalisation of such cases will be reported on in an extended version of the present paper [CFP$^+$]. By its main features, intuitionistic logic is particularly suited to account for situations like the one mentioned above, where truth is ascertained by means of a procedure (a 'proof'). Moreover, the intuitionistic environment allows for a finer-grained analysis when serving as a base for more expressive formalisms such as modal and dynamic logics. Indeed, the fact that the box-type and the diamond-type modalities are no longer interdefinable makes several mutually independent choices possible which cannot be disentangled in the classical setting. It should be remarked at this point that of course it is possible in principle to use formalisms based on classical propositional logic to analyse situations in which truth emerges as a social construct (e.g. the outcome of a procedure), and that an 'automatic' and powerful way of generating such a formalism is via Gödel-type encodings. However, the resulting treatment is significantly more cumbersome and ad hoc, and from a technical point of view such an encoding might destroy nice properties enjoyed by the original intuitionistic framework (see e.g. discussion at the end of [CGP14, Section 36.9]). Insisting on a Boolean propositional base could have been motivated by the need to rely on a well developed and solid mathematical environment. However, recent developments (cf. e.g. [CPS,CGP14,CP15,CFPS15,CC15,PSZ15a,PSZ15b,GMP$^+$15]) have made available a mathematical environment for non-classical logics[5] that is as advanced and solid as the classical one, and on which it is now possible to capitalise. Finally, these mathematical developments appear in tandem with interesting analyses on the philosophical side of formal logic (e.g. [AP14]), exploring epistemic logic in an evidentialist key, which is congenial with the kind of social situations targeted by our research programme.

Our methodology follows [MPS14,KP13], and is based on the dual characterization of the product update construction for standard PDEL-models as a certain construction transforming the complex algebras associated with a given model into the complex algebra associated with the updated model. This dual characterization naturally generalizes to much wider classes of algebras, which include,

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[4] cf. e.g. Darren Devine, *End to doubts over museum's Turner paintings as all found to be genuine*. Wales Online, 23 September 2012. Retrieved from http://www.walesonline.co.uk/news/wales-news/end-doubts-over-museums-turner-2024586 .

[5] By non-classical logics we mean logics the propositional base of which is weaker than classical propositional logic.

but are not limited to, arbitrary BAOs and arbitrary modal expansions of Heyting algebras (HAOs). Thanks to this construction, the benefits and the wider scope of applications given by a point-free, nonclassical theory of epistemic updates are made available: for instance, this construction makes it possible to derive the definition of product updates on topological spaces by means of an effective computation. As an application of this dual characterization, we present the axiomatization for the intuitionistic analogue of PDEL which arises semantically from this construction.
Structure of the paper: In Section 2, we give an alternative, two-step treatment of the PDEL-update on relational models. In Section 3, we expand on the methodology underlying the application of the duality toolkit. Section 4 is the main section, in which the construction of the PDEL-updates on Heyting algebras is introduced. In Section 5, we very briefly describe how the updates on algebras can be used to define the intuitionistic version of PDEL.

2 PDEL language and updates

In the present section, we report on the language of PDEL, and give an alternative, two-step account of the product update construction on PDEL-models. This account is similar to the treatment of epistemic updates in [MPS14,KP13], and as explained in Section 3, it lays the ground to the dualization procedure which motivates the construction introduced in Section 4. The specific PDEL framework we report on shares common features with those of [BCHS13,Ach14] and [vBGK09].

2.1 PDEL-formulas, event structures, and PES-models

In the remainder of the paper, we fix a countable set AtProp of proposition letters p, q and a set Ag of agents i. We let $\alpha_1, ..., \alpha_n, \beta$ denote rational numbers.

Definition 1. *The set $\mathcal{L}$ of* PDEL-formulas φ *and the class* $\mathsf{PEM}_{\mathcal{L}}$ *of* probabilistic event structures $\mathcal{E}$ *over $\mathcal{L}$ are built by simultaneous recursion as follows:*

$$\varphi ::= p \mid \bot \mid \varphi\wedge\varphi \mid \varphi\vee\varphi \mid \varphi \to \varphi \mid \Diamond_i\varphi \mid \Box_i\varphi \mid \langle\mathcal{E}, e\rangle\varphi \mid [\mathcal{E}, e]\varphi \mid (\sum_{k=1}^{n}\alpha_k\mu_i(\varphi)) \geq \beta.$$

The connectives $\top$, $\neg$, and $\leftrightarrow$ are defined by the usual abbreviations. A probabilistic event structure over $\mathcal{L}$ *is a tuple $\mathcal{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \mathsf{pre})$, such that E is a non-empty finite set, each $\sim_i$ is an equivalence relation on E, each $P_i : E \to [0,1]$ assigns a probability distribution on each $\sim_i$-equivalence class (i.e., $\sum\{P_i(e') : e' \sim_i e\} = 1$), Φ is a finite set of pairwise inconsistent $\mathcal{L}$-formulas, and* pre *assigns a probability distribution* $\mathsf{pre}(\bullet|\phi)$ *over E for every $\phi \in \Phi$.*

Informally, elements of E encode possible events, the relations $\sim_i$ encode as usual the epistemic uncertainty of the agent i, who assigns probability $P_i(e)$ to e being the actually occurring event, formulas in Φ are intended as the preconditions of the event, and $\mathsf{pre}(e|\phi)$ expresses the prior probability that the event $e \in E$ might occur in a(ny) state satisfying precondition ϕ. In what follows, we will refer to the structures $\mathcal{E}$ defined above as *event structures over* $\mathcal{L}$.

Definition 2. *A* probabilistic epistemic state model (PES-model) *is a structure* $\mathbb{M} = (S, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!])$ *such that* S *is a non-empty set, each* $\sim_i$ *is an equivalence relation on* S*, each* $P_i : S \to [0,1]$ *assigns a probability distribution on each* $\sim_i$*-equivalence class, (i.e.,* $\sum\{P_i(s') : s' \sim_i s\} = 1$*), and* $[\![\cdot]\!] : \mathsf{AtProp} \to \mathcal{P}S$ *is a valuation map.*

As usual, the map $[\![\cdot]\!]$ will be identified with its unique extension to $\mathcal{L}$, so that we will be able to write $[\![\varphi]\!]$ for every $\varphi \in \mathcal{L}$.

Notation 1. *For any probabilistic epistemic model* $\mathbb{M}$*, any probabilistic event structure* $\mathcal{E}$*, any* $s \in S$ *and* $e \in E$ *we let* $\mathsf{pre}(e \mid s)$ *denote the value* $\mathsf{pre}(e \mid \phi)$*, for the unique* $\phi \in \Phi$ *such that* $\mathbb{M}, s \Vdash \phi$ *(recall that the formulas in* Φ *are pairwise inconsistent). If no such* ϕ *exists then we let* $\mathsf{pre}(e \mid s) = 0$*.*

2.2 Epistemic updates

Throughout the present subsection, we fix a PES-model $\mathbb{M}$ and a probabilistic event structure $\mathcal{E}$ over $\mathcal{L}$. The updated model is given in two steps, the first of which is detailed in the following

Definition 3. *Let the* intermediate structure *of* $\mathbb{M}$ *and* $\mathcal{E}$ *be the tuple*

$$\coprod_{\mathcal{E}} \mathbb{M} := (\coprod_{|E|} S, (\sim_i^{\coprod})_{i\in\mathsf{Ag}}, (P_i^{\coprod})_{i\in\mathsf{Ag}}, [\![\cdot]\!]_{\coprod})$$

where $\coprod_{|E|} S \cong S \times E$ *is the* $|E|$*-fold coproduct of* S*, each binary relation* $\sim_i^{\coprod}$ *on* $\coprod_{|E|} S$ *is defined as follows:*

$$(s, e) \sim_i^{\coprod} (s', e') \quad \textit{iff } s \sim_i s' \textit{ and } e \sim_i e';$$

each map $P_i^{\coprod} : \coprod_{|E|} S \to [0,1]$ *is defined by* $(s, e) \mapsto P_i(s) \cdot P_i(e) \cdot \mathsf{pre}(e \mid s)$ *and* $[\![p]\!]_{\coprod} := \{(s, e) \mid s \in [\![p]\!]_{\mathbb{M}}\} = [\![p]\!]_{\mathbb{M}} \times E$ *for every* $p \in \mathsf{AtProp}$*.*

Remark 1. In general $P_i^{\coprod}$ does not induce probability distributions over the $\sim_i^{\coprod}$-equivalence classes. Hence, $\coprod_{\mathcal{E}} \mathbb{M}$ is not a PES-model.[6] However, the second step of the construction will yield a PES-model.

Finally, in order to define the updated model, observe that the map pre in $\mathcal{E}$ induces the map $pre : E \to \mathcal{L}$ defined by $\quad e \mapsto \bigvee\{\phi \in \Phi \mid \mathsf{pre}(e \mid \phi) \neq 0\}$.

[6] Indeed, Definition 9 will be introduced in Section 4 precisely with the purpose of capturing the dual of $P_i^{\coprod}$.

Definition 4. *For any PES-model* $\mathbb{M}$ *and any probabilistic event structure* $\mathcal{E}$ *over* $\mathcal{L}$*, let*

$$\mathbb{M}^{\mathcal{E}} := (S^{\mathcal{E}}, (\sim_i^{\mathcal{E}})_{i\in\mathsf{Ag}}, (P_i^{\mathcal{E}})_{i\in\mathsf{Ag}}, [\![\cdot]\!]_{\mathbb{M}^{\mathcal{E}}})$$

with

1. $S^{\mathcal{E}} := \{(s,e) \in \coprod_{|E|} S \mid \mathbb{M}, s \Vdash pre(e)\}$;
2. $[\![p]\!]_{\mathbb{M}^{\mathcal{E}}} := [\![p]\!]_{\coprod} \cap S^{\mathcal{E}}$;
3. $\sim_i^{\mathcal{E}} = \sim_i^{\coprod} \cap (S^{\mathcal{E}} \times S^{\mathcal{E}})$ *for any* $i \in \mathsf{Ag}$;
4. *each map* $P_i^{\mathcal{E}} : S^{\mathcal{E}} \to [0,1]$ *is defined by the assignment*

$$(s,e) \mapsto \frac{P_i^{\coprod}(s,e)}{\sum\{P_i^{\coprod}(s',e') \mid (s,e) \sim_i (s',e')\}}.$$

3 Methodology

In the present section, we expand on the methodology of the paper. In the previous section, we gave a two-step account of the *product update* construction which, for any PES-model $\mathbb{M}$ and any event model $\mathcal{E}$ over $\mathcal{L}$, yields the updated model $\mathbb{M}^{\mathcal{E}}$ as a certain *submodel* of a certain *intermediate model* $\coprod_{\mathcal{E}} \mathbb{M}$. This account is analogous to those given in [MPS14,KP13] of the product updates of models of PAL and Baltag-Moss-Solecki's dynamic epistemic logic EAK. In each instance, the original product update construction can be illustrated by the following diagram (which uses the notation introduced in the instance treated in the previous section):

$$\mathbb{M} \hookrightarrow \coprod_{\mathcal{E}} \mathbb{M} \hookleftarrow \mathbb{M}^{\mathcal{E}}.$$

As is well known (cf. e.g. [DP02]) in duality theory, coproducts can be dually characterized as products, and subobjects as quotients. In the light of this fact, the construction of *product update*, regarded as a "subobject after coproduct" concatenation, can be dually characterized on the algebras dual to the relational structures of PES-models by means of a "quotient after product" concatenation, as illustrated in the following diagram:

$$\mathbb{A} \twoheadleftarrow \prod_{\mathcal{E}} \mathbb{A} \twoheadrightarrow \mathbb{A}^{\mathcal{E}},$$

resulting in the following two-step process. First, the coproduct $\coprod_{\mathcal{E}} M$ is dually characterized as a certain *product* $\prod_{\mathcal{E}} \mathbb{A}$, indexed as well by the states of $\mathcal{E}$, and such that $\mathbb{A}$ is the algebraic dual of $\mathbb{M}$; second, an appropriate *quotient* of $\prod_{\mathcal{E}} \mathbb{A}$ is then taken, which dually characterizes the submodel step. On which algebras are we going to apply the "quotient after product" construction? The prime candidates are the algebras associated with the PES-models via standard Stone-type duality:

Definition 5. *For any PES-model* $\mathbb{M}$*, its complex algebra is the tuple*

$$\mathbb{M}^+ := \langle \mathcal{P}S, (\Diamond_i)_{i\in\mathsf{Ag}}, (\Box_i)_{i\in\mathsf{Ag}}, (P_i^+)_{i\in\mathsf{Ag}}\rangle$$

where for each $i \in \mathsf{Ag}$ *and* $X \in \mathcal{P}S$*,*

$$\begin{aligned}\Diamond_i X &= \{s \in S \mid \exists x(s \sim_i x \text{ and } x \in X)\},\\ \Box_i X &= \{s \in S \mid \forall x(s \sim_i x \implies x \in X)\},\\ \mathsf{dom}(P_i^+) &= \{X \in \mathcal{P}S \mid \exists y \forall x(x \in X \implies x \sim_i y)\}^7\\ P_i^+ X &= \textstyle\sum_{x\in X} P_i(x)\end{aligned}$$

In this setting, the "quotient after product" construction behaves exactly in the desired way, in the sense that one can check *a posteriori* that the following holds:[8]

Proposition 1. *For every PES-model* $\mathbb{M}$ *and any event structure* $\mathcal{E}$ *over* $\mathcal{L}$*, the algebraic structures* $(\mathbb{M}^+)^{\mathcal{E}}$ *and* $(\mathbb{M}^{\mathcal{E}})^+$ *can be identified.*

Moreover, the "quotient after product" construction holds in much greater generality than the class of complex algebras of PES-models, which is exactly its added value over the update on relational structures. In the following section, we are going to define it in detail in the setting of epistemic Heyting algebras.

4 Probabilistic dynamic epistemic updates on Heyting algebras

The present section aims at introducing the algebraic counterpart of the event update construction presented in Section 2.
For the sake of enforcing a neat separation between syntax and semantics, throughout the present section we will disregard the logical language $\mathcal{L}$, and work on *algebraic probabilistic epistemic structures* (APE-structures, cf. Definition 10) rather than on APE-models (i.e. APE-structures endowed with valuations). To be able to define the update construction, we will need to base our treatment on the following, modified definition of event structure over an algebra, rather than over $\mathcal{L}$:

Definition 6. *For any epistemic Heyting algebra* $\mathbb{A}$ *(cf. Definition 7), a* probabilistic event structure over $\mathbb{A}$ *is a tuple* $\mathbb{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \mathsf{pre})$ *such that* E, $\sim_i$, P_i *are as in Definition 1;* Φ *is a finite subset of* $\mathbb{A}$ *such that* $a_j \wedge a_k = \bot$ *for all* $a_i, a_j \in \Phi$ *such that* $a_i \neq a_j$*;* pre *assigns a probability distribution* $\mathsf{pre}(\bullet|a)$ *over* E *for every* $a \in \Phi$*.*

In what follows, we will typically refer to the structures defined above as *event structures*. In the next subsection, we introduce APE-structures based on epistemic Heyting algebras. In Subsection 4.2 we introduce the first step of the two-step update, namely, the 'product' construction. In Subsection 4.3, we introduce the second and final step, the 'quotient' construction.

[7] i.e. the domain of P_i^+ consists of all the subsets of the equivalence classes of $\sim_i$.
[8] Caveat: we are abusing notation here. Proposition 1 should be formulated using Definition 13 and Fact 2.

4.1 Algebraic probabilistic epistemic structures

Definition 7. *An* epistemic Heyting algebra *is a tuple* $\mathbb{A} := \langle \mathbb{L}, (\Diamond_i)_{i\in\mathsf{Ag}}, (\Box_i)_{i\in\mathsf{Ag}}\rangle$ *such that* $\mathbb{L}$ *is a Heyting algebra, and each* $\Diamond_i$ *and* $\Box_i$ *is a monotone unary operation on* $\mathbb{L}$ *such that for all* $a, b \in \mathbb{L}$,

$$\begin{array}{ll}
\Diamond_i(a \to b) \leq \Box_i a \to \Diamond_i b & \Diamond_i a \to \Box_i b \leq \Box_i(a \to b)\\
\Diamond_i a \wedge b \leq \Diamond_i(a \wedge \Diamond_i b) & \Box_i(a \vee \Box_i b) \leq \Box_i a \vee b\\
a \leq \Diamond_i a & \Box_i a \leq a\\
\Diamond_i \Diamond_i a \leq \Diamond_i a & \Box_i a \leq \Box_i \Box_i a.
\end{array}$$

In what follows, $\mathbb{A}$ will denote an epistemic Heyting algebra.

Definition 8. *An element* $a \in \mathbb{A}$ *is* i*-minimal if*

1. $a \neq \bot$,
2. $\Diamond_i a = a$ *and*
3. *if* $b \in \mathbb{A}$, $b < a$, *and* $\Diamond_i b = b$, *then* $b = \bot$.

Let $\mathsf{Min}_i(\mathbb{A})$ *denote the set of the* i*-minimal elements of* $\mathbb{A}$.

Notice that for any $b \in \mathbb{A} \setminus \{\bot\}$ there exists at most one $a \in \mathsf{Min}_i(\mathbb{A})$ such that $b \leq a$. Indeed every such a must coincide with $\Diamond_i b$. The next definition uses insights from [Wea03].

Definition 9. *A partial function* $\mu : \mathbb{A} \to \mathbb{R}^+$ *is an* i-premeasure *on* $\mathbb{A}$ *if* $\mathsf{dom}(\mu) = \mathsf{Min}_i(\mathbb{A}){\downarrow}$, *and* μ *is order-preserving,* $\mu(\bot) = 0$ *if* $\mathsf{dom}(\mu) \neq \varnothing$ *and for every* $a \in \mathsf{Min}_i(\mathbb{A})$ *and all* $b, c \in a{\downarrow}$ *it holds that* $\mu(b \vee c) = \mu(b) + \mu(c) - \mu(b \wedge c)$. *An* i*-premeasure on* $\mathbb{A}$ *is an* i-measure *if* $\mu(a) = 1$ *for every* $a \in \mathsf{Min}_i(\mathbb{A})$.

Definition 10. *An* algebraic pre-probabilistic epistemic structure (ApPE-structure) *is a tuple* $\mathcal{F} := \langle \mathbb{A}, (\mu_i)_{i\in\mathsf{Ag}}\rangle$ *such that* $\mathbb{A}$ *is an epistemic Heyting algebra (cf. Definition 8), and each* μ_i *is an* i*-premeasure on* $\mathbb{A}$. *An ApPE-structure* $\mathcal{F}$ *is an* algebraic probabilistic epistemic structure (APE-structure) *if each* μ_i *is a* i*-measure on* $\mathbb{A}$. *We refer to* $\mathbb{A}$ *as the* support *of* $\mathcal{F}$.

Lemma 1. *For any PES-model* $\mathbb{M}$, *the* i*-minimal elements of its complex algebra* $\mathbb{M}^+$ *are exactly the equivalence classes of* $\sim_i$.

Proposition 2. *For any PES-model* $\mathbb{M}$, *the complex algebra* $\mathbb{M}^+$ *(cf. Definition 5) is an APE-structure.*

4.2 The intermediate (pre-)probabilistic epistemic structure

In the present subsection, we define the intermediate ApPE-structure $\prod_{\mathbb{E}} \mathcal{F}$ associated with any APE-structure $\mathcal{F}$ and any event structure $\mathbb{E}$ over the support of $\mathcal{F}$ (cf. Definition 10 for the definition of support):

$$\prod_{\mathbb{E}} \mathcal{F} := \langle \prod_{|E|} \mathbb{A}, (\Diamond'_i)_{i\in\mathsf{Ag}}, (\Box'_i)_{i\in\mathsf{Ag}}, (\mu'_i)_{i\in\mathsf{Ag}}\rangle \tag{4.1}$$

Let us start by defining the algebra which will become the support of the intermediate APE-structure above:

Definition 11. *For every epistemic Heyting algebra* $\mathbb{A} = (\mathbb{L}, (\Diamond_i)_{i\in\mathsf{Ag}}, (\Box_i)_{i\in\mathsf{Ag}})$ *and every event structure* $\mathbb{E}$ *over* $\mathbb{A}$*, let*

$$\prod_{\mathbb{E}} \mathbb{A} := (\prod_{|E|} \mathbb{L}, (\Diamond'_i)_{i\in\mathsf{Ag}}, (\Box'_i)_{i\in\mathsf{Ag}}),$$

where

1. $\prod_{|E|} \mathbb{L}$ *is the* $|E|$*-fold power of* $\mathbb{L}$*, the elements of which can be seen either as* $|E|$*-tuples of elements in* $\mathbb{A}$*, or as maps* $f : E \to \mathbb{A}$*.*
2. *For any* $f : E \to \mathbb{A}$*, the map* $\Diamond'_i(f) : E \to \mathbb{A}$ *is defined by the assignment* $e \mapsto \bigvee\{\Diamond_i f(e') \mid e' \sim_i e\}$*;*
3. *For any* $f : E \to \mathbb{A}$*, the map* $\Box'_i(f) : E \to \mathbb{A}$ *is defined by the assignment* $e \mapsto \bigwedge\{\Box_i f(e') \mid e' \sim_i e\}$*.*

Below, the algebra $\prod_{\mathbb{E}} \mathbb{A}$ *will be sometimes abbreviated as* $\mathbb{A}'$*.*

We refer to [KP13, Section 3.1] for an extensive justification of the definition of the operations $\Diamond'_i$ and $\Box'_i$.

Proposition 3. *For every epistemic Heyting algebra* $\mathbb{A}$ *and every event structure* $\mathbb{E}$ *over* $\mathbb{A}$*, the algebra* $\mathbb{A}'$ *is an epistemic Heyting algebra.*

Proposition 4. *For every* $\mathbb{A}$ *and* i*,* $\mathsf{Min}_i(\mathbb{A}') = \{f_{e,a} \mid e \in E \text{ and } a \in \mathsf{Min}_i(\mathbb{A})\}$*, where for any* $e \in E$ *and* $a \in \mathsf{Min}_i(\mathbb{A})$*, the map* $f_{e,a} : E \to \mathbb{A}$ *is defined by the following assignment:*

$$e' \mapsto \begin{cases} a & \text{if } e' \sim_i e \\ \bot & \text{otherwise.} \end{cases}$$

Definition 12. *For any APE-structure* $\mathcal{F}$ *and any event structure* $\mathbb{E}$ *over the support of* $\mathcal{F}$*, let*

$$\prod_{\mathbb{E}} \mathcal{F} := \langle \prod_{\mathbb{E}} \mathbb{A}, (\mu'_i)_{i\in\mathsf{Ag}} \rangle$$

where

1. $\prod_{\mathbb{E}} \mathbb{A} = \mathbb{A}'$ *is defined as in Definition 11;*
2. *each* $\mu'_i : \mathbb{A}' \to [0,1]$ *is defined as follows:*

$$\mathsf{dom}(\mu'_i) = \mathsf{Min}_i(\mathbb{A}')\!\downarrow$$
$$\mu'_i(f) = \textstyle\sum_{e\in E} \sum_{a\in\Phi} P_i(e) \cdot \mu_i(f(e) \wedge a) \cdot \mathsf{pre}(e \mid a).$$

Proposition 5. *For every APE-structure* $\mathcal{F}$ *and every event structure* $\mathbb{E}$ *over the support of* $\mathcal{F}$*, the intermediate structure* $\prod_{\mathbb{E}} \mathcal{F}$ *is an ApPE-structure (cf. Definition 10).*

Proof. The proof that $\prod_{\mathbb{E}} \mathbb{A}$ is an epistemic Heyting algebra is entirely analogous to the proof of [KP13, Proposition 8], and is omitted. Let us assume that the domain of μ'_i is non-empty. By definition, μ'_i is order-preserving and $\mu'_i(\bot) = 0$. Finally, by Proposition 4, i-minimal elements of $\mathbb{A}'$ are of the form $f_{e,a} : E \to \mathbb{A}$

for some $e \in E$ and some i-minimal element $a \in \mathbb{A}$. Fix one such element, and let $g, h : E \to \mathbb{A}$ such that $g \vee h \leq f_{e,a}$. By definition, $f \leq f_{e,a}$ can be rewritten as $f(e') \leq f_{e,a}(e')$ for any $e' \in E$. Since $f_{e,a}(e') = \bot$ for any $e' \nsim_i e$, we can deduce that $g(e') = h(e') = \bot$ for any $e' \nsim_i e$. Hence,

$$
\begin{aligned}
& \mu'_i(g \vee h) \\
= & \sum_{e' \in E} \sum_{a \in \Phi} P_i(e') \cdot \mu_i((g(e') \vee h(e')) \wedge a) \cdot \mathsf{pre}(e' \mid a) && \text{(by definition)} \\
= & \sum_{e' \sim_i e} \sum_{a \in \Phi} P_i(e') \cdot \mu_i((g(e') \vee h(e')) \wedge a) \cdot \mathsf{pre}(e' \mid a) \\
& && (g(e') = h(e') = \bot \text{ for any } e' \nsim_i e \text{ and } \mu_i(\bot) = 0) \\
= & \sum_{e' \sim_i e} \sum_{a \in \Phi} P_i(e') \cdot \mu_i((g(e') \wedge a) \vee (h(e') \wedge a)) \cdot \mathsf{pre}(e' \mid a) && \text{(distributivity)} \\
= & \sum_{e' \sim_i e} \sum_{a \in \Phi} P_i(e') \cdot (\mu_i(g(e') \wedge a) + \mu_i(h(e') \wedge a) - \mu_i(g(e') \wedge h(e') \wedge a)) \cdot \mathsf{pre}(e' \mid a) \\
= & \sum_{e \in E} \sum_{a \in \Phi} P_i(e) \cdot (\mu_i(g(e) \wedge a) + \mu_i(h(e) \wedge a) - \mu_i(g(e) \wedge h(e) \wedge a)) \cdot \mathsf{pre}(e \mid a) \\
& && (\mu_i(\bot) = 0 \text{ by Definition 16 and } g(e') = h(e') = \bot \text{ for any } e' \nsim_i e) \\
= & \mu'_i(g) + \mu'_i(h) - \mu'_i(g \wedge h) && \text{(by definition)}
\end{aligned}
$$

Definition 13. *For any PES-model* $\mathbb{M}$ *and any event structure* $\mathcal{E} = (E, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, \Phi, \mathsf{pre})$ *over* $\mathcal{L}$*, let* $\mathbb{E}_{\mathcal{E}} := (E, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, \Phi_{\mathbb{M}}, \mathsf{pre}_{\mathbb{M}})$*, where* $\Phi_{\mathbb{M}} := \{[\![\phi]\!]_{\mathbb{M}} \mid \phi \in \Phi\}$*, and* $\mathsf{pre}_{\mathbb{M}}$ *assigns a probability distribution* $\mathsf{pre}(\bullet | a)$ *over* E *for every* $a \in \Phi_{\mathbb{M}}$*.*

Fact 2. *For any PES-model* $\mathbb{M}$ *and any event structure* $\mathcal{E}$ *over* $\mathcal{L}$*, the tuple* $\mathbb{E}_{\mathcal{E}}$ *is an event structure over the epistemic Heyting algebra underlying* $\mathbb{M}^+$*.*

Proposition 6. *For every PES-model* $\mathbb{M}$ *and any event structure* $\mathcal{E}$ *over* $\mathcal{L}$*,*

$$(\coprod_{\mathcal{E}} \mathbb{M})^+ \cong \prod_{\mathbb{E}_{\mathcal{E}}} \mathbb{M}^+.$$

4.3 The pseudo-quotient and the updated APE-structure

In the present subsection, we define the APE-structure $\mathcal{F}^{\mathbb{E}}$, resulting from the update of the APE-structure $\mathcal{F}$ with and the event structure $\mathbb{E}$ over the support of $\mathcal{F}$, by taking a suitable pseudo-quotient of the intermediate APE-structure $\prod_{\mathbb{E}} \mathcal{F}$. Some of the results which are relevant for the ensuing treatment (such as the characterization of the i-minimal elements in the pseudo-quotient) are independent of the fact that we will be working with the intermediate algebra. Therefore, in what follows, we will discuss them in the more general setting of arbitrary epistemic Heyting algebras $\mathbb{A}$:

Definition 14. *(cf. [MPS14, Sections 3.2, 3.3]) For any* $\mathbb{A}$ *and any* $a \in \mathbb{A}$, *let* $\mathbb{A}^a := (\mathbb{L}/{\cong_a}, (\Diamond_i^a)_{i\in\mathsf{Ag}}, (\Box_i^a)_{i\in\mathsf{Ag}})$, *where* $\cong_a$ *is defined as follows:* $b \cong_a c$ *iff* $b \wedge a = c \wedge a$ *for all* $b, c \in \mathbb{L}$, *each operation* $\Diamond_i^a$ *is defined by the assignment* $\Diamond_i^a[b] := [\Diamond_i(b \wedge a)]$ *and each operation* $\Box_i^a$ *is defined by the assignment* $\Box_i^a[b] := [\Box_i(a \to b)]$, *where* $[c]$ *denotes the* $\cong_a$*-equivalence class of any given* $c \in \mathbb{L}$.

Proposition 7. *(cf. [MPS14, Fact 12]) The algebra* $\mathbb{A}^a$ *of Definition 14 is an epistemic Heyting algebra.*

Proposition 8. *The following are equivalent for any* $\mathbb{A}$ *and any* $a \in \mathbb{A}$:

1. $[b] \in \mathsf{Min}_i(\mathbb{A}^a)$;
2. $[b] = [b']$ *for a unique* $b' \in \mathsf{Min}_i(\mathbb{A})$ *such that* $b' \wedge a \neq \bot$.

Hence, in what follows, whenever $[b] \in \mathsf{Min}_i(\mathbb{A}^a)$, we will assume w.l.o.g. that $b \in \mathsf{Min}_i(\mathbb{A})$ is the "canonical" (in the sense of Proposition 8) representant of $[b]$. For any APE-structure $\mathcal{F}$ and any event structure $\mathbb{E}$ over the support $\mathbb{A}$ of $\mathcal{F}$, the map pre in $\mathbb{E}$ induces the map $\overline{pre} : E \to \mathbb{A}$ defined by $e \mapsto \bigvee_{\substack{a\in\Phi \\ \mathsf{pre}(e|a)\neq 0}} a$.

It immediately follows from Propositions 4 and 8 that the i-minimal elements of $\mathbb{A}^{\mathbb{E}}$ are exactly the elements $[f_{e,a}]$ for $e \in E$ and $a \in \mathsf{Min}_i(\mathbb{A})$ such that $a \wedge \overline{pre}(e') \neq \bot$ for some $e' \sim_i e$.

Definition 15. *For any APE-structure* $\mathcal{F}$ *and any event structure* $\mathbb{E}$ *over the support of* $\mathcal{F}$, *the* updated APE-structure *is the tuple* $\mathcal{F}^{\mathbb{E}} := (\mathbb{A}^{\mathbb{E}}, (\mu_i^{\mathbb{E}})_{i\in\mathsf{Ag}})$, *s.t.:*

1. $\mathbb{A}^{\mathbb{E}} := (\prod_{\mathbb{E}} \mathbb{A})^{\overline{pre}}$, *i.e.* $\mathbb{A}^{\mathbb{E}}$ *is obtained by instantiating Definition 14 to* $\prod_{\mathbb{E}} \mathbb{A}$ *and* $\overline{pre} \in \prod_{\mathbb{E}} \mathbb{A}$;
2. $\mathsf{dom}(\mu_i^{\mathbb{E}}) = \mathsf{Min}_i(\mathbb{A}^{\mathbb{E}}){\downarrow}$ *for each partial map* $\mu_i^{\mathbb{E}} : \mathbb{A}^{\mathbb{E}} \to [0,1]$ *and* $\mu_i^{\mathbb{E}}([g]) := \frac{\mu_i'(g)}{\mu_i'(f)}$ *for every* $[g] \in \mathsf{dom}(\mu_i^{\mathbb{E}})$ *where* $[g] \leq [f]$ *for some* $[f] \in \mathsf{Min}_i(\mathbb{A}^{\mathbb{E}})$.

Notice that if $[g] \neq \bot$ then $[f]$ is unique (cf. discussion after Definition 8). If $[g] = \bot$ then $\mu'(g) = 0$. Hence the above is well-defined.

Proposition 9. *For any APE-structure* $\mathcal{F}$ *and any event structure* $\mathbb{E}$ *over the support of* $\mathcal{F}$, *the tuple* $\mathcal{F}^{\mathbb{E}}$ *is an APE-structure.*

Proof. By Proposition 7, $\mathbb{A}^{\mathbb{E}}$ is an epistemic Heyting algebra. Let us assume that the domain of $\mu_i^{\mathbb{E}}$ is non-empty. To finish the proof, it remains to be shown that each partial map $\mu_i^{\mathbb{E}}$ satisfies the conditions of Definition 9. Clearly, $\mu_i^{\mathbb{E}}(\bot) = 0$ and $\mu_i^{\mathbb{E}}([f]) = 1$ for all $[f] \in \mathsf{Min}_i(\mathbb{A}^{\mathbb{E}})$.
To argue that $\mu_i^{\mathbb{E}}$ is monotone, observe preliminarily that $\mu_i'(g) = \mu_i'(g \wedge \overline{pre})$. This follows by the definition of μ_i' and the fact that if $\mathsf{pre}(e \mid a) \neq 0$ then $a \leq \overline{pre}(e)$. Assume that $[g_1] \leq [g_2] \leq [f_{e,a}]$. This means that $g_1 \wedge \overline{pre} \leq g_2 \wedge \overline{pre}$. Since μ_i' is monotone, $\mu_i'(g_1) = \mu_i'(g_1 \wedge \overline{pre}) \leq \mu_i'(g_2 \wedge \overline{pre}) = \mu_i(g_2)$. This implies that

$$\frac{\mu_i'(g_1)}{\mu_i'(f_{e,a})} \leq \frac{\mu_i'(g_2)}{\mu_i'(f_{e,a})}$$

that is, $\mu_i^{\mathbb{E}}([g_1]) \leq \mu_i^{\mathbb{E}}([g_2])$.

As for the last condition, let $[g_1]$ and $[g_2]$ in $\mathcal{F}^{\mathbb{E}}$ such that $[g_1] \leq [f_{e,a}]$ and $[g_2] \leq [f_{e,a}]$. We have:

$$
\begin{aligned}
\mu_i^{\mathbb{E}}([g_1] \vee [g_2]) &= \frac{\mu_i'((g_1 \wedge \overline{pre}) \vee (g_2 \wedge \overline{pre}))}{\mu_i'(f_{e,a})} \\
&= \frac{\mu_i'(g_1 \wedge \overline{pre}) + \mu_i'(g_2 \wedge \overline{pre}) - \mu_i'((g_1 \wedge g_2) \wedge \overline{pre})}{\mu_i'(f_{e,a})} \\
&= \frac{\mu_i'(g_1 \wedge \overline{pre})}{\mu_i'(f_{e,a})} + \frac{\mu_i'(g_2 \wedge \overline{pre})}{\mu_i'(f_{e,a})} - \frac{\mu_i'((g_1 \wedge g_2) \wedge \overline{pre})}{\mu_i'(f_{e,a})} \\
&= \frac{\mu_i'(g_1)}{\mu_i'(f_{e,a})} + \frac{\mu_i'(g_2)}{\mu_i'(f_{e,a})} - \frac{\mu_i'(g_1 \wedge g_2)}{\mu_i'(f_{e,a})} \\
&= \mu_i^{\mathbb{E}}([g_1]) + \mu_i^{\mathbb{E}}([g_2]) - \mu_i^{\mathbb{E}}([g_1 \wedge g_2]).
\end{aligned}
$$

Lemma 2. *For any PES-model $\mathbb{M}$ and any event structure $\mathcal{E}$ over $\mathcal{L}$,*

$$(P_i^+)^{\mathbb{E}_\mathcal{E}} = (P_i^\mathcal{E})^+.$$

Proposition 1 follows from the above lemma and [KP13, Proposition 3.6].

5 PDEL, intuitionistically

In the present section, we apply the update construction on algebras introduced in the previous section to the definition of the intuitionistic counterpart of PDEL.

Definition 16. Algebraic probabilistic epistemic models (APE-models) *are tuples $\mathcal{M} = \langle \mathcal{F}, v\rangle$ s.t. $\mathcal{F} = \langle \mathbb{A}, (\mu_i)_{i\in\mathsf{Ag}}\rangle$ is an APE-structure, and $v : \mathsf{AtProp} \to \mathbb{A}$.*

The update construction of Section 4 extends from APE-structures to APE-models. Indeed, for any APE-model $\mathcal{M}$ and any event structure $\mathcal{E}$ over $\mathcal{L}$ (cf. Definition 1), the following tuple is an event structure over $\mathbb{A}$:

$$\mathbb{E}_\mathcal{E} := (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi_\mathcal{M}, \mathsf{pre}_\mathcal{M}),$$

where $\Phi_\mathcal{M} := \{[\![\phi]\!]_\mathcal{M} \mid \phi \in \Phi\}$[9], and $\mathsf{pre}_\mathcal{M}$ assigns a probability distribution $\mathsf{pre}(\bullet|a)$ over E for every $a \in \Phi_\mathcal{M}$. Then,

$$\mathcal{M}^\mathcal{E} := \langle \mathcal{F}^\mathcal{E}, v^\mathcal{E}\rangle,$$

where $\mathcal{F}^\mathcal{E} := \mathcal{F}^{\mathbb{E}_\mathcal{E}}$ as in Definition 15, and $v^\mathcal{E}(p) = [v^\Pi(p)]$ for every $p \in \mathsf{AtProp}$, where $v^\Pi(p) : E \to \mathbb{A}$ is defined by the assignment $e \mapsto v(p)$. For every $e \in E$, let $\pi_e : \prod_{\mathbb{E}_\mathcal{E}} \mathbb{A} \to \mathbb{A}$ be the eth projection; also, let $\pi : \prod_{\mathbb{E}_\mathcal{E}} \mathbb{A} \to \mathbb{A}^{\mathbb{E}_\mathcal{E}}$ be the quotient map. As explained in [MPS14, Section 3.2], the map $\iota : \mathbb{A}^{\mathbb{E}_\mathcal{E}} \to \prod_{\mathbb{E}_\mathcal{E}} \mathbb{A}$ defined by the assignment $[g] \mapsto g \wedge \overline{pre}$ is well defined.

[9] Caveat: the definition of $\mathbb{E}_\mathcal{E}$ should more appropriately be given by simultaneous induction together with the interpretation of formulas.

Definition 17. *The interpretation of $\mathcal{L}$-formulas on any APE-model $\mathcal{M}$ is defined recursively as follows:*

$$
\begin{array}{ll}
[\![p]\!]_{\mathcal{M}} = v(p) & [\![\varphi \to \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \to^{\mathbb{A}} [\![\psi]\!]_{\mathcal{M}} \\
[\![\bot]\!]_{\mathcal{M}} = \bot^{\mathbb{A}} & [\![\top]\!]_{\mathcal{M}} = \top^{\mathbb{A}} \\
[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \wedge^{\mathbb{A}} [\![\psi]\!]_{\mathcal{M}} & [\![\varphi \vee \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \vee^{\mathbb{A}} [\![\psi]\!]_{\mathcal{M}} \\
[\![\Diamond_i \varphi]\!]_{\mathcal{M}} = \Diamond_i [\![\varphi]\!]_{\mathcal{M}} & [\![\Box_i \varphi]\!]_{\mathcal{M}} = \Box_i [\![\varphi]\!]_{\mathcal{M}} \\
[\![\langle \mathcal{E}, e\rangle \varphi]\!]_{\mathcal{M}} = [\![\overline{pre}(e)]\!]_{\mathcal{M}} \wedge^{\mathbb{A}} \pi_e \circ \iota([\![\varphi]\!]_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}}) & [\![[\mathcal{E}, e]\varphi]\!]_{\mathcal{M}} = [\![\overline{pre}(e)]\!]_{\mathcal{M}} \to^{\mathbb{A}} \pi_e \circ \iota([\![\varphi]\!]_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}})
\end{array}
$$

$$[\![(\textstyle\sum_{k=1}^n \alpha_k \mu_i(\varphi_k)) \geq \beta]\!]_{\mathcal{M}} = \bigvee \{a \in \mathbb{A} \mid a \in \mathsf{Min}_i(\mathbb{A}) \textit{ and } (\textstyle\sum_{k=1}^n \alpha_k \mu_i([\![\varphi_k]\!]_{\mathcal{M}} \wedge a)) \geq \beta\}$$

The following axioms are sound on APE-models under the interpretation above:

$$\langle \mathcal{E}, e\rangle (\sum_{k=1}^{n} \alpha_k \mu_i(\varphi_k) \geq \beta) \leftrightarrow Pre(e) \wedge \Big(\sum_{k=1}^{n} \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \alpha_k \cdot P_i(e') \cdot pre(e' \mid \phi) \mu_i(\phi \wedge \langle \mathcal{E}, e'\rangle \varphi_k) + \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta \cdot P_i(e') \cdot pre(e' \mid \phi) \mu_i(\phi) \geq 0 \Big)$$

$$[\mathcal{E}, e] (\sum_{k=1}^{n} \alpha_k \mu_i(\varphi_k) \geq \beta) \leftrightarrow Pre(e) \to \Big(\sum_{k=1}^{n} \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \alpha_k \cdot P_i(e') \cdot pre(e' \mid \phi) \mu_i(\phi \wedge [\mathcal{E}, e'] \varphi_k) + \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta \cdot P_i(e') \cdot pre(e' \mid \phi) \mu_i(\phi) \geq 0 \Big)$$

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