# Chapter 1
# Unified Correspondence

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**Abstract** The present paper is aimed at giving a conceptual exposition of the mathematical principles underlying Sahlqvist correspondence theory. These principles are argued to be inherently algebraic and order-theoretic. They translate naturally on relational structures thanks to Stone-type duality theory. The availability of this analysis in the setting of the algebras dual to relational models leads naturally to the definition of an *expanded* (object) language in which the well-known 'minimal valuation' meta-arguments can be encoded, and of a *calculus for correspondence* of a proof-theoretic style in the expanded language, mechanically computing the first-order correspondent of given propositional formulas. The main advantage brought about by this formal machinery is that correspondence theory can be ported in a uniform way to families of nonclassical logics, ranging from substructural logics to mu-calculi, and also to different semantics for the same logic, paving the way to a uniform correspondence theory.

Keywords: Sahlqvist Correspondence theory, Duality, algorithmic correspondence, intuitionistic modal logic, mu-calculus

> Correspondence Theory may be applied to any kind of semantic entity.
> [J. van Benthem, Correspondence theory, Handbook of Philosophical Logic, p. 381]

## 1.1 Introduction

Correspondence theory has been a core interest of van Benthem's since early in his career, and is the field to which his most celebrated result in mathematical logic—the *van Benthem Characterization Theorem*—belongs. Throughout his subsequent career, he has been pointing out various correspondence phenomena embedded in his many research interests, which he collected e.g. in [5], [6], and [7]. Most recently, [8] ties in with his current interests in information flow. van Benthem has always been eager to point out unexplored research directions, and these papers are no exception. The correspondence phenomena he identified are often *fringe* phenomena, in the sense that they are clearly recognizable as *instances* of correspondence, but are not embedded in a systematic theory, see especially [5]. We are now in a position to bring the fringe to the core and build a unifying theory around these scattered instances. Clearly, such an encompassing theory cannot be unfolded in the scope of the present paper; our objectives are more modest, and are:
(a) to give a conceptual exposition of the mathematical principles underlying the correspondence mechanism, and how these principles work uniformly across different logics and also across different semantics for the same logic;
(b) to give pointers to the recent literature, and to mention the most important directions in which correspondence theory has been extended;
(c) to give a second reading to van Benthem's fringe examples, to show how the general principles identified in item (a) are still at work in these examples, and to point at ways in which the general theory accounts for them.

## 1.2 Correspondence via Duality

Relational semantics for modal logic provides a very clear understanding of what modal axioms mean in many different contexts of application, and is the essential reason why modal logic has become the successful formalism it is. With the introduction of Kripke semantics in the early 1960s, modal logic found itself in a very special position among non-classical logic, thanks to the fact that relational structures can be used *both* as semantics for modal logic *and* for classical first-order logic. This common semantic ground immediately elicited a whole research programme in the model theory of modal logic, focusing on its *expressivity*. A high point of this programme was of course van Benthem's theorem characterizing modal logic as the bisimulation invariant fragment of first-order logic [4].

A host of simple but insightful connections started to pop up between modal axioms which have been previously and independently studied (e.g. in formal philosophy), and basic properties of relational structures, such as reflexivity or transitivity. These connections are established via the notion of *local validity* of a modal formula in a relational structure, i.e., of that formula being satisfied at a given state *for every* valuation. The style of argument used to establish each of these connections is fairly uniform, so let us briefly review how this is done by way of one such example.

*Example 1.* The following are equivalent for any relational structure $\mathcal{F} = (W, R)$ and any $w \in W$:

1. The modal formula $\Box p \to \Box\Box p$ is true in $\mathcal{F}$ at $w$ under all assignments $V(p) \subseteq W$;
2. $\mathcal{F}$ satisfies the first-order formula $\alpha(x)$ expressing the inclusion $R[R[x]] \subseteq R[x]$ whenever the free variable $x$ is interpreted as $w$.[1]

*Proof.* For the interesting direction, i.e., assume 1 and prove 2, we need to assume that there are states $u$ and $v$ s.t. $wRu$ and $uRv$, and show that $wRv$. Consider the assignment $V^*(p) := R[w]$; this is the smallest assignment of $p$ under which the antecedent of $\Box p \to \Box\Box p$ is true. Hence, by modus ponens $w$ must satisfy also the conclusion $\Box\Box p$ under the same assignment, which implies that $u$ satisfies $\Box p$ under $V^*$, which implies that $v \in V^*(p) = R[w]$.

The Sahlqvist formulas, introduced by Hendrik Sahlqvist [45] and further developed by van Benthem [4] and others, form the best known class of modal formulas whose syntactic shape makes it possible for similar proof arguments to succeed.

*New perspective.* So what is special about the 'Sahlqvist shape', and how can we systematically recognize and reproduce it in the syntax of other, non-modal logics? The aim of the present paper is illustrating that the answers to these questions are inherently *algebraic* and *order-theoretic*. Taking this perspective has the advantage of endowing correspondence results with greater generality between logics and enhanced portability to different semantics. Such a claim of course requires elaborate justification, and it is our hope that the reader will be convinced of this by the end of the paper. For now, let us say the following: modal logic, like all propositional logics, can be interpreted into *algebras* in a *canonical* way, in the same sense in which first-order logic is interpreted into relational structures in a canonical way. On the other hand, the interpretation of modal formulas into relational structures seems to offer some degrees of choice; for instance, one could use either the forward or backward direction of the relations to interpret the modal operators. The relational models alone do not seem to provide enough justification to establish that the usual interpretation of modal formulas into relational structures is canonical in the informal sense. This looks like a fundamental asymmetry between the algebraic and relational semantics of modal logic. *Symmetry* is restored, in a sense, if we allow *Stone duality* (between complete atomic modal algebras and Kripke frames) to enter the picture: indeed, the relational interpretation of modal formulas is uniquely identified as the *dual characterization* of its interpretation on algebras. Hence, its being canonical can be derived as a consequence of this strong link, and of the canonicity of the algebraic interpretation. This is pictured in Figure 1.1(b).

This discussion provides a general illustration of how, thanks to duality, the advantages of the algebraic perspective on modal logic can be transferred to Kripke frames. But more specifically, the link between the relational and the algebraic interpretation of modal logic directly invests correspondence theory: indeed, thanks

[1] For $x \in W$ we let $R[x] = \{v \in W \mid Rxv\}$, and for $X \subseteq W$ we let $R[X] = \bigcup\{R[x] \mid x \in X\}$.

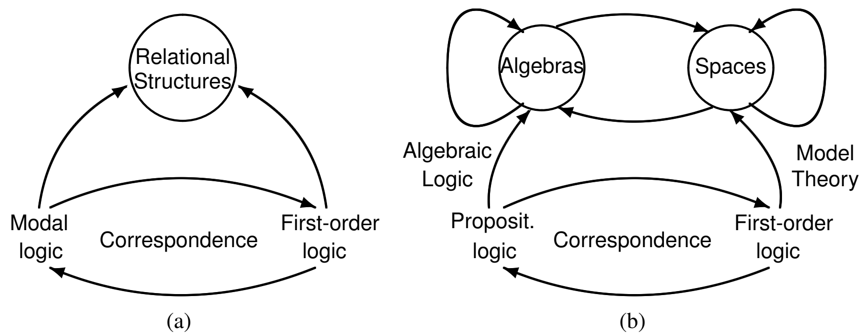


Fig. 1.1: From a model-theoretic (a) to a duality-based (b) approach to correspondence.

to it, we will be able to show that the Sahlqvist-style correspondence mechanism is driven by properties which naturally live in the *algebraic* side of diagram (b).

Finally, having been able to recognize modal correspondence theory as part of the logical fallout of the specific duality between the algebraic and the set-based semantics of modal logic, it will also become clear that correspondence theory is by no means unique to modal logic, and is uniformly available in great generality to all (classes of) propositional logics for which such dualities are available. Before being able to motivate these conclusions, let us take a step back, and resume the example we started with.

*Example, continued.* The proof in Example 1 gives an illustration of the so-called *minimal valuation argument*: assuming that a modal formula $\varphi \to \psi$ is locally valid at a given state $w$, we instantiate with the *minimal valuation* which satisfies $\varphi$ at $w$. In fact, this argument is a special case of a more general reasoning pattern, which is typically employed when proving the equivalence of the following statements:

(1) for every assignment $V$, if $\mathcal{F}, V, w \Vdash \varphi$ then $\mathcal{F}, V, w \Vdash \psi$;

(2) for every assignment $V^*$ ranging in a given subclass $K$, if $\mathcal{F}, V^*, w \Vdash \varphi$ then $\mathcal{F}, V^*, w \Vdash \psi$.

The equivalence between (1) and (2), for a suitable choice of $K$, is the crucial requirement on which the local correspondence mechanism is grounded. Indeed, (1) is just a reformulation of $\varphi \to \psi$ being locally valid at a given state $w$. If $K$ is a class of assignments $V^*$ mapping each proposition variable to a subset of $W$ which admits a *uniform description* (for instance, in the case of our example above, $K$ can be taken as the set of all the assignments such that either $V(p) = \varnothing$ or $V(p) = R[v]$ for some $v \in W$), and if further there are only finitely many members of $K$ relevant for any given state $w$, then (2) can be further manipulated[2] into an equivalent condition in the language to which this uniform description belongs. This is done by orderly substituting the predicate variables (ranging over *arbitrary* subsets) with

[2] This aspect of the story deserves a separate account, which will be given in section 1.6.

formal descriptions, in the target language, of *definable* subsets. So, for instance, if these uniform descriptions are expressible in the first-order language of $\mathcal{F}$ (as is the case of the example above), then the equivalent condition will yield a local first-order correspondent of $\varphi \to \psi$; if the descriptions are expressible in the first-order language of $\mathcal{F}$ enriched with fixed points (as is the case of the Löb formula), then we will have local correspondence with first-order logic with least fixed points. Thus, in each context, this uniform descriptions for the assignments in $K$ targets the language we want to establish modal correspondence with, which explains why we will refer to the class $K$ which we choose in each particular context as the class of *domesticated* assignments, as opposed to the *arbitrary* assignments, which roam wildly and for which no such description is available.

Having indicated that the equivalence between (1) and (2) is the crux of the matter, let us take a closer look at it. It is immediate that (1) always implies (2). It is also clear that the converse direction is false in its full generality, and our being able to prove it depends on our being able to find, for a given arbitrary assignment $V$ such that $w \in [\![\varphi]\!]_V$ (where $[\![\varphi]\!]_V$ denotes the extension of $\varphi$ in $\mathcal{F}$ under the assignment $V$), a *domesticated* assignment $V^*$ such that $w \in [\![\varphi]\!]_{V^*}$ and $[\![\psi]\!]_{V^*} \subseteq [\![\psi]\!]_V$. The latter requirement is typically achieved by assuming that the extension function induced by $\psi$ is *monotone*, and defining $V^*$ so that $V^*(p) \subseteq V(p)$ for all the relevant proposition variables. Therefore, the two sufficient requirements on $V^*$ in order for the equivalence between (1) and (2) to go through are:

$$w \in [\![\varphi]\!]_{V^*} \quad \text{and} \quad V^*(p) \subseteq V(p). \tag{1.1}$$

In all the different contexts in which (both the scattered instances of and the systematic) correspondence results hold, the *general strategy* to find this domesticated assignment $V^*$ can be described as follows[3]: for each relevant variable $p$ and every $w \in [\![\varphi]\!]_V$, the required domesticated $V^*$ is defined by stipulating $V^*(p) := [\![\alpha]\!]_V \subseteq V(p)$ for some suitable (modal) formula $\alpha$. It is often the case that $\alpha$ does *not* belong to the original language. To fix ideas, let us review what happens in the proof that (2) implies (1) when $\varphi \to \psi$ is the formula $\Box p \to \Box\Box p$ in the Example 1: fix an arbitrary $V$ such that $w \in [\![\Box p]\!]_V$; then the following chain of equivalences holds:

$$\begin{aligned} w \in [\![\Box p]\!]_V \ & \text{iff } \{w\} \subseteq \Box_R V(p) \\ & \text{iff } \{w\} \subseteq (R^{-1}[V(p)^c])^c \\ & \text{iff } \{w\} \cap R^{-1}[V(p)^c] = \varnothing \\ & \text{iff } R[\{w\}] \cap V(p)^c = \varnothing \\ & \text{iff } R[w] \subseteq V(p). \end{aligned} \tag{1.2}$$

The above chain of equivalences effectively rewrites our assumptions into a workable choice of domesticated valuation: indeed, it says that a valuation satisfies the antecedent of our given formula at $w$ iff it assigns $p$ to a superset of $R[w]$. This immediately implies that the set of valuations satisfying the antecedent of our given

[3] It is certainly not the only way to describe the correspondence mechanism, but it is useful for our purposes.

formula at $w$ (ordered pointwise) has a *minimum*, given by the valuation $V^*$ assigning $V^*(p) := R[w]$ and $\varnothing$ to all the other variables. This valuation clearly satisfies both requirements in clause (1.1), which as discussed, are sufficient condition for the equivalence between (1) and (2) to be established.

But, more interestingly, for which $\alpha$ can we identify the set $R[w]$ as $[\![\alpha]\!]_V$? Or more precisely, how should we expand our base language (and hence also our original assignments $V$) so that we get $V^*(p) = [\![\alpha]\!]_V$? This example shows that we certainly need to expand our language with at least the following two types of syntactic ingredients:
(a) ingredients which enable us to speak about *singletons*;
(b) ingredients which enable us to speak about *direct R-images of subsets*.

As to (b), it is well known that, for every subset $X$ (which might be in particular a singleton), the assignment $X \mapsto R[X]$ provides the interpretation for the *backward-looking diamond* $\blacklozenge$, which is interpreted by the semantic diamond associated with $R^{-1}$. This is a well known situation in modal *tense* logic, where the backward-looking modalities belong to the base language; for the modal languages in which this is not the case, the backward-looking modalities will be added to the expanded language.
As to (a), the most convenient way for us to speak about singletons is to introduce a special sort of variables $\mathbf{h}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \ldots$ in the extended modal language, which are to be interpreted as singletons; we call them *nominals*, after the analogous devices adopted in hybrid logic. In sections 1.3 and 1.4, the expanded language will be discussed more formally and generally. For the moment, we only remark that nominals, interpreted as singletons, make it possible to encode *local* satisfaction of modal formulas as *global* satisfaction of certain inequalities, as follows:

$$\mathcal{F}, V, w \Vdash \varphi \quad \text{iff} \quad \mathcal{F}, V_{\mathbf{j}:=w} \Vdash (\mathbf{j} \leq \varphi), \tag{1.3}$$

where $V_{\mathbf{j}:=w}$ is the extended $\mathbf{j}$-variant of $V$ sending $\mathbf{j}$ to $\{w\}$, and for every valuation $V$ and formulas $\psi$ and $\chi$ we write $\mathcal{F}, V \Vdash \psi \leq \chi$ to indicate that $[\![\psi]\!]_V \subseteq [\![\chi]\!]_V$.

Given both types of syntactic ingredients, we can stipulate in the example above $V^*(p) := [\![\alpha]\!]_V$ for $\alpha = \blacklozenge\mathbf{j}$. Notice that the introduction of this language expansion is harmless w.r.t. our target language: the standard translation of formulas in the language expanded with both nominals and backward-looking modalities falls within the basic first-order frame language. But more interestingly, which advantages does this expanded language bring to us?

Firstly, we have gained a better calculus: for instance, the equivalence between the beginning and the end of the (rather clumsy) chain of set-theoretic equivalences (1.2) above can be justified in one line as the following instance of the well known tense axiomatics:

$$\mathbf{j} \leq \Box p \quad \text{iff} \quad \blacklozenge\mathbf{j} \leq p. \tag{1.4}$$

Secondly and more importantly, we have defined a formal setting in which the computation of the minimal valuation is internalized at the level of a suitable object language. Indeed, the left-to-right direction of (1.4) provides us with the minimal valuation $V^*(p)$ which is expressed in the extended language as $\blacklozenge\mathbf{j}$. This enables us to proceed to a full *mechanization of the minimal valuation argument*. Indeed, after computing the needed minimal valuation as above, the actual instantiation with this valuation is facilitated and justified by the following version of *Ackermann's lemma* [2]:

**Lemma 1.** *Let $\alpha, \beta(p), \gamma(p)$ be formulas of a (modal) language $\mathcal{L}^+$ over the set of variables* $\mathsf{PROP}$*; let $p \in \mathsf{PROP}$ such that $p$ does not occur free in $\alpha$, $\beta$ is negative in $p$ and $\gamma$ is positive in $p$, then the following are equivalent for every $\mathcal{L}^+$-Kripke frame $\mathcal{F}$:*

(a) $\mathcal{F} \Vdash (\alpha \leq p \Rightarrow \beta(p) \leq \gamma(p))$;
(b) $\mathcal{F} \Vdash \beta(\alpha/p) \leq \gamma(\alpha/p)$.

*Proof.* For every formula $\varphi$ and valuation $V$, let $[\![\varphi[p]]\!]_V$ be the unary operation on $\mathcal{P}(W)$ sending $X \in \mathcal{P}(W)$ to $[\![\varphi]\!]_{V'}$ where $V'$ is the $p$-variant of $V$ sending $p$ to $X$.

As to the direction from (a) to (b): assume contrapositively that $[\![\beta(\alpha/p)]\!]_V \nsubseteq [\![\gamma(\alpha/p)]\!]_V$ for some valuation $V$. Let $V^*$ be the $p$-variant of $V$ such that $V^*(p) := [\![\alpha]\!]_V$. Then, because the variable $p$ does not occur in $\alpha$, we have $[\![\alpha]\!]_{V^*} = [\![\alpha]\!]_V = V^*(p)$, which proves that $\mathcal{F}, V^* \Vdash \alpha \leq p$. However, for every formula $\xi$, the following chain of equalities holds: $[\![\xi(p)]\!]_{V^*} = [\![\xi[p]]\!]_{V^*}(V^*(p)) = [\![\xi[p]]\!]_{V^*}([\![\alpha]\!]_V) = [\![\xi(\alpha/p)]\!]_V$. This and the contrapositive assumption prove that $\mathcal{F}, V^* \nVdash \beta(p) \leq \gamma(p)$. Conversely, assume that $\mathcal{F} \Vdash \beta(\alpha/p) \leq \gamma(\alpha/p)$, and let $V$ be such that $[\![\alpha]\!]_V \subseteq V(p)$. Then, since $\beta$ and $\gamma$ are respectively negative and positive in $p$, we have: $[\![\beta(p)]\!]_V \subseteq [\![\beta(\alpha/p)]\!]_V \subseteq [\![\gamma(\alpha/p)]\!]_V \subseteq [\![\gamma(p)]\!]_V$, which proves that $\mathcal{F}, V \Vdash \beta(p) \leq \gamma(p)$.

The proof of the direction (a) $\Rightarrow$ (b) in the lemma above encodes the minimal valuation argument in a very general way. This provides us with a crucial step towards mechanizing the correspondence process via the elimination of variables, as we can now simply appeal to the lemma instead of making an ad hoc minimal valuation argument. Notice also that the lemma does not depend on the particular choice of language $\mathcal{L}^+$.[4] Besides the assumptions of monotonicity/antitonicity of the interpretation of formulas, the only requirement encoded in the proof is that the minimal valuation be defined in terms of the resources of $\mathcal{L}^+$.

*Towards a calculus for correspondence.* Using the resources of the expanded language, and the stipulations made in clauses (1.3) and (1.4), it is not difficult to check the soundness on Kripke frames of the following chain of equivalences:

$$\begin{array}{rll} \forall p[\Box p \leq \Box\Box p] & \text{iff } \forall p \forall \mathbf{j}[\mathbf{j} \leq \Box p \Rightarrow \mathbf{j} \leq \Box\Box p] & \\ & \text{iff } \forall p \forall \mathbf{j}[\blacklozenge\mathbf{j} \leq p \Rightarrow \mathbf{j} \leq \Box\Box p] & \\ & \text{iff } \forall \mathbf{j}[\mathbf{j} \leq \Box\Box\blacklozenge\mathbf{j}] & \text{(lemma 1.1)} \\ & \text{iff } \forall \mathbf{j}[\blacklozenge\blacklozenge\mathbf{j} \leq \blacklozenge\mathbf{j}]. & \end{array}$$

[4] In fact, it works also when $\mathcal{F}$ is an ordered algebra where the operations interpret the $\mathcal{L}^+$-connectives.

indeed, thanks to the above stipulations on the interpretation of nominals, the condition $\mathcal{F} \Vdash \Box p \leq \Box\Box p$ can be equivalently rewritten as $\mathcal{F} \Vdash \forall \mathbf{j}[\mathbf{j} \leq \Box p \Rightarrow \mathbf{j} \leq \Box\Box p]$; by the stipulations on the additional modal operators, this clause can equivalently be rewritten as $\mathcal{F} \Vdash \forall \mathbf{j}[\blacklozenge \mathbf{j} \leq p \Rightarrow \mathbf{j} \leq \Box\Box p]$. Hence, by Ackermann's lemma applied to $\alpha := \blacklozenge \mathbf{j}$, $\beta(p) := \mathbf{j}$, $\gamma(p) := \Box\Box p$, we get that $\mathcal{F} \Vdash \forall \mathbf{j}[\mathbf{j} \leq \Box\Box\blacklozenge \mathbf{j}]$, which can be rewritten as $\mathcal{F} \Vdash \forall \mathbf{j}[\blacklozenge\blacklozenge \mathbf{j} \leq \blacklozenge \mathbf{j}]$.

To sum up, what we are heading towards is introducing a formal (*object*) language and a *syntactic* machinery in which the *semantic* 'minimal valuation' *meta*-argument given in Example 1 can be encoded. This small copernican revolution can be traced back to [37]. As to the benefits it brings: once we are dealing with syntax, we are free to interpret these strings of symbols and transformation rules in all sorts of models which happen to soundly interpret them; for instance, *atomistic*[5] tense Boolean algebras, and more specifically, the *complex algebras*, i.e. the modal algebras dually associated with relational structures, are obvious sound models. In the latter, nominals would then be interpreted as atoms of the algebra, and it is easy to see that the first equivalence is sound precisely because of atomicity. In fact, thanks to duality, the soundness of the chain of equivalences above w.r.t. complex algebras is the equivalent counterpart of the soundness proof on frames. Interpreting $\forall \mathbf{j}[\blacklozenge\blacklozenge \mathbf{j} \leq \blacklozenge \mathbf{j}]$ on complex algebras, where $\mathbf{j}$ ranges over the singletons, we readily obtain the well known first-order condition

$$\forall x(R[R[x]] \subseteq R[x]),$$

which standardly abbreviates the usual transitivity condition.

But there is more. In fact, we can do just as well with much more general algebras than the complex algebras of Kripke frames. All we need of an algebraic model for this (very simple) proof to be sound is its being a poset endowed with a pair of adjoint operations $\blacklozenge \dashv \Box$, and its being join-generated by some designated subset $J$ (which will provide the interpretation for nominals). Of course, for the sake of finding a suitable environment for *classes* of logics, we need to assume more: in particular, we want to assume the existence of a rich enough algebraic environment, able to provide interpretation to logical connectives; certain complete (distributive) lattice expansions which we will introduce below are adequate for most purposes. This enables us to explore the full domain of applicability of correspondence arguments, which turns out to be much wider than classical modal logic.

This concludes the informal presentation of the view on correspondence theory pursued in the present paper. In the following section, we will expand on some of the technical details supporting this perspective.

[5] A lattice is atomistic if every element is the supremum of a set of atoms.

## 1.3 A calculus for correspondence

Let us start by formally introducing the expanded syntax we mentioned in the previous section: it will include the backward-looking box corresponding to the diamond taken as a primitive operator, as well as a denumerably infinite set of sorted variables $\mathsf{NOM}$ called *nominals*, the elements of which will be denoted with $\mathbf{i}, \mathbf{j}$, possibly indexed.

The *formulas* of $\mathcal{L}^+$ are given by the following recursive definition:

$$\varphi ::= \bot \mid p \mid \mathbf{j} \mid \varphi \vee \psi \mid \neg\varphi \mid \Diamond\varphi \mid \blacksquare\varphi,$$

where $p \in \mathsf{PROP}$ and $\mathbf{j} \in \mathsf{NOM}$. The derived connectives $\wedge$, $\Box$, $\rightarrow$, $-$,... are defined in the standard way. In order to formalize the correspondence arguments, we will have to expand $\mathcal{L}^+$ to accommodate inequalities and quasi-inequalities. To be precise, if $\varphi, \varphi_1, \ldots, \varphi_n, \psi, \psi_1, \ldots, \psi_n \in \mathcal{L}^+$ then $\varphi \leq \psi$ is an inequality and $\varphi_1 \leq \psi_1 \,\&\cdots\&\, \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ is a quasi-inequality. Disjunctions $\varphi \leq \psi \;⅋\; \chi \leq \xi$ between inequalities will be sometimes considered.

Formulas, inequalities and quasi-inequalities not containing any propositional variables (but possibly containing nominals) will be called *pure*. As we will see next, these can be readily translated into the first-order frame correspondence language; hence we aim to introduce rules for a calculus of syntactic transformations of quasi-inequalities, by means of which quasi-inequalities in $\mathcal{L}^+$ can be transformed into *pure* ones, so as to preserve logical equivalence. In order to motivate this calculus, let us introduce the intended interpretation of the expanded language.

A *valuation for* $\mathcal{L}^+$ on a Kripke frame $\mathcal{F} = (W, R)$ is any map $V$ from the set $\mathsf{PROP} \cup \mathsf{NOM}$ of propositional variables and nominals into the powerset $\mathcal{P}(W)$, such that each $\mathbf{i} \in \mathsf{NOM}$ is assigned to the singleton subset $\{x\}$ for some $x \in W$. A *model* for $\mathcal{L}^+$ is a tuple $\mathcal{M} = (\mathcal{F}, V)$ such that $\mathcal{F}$ is a Kripke frame and $V$ is a valuation for $\mathcal{L}^+$. For any such model, the satisfaction relation for formulas in $\mathcal{L}^+$ is recursively defined as follows (here we report only the new connectives):

$$\begin{array}{rll}
\mathcal{M}, w \Vdash \mathbf{i} & \text{iff} & V(\mathbf{i}) = \{w\}, \\
\mathcal{M}, w \Vdash \blacksquare\varphi & \text{iff} & \text{for every } v, \text{ if } vRw \text{ then } \mathcal{M}, v \Vdash \varphi.
\end{array}$$

The local satisfaction relation extends to inequalities and quasi-inequalities as follows:

$$\begin{array}{rll}
\mathcal{M}, w \Vdash \varphi \leq \psi & \text{iff} & \text{if } \mathcal{M}, w \Vdash \varphi \text{ then } \mathcal{M}, w \Vdash \psi, \\
\mathcal{M}, w \Vdash (\&_{i=1}^n \varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi & \text{iff} & \text{if } \mathcal{M}, w \Vdash \varphi_i \leq \psi_i \text{ for } 1 \leq i \leq n \\
& & \text{then } \mathcal{M}, w \Vdash \varphi \leq \psi.
\end{array}$$

From the clauses above, the global satisfaction relation for inequalities and quasi-inequalities is defined in the usual way, by universally quantifying over $w$; namely,

$$\begin{array}{rll}
\mathcal{M} \Vdash \varphi \leq \psi & \text{iff} & \text{for any } w, \text{ if } \mathcal{M}, w \Vdash \varphi \text{ then } \mathcal{M}, w \Vdash \psi, \\
\mathcal{M} \Vdash (\&_{i=1}^n \varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi & \text{iff} & \text{for any } w, \text{ if } \mathcal{M}, w \Vdash \varphi_i \leq \psi_i \text{ for } 1 \leq i \leq n \\
& & \text{then } \mathcal{M}, w \Vdash \varphi \leq \psi.
\end{array}$$

For every model $\mathcal{M} = (\mathcal{F}, V)$ and every $\varphi \in \mathcal{L}^+$, the symbol $[\![\varphi]\!]_{\mathcal{M}}$ denotes as usual the set of states of $\mathcal{M}$ at which $\varphi$ is satisfied. When there could be no confusion about $\mathcal{F}$, the symbol $[\![\varphi]\!]_V$ will alternatively be used.

As mentioned in section 1.2, *local* satisfaction of formulas can be encoded as a special case of the *global* satisfaction of inequalities, as reported in the following proposition:

**Proposition 1** *For any Kripke frame $\mathcal{F}$, any valuation $V$ for $\mathcal{L}^+$ and any $\varphi \in \mathcal{L}$,*

$$\mathcal{F}, V, w \Vdash \varphi \quad \textit{iff} \quad \mathcal{F}, V' \models \mathbf{j} \leq \varphi \textit{ and } V'(\mathbf{j}) = \{w\},$$

*with $V' \sim_{\mathbf{j}} V$ and $\mathbf{j}$ a nominal not occurring in $\varphi$.*

The Ackermann lemma (lemma 1.2) implies that the following rules are sound and invertible w.r.t. the standard Kripke semantics:

$$\frac{\forall p[(\alpha \leq p \ \& \ \&_{1 \leq i \leq n}(\gamma_i(p) \leq \delta_i(p))) \Rightarrow \varphi(p) \leq \psi(p)]}{\&_{1 \leq i \leq n}(\gamma_i(\alpha/p) \leq \delta_i(\alpha/p)) \Rightarrow \varphi(\alpha/p) \leq \psi(\alpha/p)} \text{(LA)} \qquad \frac{\forall p[\varphi(p) \leq \psi(p)]}{\varphi(\bot/p) \leq \psi(\bot/p)} (\bot)$$

subject to the restrictions that $\alpha$ is $p$-free, and that $\varphi$ and the $\delta_i$ are negative in $p$, while $\psi$ and the $\gamma_i$ are positive in $p$. Notice that the rule $(\bot)$ can be regarded as the special case of (LA) in which $\alpha := \bot$. Likewise, a mirror-image version of lemma 1.2 implies that the following rules are sound and invertible w.r.t. the standard Kripke semantics:

$$\frac{\forall p[(p \leq \alpha \ \& \ \&_{1 \leq i \leq n}(\gamma_i(p) \leq \delta_i(p))) \Rightarrow \varphi(p) \leq \psi(p)]}{[\&_{1 \leq i \leq n}(\gamma_i(\alpha/p) \leq \delta_i(\alpha/p)) \Rightarrow \varphi(\alpha/p) \leq \psi(\alpha/p)]} \text{(RA)} \qquad \frac{\forall p[\varphi(p) \leq \psi(p)]}{\varphi(\top/p) \leq \psi(\top/p)} (\top)$$

subject to the restrictions that $\alpha$ is $p$-free, and that $\varphi$ and the $\delta_i$ are positive in $p$, while $\psi$ and the $\gamma_i$ are negative in $p$. In addition to this, the following proposition is an immediate consequence of the stipulations above:

**Proposition 2** *For every model $\mathcal{M} = (\mathcal{F}, V)$ for $\mathcal{L}^+$, every $\mathbf{j} \in \mathsf{NOM}$, and all $\varphi, \psi, \chi \in \mathcal{L}^+$,*

1. *$\mathcal{F}, V \Vdash \varphi \leq \psi$ iff $\mathcal{F}, V \models \forall \mathbf{j}[\mathbf{j} \leq \varphi \Rightarrow \mathbf{j} \leq \psi]$, for any nominal $\mathbf{j}$ not occurring in $\varphi \leq \psi$.*
2. *$\mathcal{F}, V \Vdash \varphi \vee \chi \leq \psi$ iff $\mathcal{F}, V \Vdash \varphi \leq \psi$ and $\mathcal{F}, V \Vdash \chi \leq \psi$.*
3. *$\mathcal{F}, V \Vdash \varphi \leq \chi \vee \psi$ iff $\mathcal{F}, V \Vdash \varphi - \chi \leq \psi$, where $\varphi - \chi := \varphi \wedge \neg\chi = \neg(\neg\varphi \vee \chi)$.*
4. *$\mathcal{F}, V \Vdash \Diamond\varphi \leq \psi$ iff $\mathcal{F}, V \Vdash \varphi \leq \blacksquare\psi$.*
5. *$\mathcal{F}, V \Vdash \mathbf{j} \leq \Diamond\psi$ iff $\mathcal{F}, V \models \exists \mathbf{i}[\mathbf{j} \leq \Diamond\mathbf{i} \ \& \ \mathbf{i} \leq \psi]$, for any nominal $\mathbf{i}$ not occurring in $\mathbf{j} \leq \Diamond\psi$.*
6. *$\mathcal{F}, V \Vdash \psi \leq \neg\varphi$ iff $\mathcal{F}, V \Vdash \varphi \leq \neg\psi$.*
7. *$\mathcal{F}, V \Vdash \neg\varphi \leq \psi$ iff $\mathcal{F}, V \Vdash \neg\psi \leq \varphi$.*

The proposition above essentially says that the following rules are sound and invertible w.r.t. the standard Kripke semantics:

$$\frac{\varphi \leq \psi}{\forall \mathbf{j}[\mathbf{j} \leq \varphi \Rightarrow \mathbf{j} \leq \psi]}\ (\mathrm{FA})^{*} \qquad \frac{\varphi \vee \chi \leq \psi}{\varphi \leq \psi \quad \chi \leq \psi}\ (\vee\text{-}\Delta) \qquad \frac{\varphi \leq \chi \vee \psi}{\varphi - \chi \leq \psi}\ (\vee\mathrm{RR})$$

$$\frac{\Diamond\varphi \leq \psi}{\varphi \leq \blacksquare\psi}\ (\Diamond\mathrm{LA}) \qquad \frac{\mathbf{j} \leq \Diamond\psi}{\exists \mathbf{i}(\mathbf{j} \leq \Diamond\mathbf{i}\ \&\ \mathbf{i} \leq \psi)}\ (\mathbf{j}\mathrm{CJP})^{\dagger} \qquad \frac{\varphi \leq \neg\psi}{\psi \leq \neg\varphi}\ (\neg\mathrm{RGA}) \qquad \frac{\neg\varphi \leq \psi}{\neg\psi \leq \varphi}\ (\neg\mathrm{LGA})$$

*where the introduced nominal **j** does not occur in derivation so far.

†where the introduced nominal **i** does not occur in derivation so far.

It is easy to show that the calculus admits derived invertible rules such as the following:

$$\frac{\varphi \leq \chi \wedge \psi}{\varphi \leq \chi \quad \varphi \leq \psi}\ (\wedge\text{-}\Delta) \qquad \frac{\varphi \wedge \chi \leq \psi}{\varphi \leq \chi \rightarrow \psi}\ (\wedge\mathrm{LR})$$

$$\frac{\varphi \leq \Box\psi}{\blacklozenge\varphi \leq \psi}\ (\Box\mathrm{RA}) \qquad \frac{\Box\varphi \leq \neg\mathbf{j}}{\exists \mathbf{i}(\Box\neg\mathbf{i} \leq \neg\mathbf{j}\ \&\ \varphi \leq \neg\mathbf{i})}\ (\neg\mathbf{j}\mathrm{CMP})^{\dagger}$$

The calculus introduced above can be used to derive first-order correspondents of formulas, inequalities, and quasi-inequalities; formal derivations in this calculus can be semantically interpreted as 'minimal valuation' meta-arguments, which justifies the statement that this calculus indeed mechanizes these meta-arguments. Several algorithms have been introduced in the literature (see, e.g., [18], [30], [22]) which specify how these derivations should proceed; these algorithms are also shown to be successful for classes of formulas which significantly extend the class of Sahlqvist formulas. Reporting in detail on these algorithms and their properties is certainly beyond the aims of this paper; however we conclude the present section by discussing examples, since we believe that this, rather than the extensive theory, will give the reader a better idea on how to proceed in practice.

*Example 2.* In [39] Goranko and Vakarelov show that the formula $p \wedge \Box(\Diamond p \rightarrow \Box q) \rightarrow \Diamond\Box\Box q$, which falls in their class of Inductive formulas, has a first-order frame correspondent which does not correspond to any Sahlqvist formula in the basic modal language. For the sake of a smoother application of the rules introduced above, we rewrite this formula as an inequality and proceed as follows:

$$\begin{array}{lll}
& \forall p \forall q(p \wedge \Box(\Diamond p \rightarrow \Box q) \leq \Diamond\Box\Box q) & \\
\text{iff} & \forall p \forall q \forall \mathbf{j}(\mathbf{j} \leq p \wedge \Box(\Diamond p \rightarrow \Box q) \Rightarrow \mathbf{j} \leq \Diamond\Box\Box q) & (\mathrm{FA}) \\
\text{iff} & \forall p \forall q \forall \mathbf{j}(\mathbf{j} \leq p\ \&\ \mathbf{j} \leq \Box(\Diamond p \rightarrow \Box q) \Rightarrow \mathbf{j} \leq \Diamond\Box\Box q) & (\wedge\mathrm{RA}) \\
\text{iff} & \forall q \forall \mathbf{j}(\mathbf{j} \leq \Box(\Diamond \mathbf{j} \rightarrow \Box q) \Rightarrow \mathbf{j} \leq \Diamond\Box\Box q) & (\mathrm{LA}) \\
\text{iff} & \forall q \forall \mathbf{j}(\blacklozenge(\blacklozenge\mathbf{j} \wedge \Diamond\mathbf{j}) \leq q \Rightarrow \mathbf{j} \leq \Diamond\Box\Box q) & (\Box\mathrm{RA}), (\wedge\mathrm{LR}), (\Box\mathrm{RA}) \\
\text{iff} & \forall \mathbf{j}(\mathbf{j} \leq \Diamond\Box\Box\blacklozenge(\blacklozenge\mathbf{j} \wedge \Diamond\mathbf{j})). & (\mathrm{LA})
\end{array}$$

Note that the last application of (LA) yields an empty & in the antecedent. Now the last quasi-inequality is pure, and translates, after some slight simplification, into the expected first-order local frame condition $\exists y(Rxy \wedge \forall z(R^2yz \rightarrow \exists u(Ruz \wedge Rux \wedge Rxu)))$.

## 1.4 Algebraic soundness of the calculus for correspondence

Discrete Stone duality for Kripke frames guarantees that the interpretation of the expanded language on Kripke frames systematically translates to complete atomic modal algebras.

$\mathcal{L}^+$-valuations on Kripke frames translate as assignments on the dual algebras, under which, nominals are interpreted as *atoms*. Inequalities and quasi-inequalities are interpreted in algebras using their natural lattice order, and satisfaction and validity naturally carry over to algebras as well. In particular, it is not difficult to show that both lemma 1 and proposition 2 hold if Kripke frames are replaced by complete atomic modal algebras, which again means that the calculus for correspondence defined in the previous section is sound w.r.t. the algebraic duals of Kripke frames. However, this is neither surprising nor does it give us anything more than we had before.

The algebraic perspective starts to become interesting when noticing that, as we had mentioned in section 1.2, almost all the rules of the calculus for correspondence are sound w.r.t. a *significantly larger* class of algebras than complete atomic modal algebras:

**Definition 1.** A *perfect distributive lattice* (cf. [27, Def. 2.9]) is a complete lattice $\mathbb{C}$ such that the set $J^\infty(\mathbb{C})$ of the completely join-prime elements[6] is join-dense in $\mathbb{C}$ (meaning that $a = \bigvee\{j \in J^\infty(\mathbb{C}) \mid j \leq a\}$ for every $a \in \mathbb{C}$) and the set $M^\infty(\mathbb{C})$ of the completely meet-prime elements is meet-dense in $\mathbb{C}$ (meaning that $a = \bigwedge\{m \in M^\infty(\mathbb{C}) \mid a \leq m\}$ for every $a \in \mathbb{C}$).

Analogously to the duality between complete atomic Boolean algebras and sets, a Stone-type duality holds between perfect distributive lattices and *posets*, as a consequence of which, perfect distributive lattices can be equivalently characterized (cf. [33, Def. 2.14]) as those lattices each of which is isomorphic to the lattice $\mathcal{P}^\uparrow(X)$ of the upward-closed subsets of some poset $X$. In particular, the role atoms had in the Boolean algebra setting is taken over, in this generalized duality, by the completely join-prime elements.

**Definition 2 (Perfect distributive lattice with operators).** (cf. [32]) A distributive lattice with operators (DLO) $\mathbb{A}$ is *perfect* if its lattice reduct is a perfect distributive lattice and every additional operation is, in each coordinate, either completely join- or meet-preserving or completely join- or meet-reversing.

So for instance, the unary additional operations in a DLO need to satisfy at least one property in the following array: for every $S \subseteq A$,

$$\begin{array}{ll} \Diamond(\bigvee S) = \bigvee\{\Diamond s \mid s \in S\} & \Box(\bigwedge S) = \bigwedge\{\Box s \mid s \in S\} \\ \rhd(\bigvee S) = \bigwedge\{\rhd s \mid s \in S\} & \lhd(\bigwedge S) = \bigvee\{\lhd s \mid s \in S\}. \end{array} \tag{1.1}$$

[6] An element $c$ of a complete lattice is *completely join-prime* if $c \neq \bot$ and, for every subset $S$ of the lattice, $c \leq \bigvee S$ iff $c \leq s$ for some $s \in S$, and is *completely meet-prime* if $c \neq \top$ and, for every subset $S$ of the lattice, $c \geq \bigwedge S$ iff $c \geq s$ for some $s \in S$.

It is not difficult to show that both lemma 1 and all items of proposition 2, with the exception of item 7, hold if $\mathcal{F}$ is replaced by a suitable perfect DLO (suitable in the sense that it has the appropriate array of operations and in particular, in it, the connective $\neg$ is interpreted e.g. as intuitionistic negation), and $\mathcal{L}^+$-valuations on frames are replaced with $\mathcal{L}^+$-assignments on perfect DLOs which map nominals to completely join-prime elements.

For instance, item 1 of proposition 2 is sound because, by definition, in a perfect DLO every element is the join of the set of completely join-prime elements below it; item 5 is sound because the following equivalence holds in every perfect DLO $(\mathbb{A}, \Diamond)$: for every $j \in J^\infty(\mathbb{A})$ and every $a \in \mathbb{A}$,

$$\begin{array}{lll} & j \leq \Diamond a & \\ & = \Diamond(\bigvee\{i \in J^\infty(\mathbb{A}) \mid i \leq a\}) & \text{(definition of perfect DLO)} \\ & = \bigvee\{\Diamond i \in J^\infty(\mathbb{A}) \mid i \leq a\} & (\Diamond \text{ is completely } \bigvee\text{-preserving}) \\ & & \\ \text{iff} & j \leq \Diamond i \text{ for some } i \in J^\infty(\mathbb{A}) \text{ s.t. } i \leq a. & (j \text{ is completely join-prime}) \end{array}$$

By general order-theoretic facts (see e.g. [31]), all the operations of a perfect DLO admit right or left residuals in each coordinate, or are adjoints[7]; this immediately proves items 2, 3, 4 and 6. This means that all the rules of the calculus given in the previous section, with the exception of ($\neg$LGA), are sound and invertible w.r.t. perfect DLOs. In fact, soundness and invertibility w.r.t. perfect DLOs can be shown for a few more rules: for instance, thanks to the fact that in a perfect DLO every element is not only the join of the set of completely join-prime elements below it, but is also the meet of the set of completely meet-prime elements above it, the language $\mathcal{L}^+$ can be further expanded by adding a new sort of variables $\mathbf{l}, \mathbf{m}, \mathbf{n} \ldots \in \mathsf{CONOM}$, referred to as *co-nominals*, ranging over the completely meet-prime elements, and it can be easily shown that the following facts hold in every perfect DLO $\mathbb{A}$, which can be added to (the DLO-version of) the list of proposition 2: for all $a, b \in \mathbb{A}$,

8. $a \leq b$ iff for every $m \in M^\infty(\mathbb{A})$, if $b \leq m$ then $a \leq m$;
9. $a \leq b$ iff for every $j \in J^\infty(\mathbb{A})$ and every $m \in M^\infty(\mathbb{A})$, if $j \leq a$ and $b \leq m$ then $j \leq m$;

these equivalences imply that the following rules are sound and invertible w.r.t. perfect DLOs:

$$\frac{\varphi \leq \psi}{\forall \mathbf{m}[\psi \leq \mathbf{m} \Rightarrow \varphi \leq \mathbf{m}]}\ (\text{UA}) \qquad \frac{\varphi \leq \psi}{\forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq \varphi \ \&\ \psi \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}]}\ (\text{ULA})$$

It can also be shown that the *derived* rules ($\wedge$RA), ($\wedge$LR), ($\Box$RA), and ($\neg\mathbf{j}$CMP) introduced in the previous section are sound and invertible w.r.t. DLOs, except that they cannot be soundly derived anymore, but need to be added to the calculus as

[7] Notice for instance that the defining clause of the least upper bound, i.e. $a \vee b \leq c$ iff $a \leq c$ and $b \leq c$ for all $a, b, c \in \mathbb{A}$ can be equivalently restated by saying that $\vee : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is left adjoint to the diagonal map $\Delta : \mathbb{A} \to \mathbb{A} \times \mathbb{A}$ defined by the assignment $a \mapsto (a, a)$. Likewise, $\wedge$ is the right adjoint of $\Delta$. This is why we refer to the corresponding rules as *$\Delta$-rules*. More on adjoints and residuals can be found in the appendix.

primitive rules, and their soundness and invertibility should be proved from first principles. Indeed, they can be shown to be sound and invertible for $\Box$ taken as a primitive connective, the implication $\rightarrow$ and subtraction $-$ respectively interpreted by means of the Heyting and the dual Heyting implications, and $\neg\mathbf{i}$ in $(\neg\mathbf{j}\mathrm{CMP})$ replaced by $\mathbf{m} \in \mathsf{CONOM}$. In which case, the following rules can also be shown to be sound and invertible on perfect DLOs, using the fact that the Heyting implication is completely join-reversing in its first coordinate and completely meet-preserving in its second one, and the dual Heyting implication is completely join-preserving in its first coordinate and completely meet-reversing in its second one:

$$\frac{\varphi \rightarrow \chi \leq \mathbf{m}}{\exists \mathbf{j} \exists \mathbf{n}[\mathbf{j} \leq \varphi \ \&\ \chi \leq \mathbf{n} \ \&\ \mathbf{j} \rightarrow \mathbf{n} \leq \mathbf{m}]}\ (\rightarrow\text{Appr}) \qquad \frac{\mathbf{j} \leq \chi - \psi}{\exists \mathbf{i} \exists \mathbf{m}[\mathbf{i} \leq \chi \ \&\ \psi \leq \mathbf{m} \ \&\ \mathbf{j} \leq \mathbf{i} - \mathbf{m}]}\ (-\text{Appr})$$

By now, the reader may have realized that the way rules are introduced easily and uniformly generalizes to any additional operation in a DLO, and applies also to the algebraic interpretation of logical languages outside the scope of modal logic, such as for instance the substructural logics, many-valued logics, and so on. For instance, the following rules for the substructural *fusion* $\circ$ and its two right residuals $/_\circ$ and $\backslash_\circ$, and for *fission* $\star$ and its two left residuals $/_\star$ and $\backslash_\star$ can be shown to be sound and invertible on DLOs:

$$\cfrac{\cfrac{\varphi \circ \chi \leq \psi}{\varphi \leq \psi /_\circ \chi}}{\chi \leq \varphi \backslash_\circ \psi}\ (\circ\text{R}) \qquad \cfrac{\cfrac{\varphi \leq \chi \star \psi}{\varphi /_\star \chi \leq \psi}}{\psi \backslash_\star \varphi \leq \chi}\ (\star\text{R})$$

$$\frac{\mathbf{j} \leq \chi \circ \psi}{\exists \mathbf{i} \exists \mathbf{h}[\mathbf{i} \leq \chi \ \&\ \mathbf{h} \leq \psi \ \&\ \mathbf{j} \leq \mathbf{i} \circ \mathbf{h}]}\ (\circ\text{Appr}) \qquad \frac{\varphi \star \chi \leq \mathbf{m}}{\exists \mathbf{n} \exists \mathbf{l}[\chi \leq \mathbf{n} \ \&\ \varphi \leq \mathbf{l} \ \&\ \mathbf{n} \star \mathbf{l} \leq \mathbf{m}]}\ (\star\text{Appr})$$

$$\frac{\varphi /_\circ \chi \leq \mathbf{m}}{\exists \mathbf{n} \exists \mathbf{j}[\varphi \leq \mathbf{n} \ \&\ \mathbf{j} \leq \chi \ \&\ \mathbf{n} /_\circ \mathbf{j} \leq \mathbf{m}]}\ (/_\circ\text{Appr}) \qquad \frac{\varphi \backslash_\circ \chi \leq \mathbf{m}}{\exists \mathbf{j} \exists \mathbf{n}[\mathbf{j} \leq \varphi \ \&\ \chi \leq \mathbf{n} \ \&\ \mathbf{j} \backslash_\circ \mathbf{n} \leq \mathbf{m}]}\ (\backslash_\circ\text{Appr})$$

$$\frac{\mathbf{j} \leq \chi /_\star \psi}{\exists \mathbf{i} \exists \mathbf{m}[\mathbf{i} \leq \chi \ \&\ \psi \leq \mathbf{m} \ \&\ \mathbf{j} \leq \mathbf{i} /_\star \mathbf{m}]}\ (/_\star\text{Appr}) \qquad \frac{\mathbf{j} \leq \chi \backslash_\star \psi}{\exists \mathbf{m} \exists \mathbf{i}[\chi \leq \mathbf{m} \ \&\ \mathbf{i} \leq \psi \ \&\ \mathbf{j} \leq \mathbf{m} \backslash_\star \mathbf{i}]}\ (\backslash_\star\text{Appr})$$

*Duality, relational structures and target correspondence language.* Just in the same way in which the duality between complete atomic Boolean algebras and sets can be expanded to a duality between complete atomic modal algebras and relational structures consisting of *sets* endowed with arrays of relations, the duality between perfect distributive lattices and posets can be expanded to a duality between perfect DLOs and relational structures $\mathcal{F} = (W, \leq, \ldots)$, consisting of *posets* endowed with arrays of relations. Each relation in the array induces (and up to isomorphism is induced by) one additional operation in the usual way, i.e., $n$-ary operations correspond to $n+1$-ary relations. Examples of such structures can be found in section 1.11, where more details and references are provided. The only important detail for the sake of the present discussion is that the *complex algebras* $\mathcal{F}^+$ for these frames can be defined as in the classical setting, with the notable difference that they are based on the (perfect distributive) lattice $\mathcal{P}^{\uparrow}(W)$ of the *upward-closed* subsets of $(W, \leq)$. This is unsurprising, and perfectly fits with the well-known fact that the valuations for e.g. intuitionistic logic are to be *persistent*. As in the case of classical modal logic, these relational structures are *both* models for the extended propositional (modal) language, *and* for the first-order language(s) which are naturally interpreted on them, and which will be our target correspondence languages. The only remaining open issue is then to establish a *standard translation* of pure formulas and quasi-inequalities of the extended propositional language $\mathcal{L}^+$ into these first-order correspondence languages. How? Because of space constraints we will not give full details, which are straightforward and can be found in [22]; instead, we restrict our attention to the interpretation of the variables in $\mathsf{NOM}$ and $\mathsf{CONOM}$ in the dual relational structures, and justify why this interpretation gives rise to first-order definable conditions on any structure $\mathcal{F} = (W, \leq, \ldots)$. Duality is crucial to establish this interpretation. Indeed, there is only one solution which takes all the following facts into account:

(a) on perfect distributive lattices, nominals and co-nominals are respectively interpreted as completely join- and meet-prime elements;

(b) the complex algebra of $\mathcal{F} = (W, \leq, \ldots)$ is based on the perfect distributive lattice $\mathcal{P}^{\uparrow}(W)$;

(c) the collections of all completely join- and meet-prime elements of $\mathcal{P}^{\uparrow}(W)$ are respectively[8]

$$\{x\uparrow \mid x \in W\} \quad \text{and} \quad \{W \setminus x\downarrow \mid x \in W\};$$

(d) the unique homomorphic extension $\widehat{V}$ of each $\mathcal{L}^+$-valuation on $\mathcal{F}$ is to be an $\mathcal{L}^+$-valuation on $\mathcal{F}^+$;

(e) it should be that case that, for all models $(\mathcal{F}, V)$ and all $\varphi, \psi \in \mathcal{L}^+$,

$$\mathcal{F}, V \Vdash \varphi \leq \psi \quad \text{iff} \quad \mathcal{F}^+, \widehat{V} \Vdash \varphi \leq \psi.$$

The only way to define the interpretation of $\mathbf{j} \in \mathsf{NOM}$ and $\mathbf{m} \in \mathsf{CONOM}$ which takes all these facts into account is to stipulate that $\mathcal{L}^+$-valuations $V$ on $\mathcal{F}$ assign variables $\mathbf{j} \in \mathsf{NOM}$ to elements in $\{x\uparrow \mid x \in W\}$ and variables $\mathbf{m} \in \mathsf{CONOM}$ to elements in $\{W \setminus x\downarrow \mid x \in W\}$. As was the case in the classical setting, the interpretations of nominals and co-nominals are clearly definable in the most restricted correspondence language which the structures $\mathcal{F}$ are models of.

Stepping back from this discussion, we note two points: duality was crucial in establishing the connection of clearest practical value to our current agenda, namely being able to translate the pure fragment of the extended language $\mathcal{L}^+$ into the target first-order correspondence language. However, the reasoning used in establishing this connection illustrates a *methodological* point about dualities, namely, that they can be used not only as a proof tool, but also as a *defining* tool. For instance, in more

[8] As usual, $x\uparrow$ denotes the subset $\{y \mid y \in W \text{ and } x \leq y\}$, and $x\downarrow$ denotes the subset $\{y \mid y \in W \text{ and } y \leq x\}$.

general settings than the ones presented so far, like lattice based logics, the algebraic semantics is clear but one might be in the dark as to what an appropriate relational semantics might be, both as regards an appropriate class of relational structures and as to the appropriate interpretation of the propositional language in such a class. This is where duality can be used as a defining tool: firstly, relational structures can be extracted, as it were, from perfect lattices [27]; secondly, the interpretation of the propositional formulas in algebras transfers via the duality to these extracted structures. To mention a related but different example, in [44, 43] duality is used to semantically identify the intuitionistic counterparts of public announcement logic and of the logic of epistemic actions and knowledge.

## 1.5 Four conclusions and a question

*Conclusion 1*: thanks to the algebraic insights facilitated by duality, correspondence theory can be developed uniformly for more than modal-like logics; as we have illustrated, also substructural logics, intuitionistic logic and its fragments, MV-logics, as well as distributive and intuitionistic modal logic, and more in general, all the logics the algebraic semantics of which is given by DLOs can be encompassed. Also, $\mu$-calculus (see subsection 1.8.3), monotone modal logic [29] and their lattice-based extensions are examples of logics which can be uniformly treated by this theory.

*Conclusion 2*: the algebraic and algorithmic developments for correspondence can and have been merged. This now allows for algebraic canonicity to be treated either independently from correspondence in the style of [36], or via correspondence as in [22]. And there is more: as discussed at the end of the previous subsection, even in vastly more general settings, concrete relational structures can be extracted from the algebras. Therefore, even in these rarified algebraic settings, speaking of correspondence theory does not amount to merely establishing an elaborate social convention, or a manner of speaking, by means of which we can pretend that relational models which are not really there virtually manifest themselves by means of their algebraic ghosts. On the contrary, the obtained correspondence theory makes sense, on the extracted relational structures, in the traditional way.

*Conclusion 3*: for the sake of the present paper, we have distilled the main features of the algebraic-algorithmic approach into a more informal presentation of a calculus for correspondence, the set of rules of which can be modified, expanded or reduced, so that the calculus can be adapted to different logical languages, and so that it can be proven sound w.r.t. different semantics; however, the underlying mathematical principles which drive this calculus (as well as the algorithms, and more in general, all the Sahlqvist-style correspondence arguments) remain stable across the different settings, and are: the Ackermann lemma in any of its many forms, the residuation/adjunction properties of the operations interpreting the logical connectives, and the approximation properties of the 'states' (or co-states) of the relational semantics, which generate (and co-generate) their dual complex algebras.

*Conclusion 4*: The computation process of first-order correspondents can be neatly divided in two stages: a first stage, in which quasi-inequalities are transformed into *pure* ones, and a second stage, where pure quasi-inequalities are interpreted in the given classes of relational structures. Different relational semantics might then yield different interpretations of the same pure quasi inequality, and some instances of this will be discussed in section 1.9. The definition of this syntactic calculus and the possibility of soundly interpreting it in a generalized algebraic environment (which can then be translated, in a second stage, into several concrete relational semantics) gives some mathematical flesh to van Benthem's insight that "Correspondence Theory may be applied to any kind of semantic entity".

*Question*: How powerful is this algebraic-algorithmic procedure? In the case of classical modal logic it is state of the art, and covers syntactically characterized classes of formulas which *significantly extend* the Sahqvist class (viz. Inductive, Recursive, see [39]). But can we claim that, in all the other (e.g. lattice-based) cases, the algorithmic procedure is just as powerful? The answer to this question requires being able to recognize Sahlqvist, Inductive, Recursive classes for each logical language to which the algorithmic correspondence applies. In section 1.7 we suggest a way in which this can be done.

## 1.6 The van Benthem formulas

One aspect of the discussion in section 1.2 still needs to be justified, which concerns how to extract the correspondent of a given modal formula, provided the equivalence between clauses (1) and (2) holds (which are reported below); before moving on to what we have promised to do at the end of the previous section, in the present section we discuss this aspect briefly. As mentioned early on in section 1.2, suppose that, for a certain subclass of valuations $K$, the following are equivalent:

(1) $\mathcal{F}, V, w \Vdash \varphi(p_1, \ldots, p_n)$ for every assignment $V$;

(2) $\mathcal{F}, V^*, w \Vdash \varphi(p_1, \ldots, p_n)$ for every assignment $V^* \in K$.

Suppose moreover that each member $V^* \in K$ and $1 \leq i \leq n$, the subset $V^*(p_i)$ can be defined (possibly parametrically) by a formula $\alpha_i(w, \overline{v})$ in some extension $L'$ of the frame correspondence language $L_0$. Here we typically think of $L'$ as $L_0$ itself or some language in between $L_0$ and $L_2$ such as first-order logic with least fixed points, or perhaps a first-order logic with branching quantifiers such as information friendly logic.

Let $\Sigma$ be the set of all $L'$-formulas $\mathrm{ST}_x(\varphi)(\alpha_1(w, \overline{v}), \ldots, \alpha_n(w, \overline{v}))$ obtained by substituting in $\mathrm{ST}_x(\varphi)$ the predicate symbols $P_1, \ldots P_n$ with the $L'$-formulas $\alpha_1(w, \overline{v}), \ldots, \alpha_n(w, \overline{v})$ corresponding to the valuations in $K$. Clearly, $\forall \overline{P} \mathrm{ST}_x(\varphi) \models \Sigma[x := w]$, where $\overline{P}$ is the vector of all predicate symbols occurring in $\mathrm{ST}_x(\varphi)$. But also, because of the equivalence between (1) and (2) assumed above, $\Sigma \models \forall\overline{P}\mathrm{ST}_x(\varphi)[x := w]$. If $\Sigma$ is finite, then $\bigwedge \Sigma$ is clearly an $L'$ local frame correspondent for $\varphi$.

Even if $\Sigma$ is infinite, we can still find an $L'$ equivalent, provided $L'$ is compact: Since $\Sigma \models \forall\overline{P}\mathrm{ST}_x(\varphi)[x := w]$ we have $\Sigma \models \mathrm{ST}_x(\varphi)[x := w]$, and we may then appeal to the compactness of $L'$ to find some finite subset $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models \mathrm{ST}_x(\varphi)[x := w]$.

We claim that $\Sigma' \models \forall\overline{P}\mathrm{ST}_x(\varphi)[x := w]$. Indeed, let $\mathcal{M}$ be any $L_1$-model such that $\mathcal{M} \models \Sigma'[x := w]$. Since the predicate symbols in $\overline{P}$ do not occur in $\Sigma'$, every $\overline{P}$-variant of $\mathcal{M}$ also models $\Sigma'$, and hence also $\mathrm{ST}_x(\varphi)$. It follows that $\mathcal{M} \models \forall\overline{P}\mathrm{ST}_x(\varphi)[x := w]$. Thus we may take $\bigwedge \Sigma'$ as a local first-order frame correspondent for $\varphi$.

The case in which $L' = L_0$ and $K$ is the class of all parametrically $L_0$-definable valuations was studied by van Benthem in [4]. Under these assumptions, the class of formulas for which the equivalence between (1) and (2) holds was named the *van Benthem formulas* in [17]. All the well known syntactically characterized classes of first-order definable modal formulas (Sahlqvist, Inductive, etc.) are encompassed by the van Benthem formulas. However, in its full generality, the class of van Benthem formulas is of little practical use. Indeed, for infinite sets $\Sigma$, the above argument, relying on compactness as it does, does not enable us to explicitly calculate a correspondent for a given formula $\varphi$, or devise an algorithm which produces frame correspondents for each member of a given *class* of modal formulas. One therefore typically concentrates on cases in which the class $K$ can be described by $L'$-formulas of uniform shape and hence of bounded complexity. In [23] an account of classical correspondence is given in terms of a hierarchy of such classes $K$.

## 1.7 Characterizing the Sahlqvist formulas across different logics

As discussed at the end of subsection 1.5, being able to measure the effectiveness of the algebraic-algorithmic approach across different logics requires being able to recognize Sahlqvist, Inductive, Recursive classes for each logical language to which the algorithmic correspondence applies. In the following subsection, we give a very portable definition of Sahlqvist formulas, or rather inequalities, that is general enough to be applied unchanged across a wide variety of logics. In subsection 1.7.2, we contrast this briefly with other definitions in the literature.

### *1.7.1 The Sahlqvist inequalities: a general purpose definition*

Given a logic with DLOs as algebraic semantics, what should 'morally' be the class of Sahlqvist formulas for this logic? As glimpsed above, the reduction strategy for Sahlqvist formulas is based on the order-theoretic properties of adjunction and residuation possessed by the operations interpreting the connectives. More specifically, it is the order of alternation of connectives with these properties over certain variable occurrences which is of crucial importance, since it enables the input clause to be transformed into an equivalent one satisfying the restrictions under which (LA) or (RA) can be applied. Our answer will accordingly be couched in these terms.

To fix ideas, let us consider a logical signature containing classical negation (like that of basic modal logic) but otherwise undefined. (Negated) Sahlqvist formulas in such a signature can be described in terms of their generation trees, as illustrated in figure 1.2. Namely, the nodes in the upper part are labelled with connectives interpreted by means of left residuals or $\Delta$-adjoints. The lower parts of branches ending in positive variables are labelled with connectives interpreted by means of right adjoints. This is the basic Sahlqvist shape that we are going to reproduce across signatures.

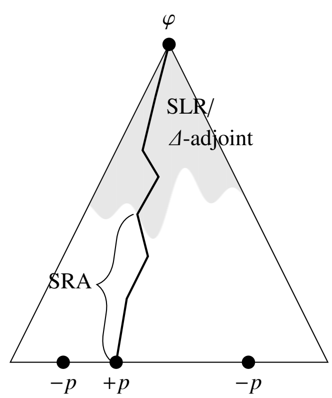


Fig. 1.2: The basic Sahlqvist shape

To port this shape to signatures without classical negation, we will have to introduce some bookkeeping machinery and the following auxiliary definitions and notation: we work with the usual notion of a *generation tree* of a formula. A *signed generation tree* (see e.g., [33]) associates with each node in a generation tree a sign, + or −, in such a way that children of nodes labelled with connectives which are order preserving (order reversing) in the appropriate coordinate have the same (opposite) sign as their parent. The *positive (negative) generation tree of* $\varphi$, denoted $+\varphi$ ($-\varphi$), is thus obtained by signing the root in the generation tree of $\varphi$ with + (−) and propagating the signs.

**Definition 3 (Order types and critical branches).** An *order type* over $n \in \mathbb{N}$ is an $n$-tuple $\epsilon \in \{1, \partial\}^n$. For any formula $\varphi(p_1, \ldots p_n)$, any order type $\epsilon$ over $n$, and any $1 \leq i \leq n$, an *$\epsilon$-critical node* in a signed generation tree of $\varphi$ is a (leaf) node $+p_i$ with $\epsilon_i = 1$ or $-p_i$ with $\epsilon_i = \partial$. An *$\epsilon$-critical branch* in the tree is a branch from an $\epsilon$-critical node.

We are now ready to reproduce the Sahlqvist shape in non-classical settings. For definiteness' sake we work in the distributive setting, and consider a signature which provides a representative sample of connectives commonly encountered in the literature, taken from intuitionistic logic, Distributive Modal logic (cf. [33, 22]), and substructural logic.

| $\Delta$-adjoints | Syntactically Right Adjoints (SRA) |
|---|---|
| $+$ $\vee$ $\wedge$ | $+$ $\wedge$ $\Box$ $\rhd$ |
| $-$ $\wedge$ $\vee$ | $-$ $\vee$ $\Diamond$ $\lhd$ |
| **Syntactically Left Residuals (SLR)** | **Syntactically Right Residuals (SRR)** |
| $+$ $\Diamond$ $\lhd$ $\circ$ | $+$ $\vee$ $\star$ $\rightarrow$ |
| $-$ $\Box$ $\rhd$ $\star$ $\rightarrow$ | $-$ $\wedge$ $\circ$ |

Table 1.1: Classification of nodes

**Definition 4.** Nodes in generation trees are classified according to table 1.1. A branch in a signed generation tree $*s$, $* \in \{+, -\}$, is *excellent* if it is the concatenation of two paths $P_1$ and $P_2$, one of which may possibly be of length 0, such that $P_1$ is a path from the leaf consisting (apart from variable nodes) only of SRA-nodes, and $P_2$ consists (apart from variable nodes) only of $\Delta$-adjoint and SLR-nodes.

**Definition 5.** For any order type $\epsilon$, the signed generation tree of a formula $\varphi$ is $\epsilon$-*Sahlqvist* if every $\epsilon$-critical branch is excellent. An inequality $\varphi \leq \psi$ is $\epsilon$-*Sahlqvist* if the trees $+\varphi$ and $-\psi$ are both $\epsilon$-Sahlqvist. An inequality is *Sahlqvist* if it is $\epsilon$-Sahlqvist for some $\epsilon$.

Notice that, according to definition 5, the generation trees of the two sides of an $\epsilon$-Sahlqvist inequality reproduce the pattern illustrated in figure 1.2, modulo the order type $\epsilon$.

We wish to stress the methodology that definition 5 aims at exemplifying. This definition is intended to serve as a template applicable to any signature via a classification of connectives such as the one of table 1.1. The place of any given logical connective in this classification is *not inherent* to the connective; rather, it entirely depends on the order-theoretic properties of the interpretation of the given connectives, *relative to a specific semantics*, and is hence bound to change when switching to a different interpretation. For instance, the classification of table 1.1 is relative to the usual interpretation of logical connectives in the setting of *distributive* lattices. When interpreted in general lattices, $+\vee$ and $-\wedge$ do not fall into the SRR category anymore, because their standard interpretations in general lattices are not residuated.

In essence, Definition 5 gives us a winning strategy which guarantees the success of our calculus, as well as of algorithms like ALBA, SQEMA and indeed the Sahlqvist-van Benthem algorithm. Success consists in eliminating all occurring variables by means of applications of the Ackermann rules (LA) or (RA). The Sahlqvist shape guarantees that, for every variable, input inequalities can be transformed into a shape to which an Ackermann rule is applicable. The order type $\epsilon$ tells us which occurrences of a given variable $p$ we need to 'display', i.e., get to occur in inequalities of the form $p \leq \alpha$ or $\alpha \leq p$ as prescribed by (LA) or (RA). The Sahlqvist shape guarantees that this is always possible. Indeed, going down a critical branch, we can surface the subtree containing the SRA part of the critical branch, by applying *approximation rules*[9] to the SLR-nodes and $\Delta$-rules (see footnote 7 on page 13) to the $\Delta$-adjoint nodes. Then the SRA-nodes on the remainder of this branch can be stripped off by means of the residuation/adjunction rules, thus surfacing the variable occurrence and simultaneously calculating the minimal valuation for it. Finally, notice that the remaining occurrences of $p$ are of the opposite order type: this guarantees that they have the right polarity to receive the calculated minimal valuations, as prescribed by (LA) or (RA).

*Example 3.* The Dunn axioms for positive modal logic $\Box p \wedge \Diamond q \leq \Diamond(p \wedge q)$ and $\Box(p \vee q) \leq \Box p \vee \Diamond q$, as well their intuitionistic counterparts $\Diamond(p \rightarrow q) \leq \Box p \rightarrow \Diamond q$ and $\Diamond p \rightarrow \Box q \leq \Box(p \rightarrow q)$ are all Sahlqvist inequalities. Specifically, the first inequality is $\epsilon$-Sahlqvist with $\epsilon(p) = 1$ and $\epsilon(q) = 1$, the second is $\epsilon$-Sahlqvist with $\epsilon(p) = \partial$ and $\epsilon(q) = \partial$, and neither is $\epsilon$-Sahlqvist for any other order type. The third and fourth inequalities are both $\epsilon$-Sahlqvist with $\epsilon(p) = 1$ and $\epsilon(q) = \partial$, and again neither is $\epsilon$-Sahlqvist for any other order type.

The Löb inequality $\Box(\Box p \rightarrow p) \leq \Box p$ is not $\epsilon$-Sahlqvist for any order type, because in the positive generation tree of the left hand side both positive and negative occurrences of $p$ have the properly SRR-node $+ \rightarrow$ as ancestor, making their corresponding branches non-excellent.

In similar way, the Frege inequality $p \rightarrow (q \rightarrow r) \leq (p \rightarrow q) \rightarrow (p \rightarrow r)$ is not $\epsilon$-Sahlqvist for any order type, because both positive and negative occurrences of $q$ have properly SRR-nodes $+ \rightarrow$ as ancestors, making their corresponding branches non-excellent.

### 1.7.2 Other approaches to syntactic characterization

Definitions of Sahlqvist-like classes generally come in two flavours: positive, or constructive, definitions that tell one how the formulas in the class can be built up, and negative definitions which define a class by banning certain alternations of connectives. While not being explicitly constructive, the definition offered in the previous subsection is clearly positive. We would like to contrast it with the negative definition used in [33]. This definition classifies the connectives of Distributive

[9] The approximation rules are those which introduce new nominals or co-nominals. All the other rules introduced so far, except (LA), (RA), ($\top$), and ($\bot$), are collectively referred to as residuation/adjunction rules.

Modal Logic (DML) as *Choice* and *Universal*, according to the Table 1.2. A signed generation tree is then declared to be $\epsilon$-Sahlqvist if on no $\epsilon$-critical branch there is a choice node with a universal node as ancestor.[10] The notion of a Sahlqvist inequality is then further defined exactly as in definition 5, above. Comparing Tables 1.1 and 1.2 will also make it clear that, in terms of adjunction and residuation, Choice and Universal have the following meaning:

$$\begin{aligned} \text{Choice} &= \text{Not a right adjoint} \\ \text{Universal} &= \text{Neither a left residual nor a } \Delta\text{-adjoint.} \end{aligned}$$

| Choice | | Universal | |
|---|---|---|---|
| $+$ | $\vee\ \Diamond\ \triangleleft$ | $+$ | $\Box\ \triangleright$ |
| $-$ | $\wedge\ \Box\ \triangleright$ | $-$ | $\Diamond\ \triangleleft$ |

Table 1.2: Universal and choice nodes.

Thus, when restricted to the signature of DML, this definition and definition 5 are equivalent. However, generalizing the Choice-Universal style definition does become problematic once binary connectives like the intuitionistic implication $\rightarrow$ is involved. Indeed, the Heyting implication is not a right adjoint, and is neither a left residual nor a $\Delta$-adjoint, and hence $+ \rightarrow$ is both Choice and Universal. Now, applying the Choice-Universal style definition, inequalities such as $p \rightarrow \Box p \leq \Diamond\Box p$, which cannot be solved, would be classified as $\epsilon$-Sahlqvist with $\epsilon(p) = 1$. One way to remedy this is to declare the ancestor relation to be reflexive, rendering an occurrence of a choice-and-universal node a violation of the rule prohibiting choice nodes in the scope of universal ones. A more elegant solution, we maintain and hope the reader would agree, would be to adopt a definition in the style of the previous subsection.

## 1.8 Three moves towards a unified correspondence theory

The present section is aimed at discussing how three recent directions in correspondence theory can be encompassed in the algebraic-algorithmic approach, based on the recognition that all these directions are predicated on the same basic order-theoretic principles we have discussed in the previous sections. The first generalization, presented in subsection 1.8.1, concerns correspondence settings in which the target language is first-order logic with *least* (or more in general *extremal*) *fixed points* (FO+LFP). In subsection 1.8.2, various syntactic generalizations of the Sahlqvist class will be discussed, which are obtained by relaxing the requirements

[10] This has been slightly paraphrased in order to exploit the terminology already introduced above.

of definition 5. Finally, subsection 1.8.3 focuses on a recent research line in which van Benthem has been active, which extends algorithmic correspondence theory to propositional logics expanded with fixed points, such as the modal mu-calculus.

### 1.8.1 Expanding the target language with fixed points

When trying to reduce an inequality with the calculus of correspondence, one reason of failure is that it is not possible to obtain a form to which (LA) or (RA) is applicable, and particularly because any obtainable $\alpha$ (as in the formulation of these rules) is not $p$-free. Consider for example the Löb inequality $\Box(\Box p \to p) \leq \Box p$. Let us apply the calculus to it:

$$\begin{aligned}
&\forall p[\Box(\Box p \to p) \leq \Box p]\\
\text{iff } &\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box(\Box p \to p) \ \&\ \Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
\text{iff } &\forall p \forall i \forall \mathbf{m}[(\blacklozenge \mathbf{i} \leq \Box p \to p \ \&\ \Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
\text{iff } &\forall p \forall i \forall \mathbf{m}[(\blacklozenge \mathbf{i} \wedge \Box p \leq p \ \&\ \Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}].
\end{aligned}$$

We would have been able to apply (LA), had it not been for the $p$ occurring on the left hand side of $\blacklozenge \mathbf{i} \wedge \Box p \leq p$. So this is how far we can get and no further, and with good reason: the Löb inequality has no first-order frame correspondent, as is well known. However, the Ackermann lemma can be strengthened to the following version:

**Lemma 2.** *Let $\alpha(p)$, $\beta(p)$, and $\gamma(p)$ be formulas of a language $\mathcal{L}^+$ interpreted on perfect DLOs, with $\alpha(p)$ and $\beta(p)$ positive in $p$ and $\gamma(p)$ negative in $p$. Then the following are equivalent for every perfect DLO $\mathbb{C}$ and variable assignment $v$:*

1. $\mathbb{C}, v \models \beta(\mu p.\alpha(p)/p) \leq \gamma(\mu p.\alpha(p))/p)$;
2. *there exists some $v' \sim_p v$ such that $\mathbb{C}, v' \models \alpha(p) \leq p$, and $\mathbb{C}, v' \models \beta(p) \leq \gamma(p)$,*

*where $\mu p.\alpha(p)$ is the least fixed point of $\alpha(p)$.*

*Proof.* We begin by noting that, since we are working in a complete lattice, least fixed points of monotone (term) functions exist by the Knaster-Tarski theorem. As regards '$1 \Rightarrow 2$', let $v'(p) := v(\mu p.\alpha(p))$. As regards '$2 \Rightarrow 1$', $\mathbb{C}, v' \models \alpha(p) \leq p$ implies that $v'(p)$ is a pre-fixed point of $\alpha(\cdot)$,[11] and hence $\mu p.\alpha(p) \leq v'(p)$. Therefore, $\beta(\mu p.\alpha(p)/p) \leq \beta(v'(p)) \leq \gamma(v'(p)) \leq \gamma(\mu p.\alpha(p)/p)$.

Lemma 2 justifies the following rule:

$$\frac{\forall p[(\alpha(p) \leq p \ \&\ \mathop{\&}_{1 \leq i \leq n} \beta_i(p) \leq \gamma_i(p)) \Rightarrow \varphi \leq \psi]}{\mathop{\&}_{1 \leq i \leq n} \beta_i(\mu p.\alpha(p)/p) \leq \gamma_i(\mu p.\alpha(p)/p) \Rightarrow \varphi \leq \psi} \quad \text{(RLA)}$$

where $\alpha, \beta_i$, and $\gamma_i$ are as in the lemma, and $\varphi$ and $\psi$ are negative and positive in $p$, respectively. Back to the Löb inequality, we can now apply RLA to eliminate $p$:

[11] Here $\alpha(\cdot)$ is obtained from the term function $\alpha$ by leaving $p$ free and fixing all other variables to the values prescribed by $v$.

$$
\begin{aligned}
&\text{iff } \forall \mathbf{i}\forall \mathbf{m}[\Box(\mu p.(\blacklozenge \mathbf{i} \wedge \Box p)) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\text{iff } \forall \mathbf{i}[\mathbf{i} \leq \Box(\mu p.(\blacklozenge \mathbf{i} \wedge \Box p))]\\
&\text{iff } \forall \mathbf{i}[\blacklozenge \mathbf{i} \leq \mu p.(\blacklozenge \mathbf{i} \wedge \Box p)].
\end{aligned}
$$

Under duality with Kripke frames, the condition above translates as $\forall w[R[w] \subseteq \mu X.(R[w] \cap (R^{-1}[X^c])^c)]$, which gives the expected condition of transitivity and converse well foundedness.

As another example, consider the van Benthem inequality $\Box\Diamond\top \leq \Box(\Box(\Box p \to p) \to p)$:

$$
\begin{aligned}
&\forall p[\Box\Diamond\top \leq \Box(\Box(\Box p \to p) \to p)]\\
&\text{iff } \forall p\forall \mathbf{i}\forall \mathbf{m}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box(\Box(\Box p \to p) \to p) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\text{iff } \forall p\forall \mathbf{i}\forall \mathbf{m}\forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box\mathbf{n} \leq \mathbf{m} \ \&\ \Box(\Box p \to p) \to p \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\text{iff } \forall p\forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box\mathbf{n} \leq \mathbf{m} \ \&\ \mathbf{j} \leq \Box(\Box p \to p) \ \&\ \mathbf{j} \to p \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\text{iff } \forall p\forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box\mathbf{n} \leq \mathbf{m} \ \&\ \blacklozenge\mathbf{j} \leq \Box p \to p \ \&\ \mathbf{j} \to p \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\text{iff } \forall p\forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box\mathbf{n} \leq \mathbf{m} \ \&\ \blacklozenge\mathbf{j} \wedge \Box p \leq p \ \&\ \mathbf{j} \to p \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
(*)\ &\text{iff } \forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box\mathbf{n} \leq \mathbf{m} \ \&\ \mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p) \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\text{iff } \forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p) \leq \mathbf{n}) \Rightarrow \forall \mathbf{m}[\Box\mathbf{n} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]]\\
&\text{iff } \forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p) \leq \mathbf{n}) \Rightarrow \blacklozenge\mathbf{i} \leq \mathbf{n}]\\
&\text{iff } \forall \mathbf{i}\forall \mathbf{j}[\mathbf{i} \leq \Box\Diamond\top \Rightarrow \forall \mathbf{n}[\mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p) \leq \mathbf{n} \Rightarrow \blacklozenge\mathbf{i} \leq \mathbf{n}]]\\
&\text{iff } \forall \mathbf{i}\forall \mathbf{j}[\mathbf{i} \leq \Box\Diamond\top \Rightarrow \blacklozenge\mathbf{i} \leq \mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p)]\\
&\text{iff } \forall \mathbf{i}\forall \mathbf{j}[\mathbf{i} \leq \Box\Diamond\top \Rightarrow \mathbf{i} \leq \Box(\mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p))]\\
&\text{iff } \forall \mathbf{j}[\Box\Diamond\top \leq \Box(\mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p))].
\end{aligned}
$$

In the equivalence marked with $(*)$, the Right Ackermann lemma has been applied with $\alpha(p) := \blacklozenge\mathbf{j} \wedge \Box p$ and $\beta(p) := \mathbf{j} \to p$ being positive in $p$, and $\gamma(p) := \mathbf{n}$ being negative in $p$.

Correspondence with FO+LFP has been studied in [7], [20], [8], [11] and other papers. It is not possible here to do justice to this work, but that is not the aim of the current paper.

### *1.8.2 Syntactic generalizations of the Sahlqvist class*

The class of Sahlqvist formulas is, quite rightly, considered to be the paradigmatic syntactically definable class of modal formulas admitting first-order correspondents. This pre-eminent status can, however, blind one to the fact that there is much interesting and systematic correspondence theory that can be done with formulas that lie strictly *outside* this class. There is indeed life beyond the Sahlqvist formulas. Some of this work is orthogonal to the Sahlqvist theme, in the sense that the arguments bear no obvious resemblance to the minimal valuation strategy: here we are thinking, for example, of the modal reduction principles interpreted over transitive frames, which all have first-order correspondents [3]. In the present section we will, however, be looking at classes of formulas that represent the natural generalization of the Sahlqvist formulas, in the sense that they are obtained by taking the order-theoretic insights underlying the Sahlqvist 'winning strategy' (see discussion following definition 5) to their natural boundaries of applicability.

A very noticeable feature of definition 5 is the fact that nodes lower down on critical branches need to be syntactically right adjoint, not, e.g., syntactically right residual. For unary connectives, residuation and adjunction are equivalent notions (see appendix), so this imposes no restriction, but for connectives of higher arity it does. For example, the Löb inequality $\Box(\Box p \to p) \leq \Box p$, considered in subsection 1.8.1, is *not* Sahlqvist, for in the generation tree $+\Box(\Box p \to p)$ both the leaves $+p$ and $-p$ have the properly SRR node $+ \to$ as ancestor, and hence for no choice of order type $\epsilon$ will the $\epsilon$-critical paths be excellent. The Löb formula belongs to the class of so-called *Recursive* or *Regular formulas*, introduced in [38], which all have frame correspondents in FO+LFP (see also [20]). We will not burden the reader with precise definitions here, but the intuition is that one firstly relaxes the definition of an excellent branch (definition 4) to that of a *good branch* by also allowing the occurrence of SRR nodes (and not just of SRA nodes) in the lower part of critical branches. Merely substituting the "good" for "excellent" in the definition of the Sahlqvist inequalities (definition 5) would be too liberal, however, for that would allow inequalities like $\Diamond(\Box p \star p) \leq \Diamond p$, on which the calculus of correspondence fails entirely, since we cannot bring them into a shape to which even the recursive Ackermann lemma is applicable. To ensure that the calculus works, it is enough to add the further requirement that at most one $\epsilon$-critical branch may pass through any given properly SRR-node; this yields precisely the *$\epsilon$-Recursive inequalities*.

But here the reader may very well protest that we have promised extensions of the Sahlqvist class for which first-order correspondence holds, while the Recursive formulas are only guaranteed to have correspondents in FO+LFP. Indeed, the Recursive inequalities as a generalization of the Sahlqvist class is still too liberal. In order to guarantee first-order correspondence, the ordinary non-recursive Ackermann lemma will have to be applicable for each variable elimination. In order to ensure this, one needs to impose upon the variables in Recursive inequalities a partial ordering, and demand not only that at most one $\epsilon$-critical branch pass through any given properly SRR-node, but also that if an $\epsilon$-critical branch passes through a properly SRR-node, all variables occurring on other branches passing through it have to be strictly less (according to the ordering) than the variable on the critical branch. This gives rise to the classes of *Inductive formulas* and *inequalities*, for formal definitions of which the reader is referred to [38], [22], and [21]. As an example, the Frege inequality $p \to (q \to r) \leq (p \to q) \to (p \to r)$ from the implicative fragment of intuitionistic logic is Inductive; however, it is not Sahlqvist, as shown in example 3. For an ALBA reduction of this inequality see [22, example 7.5].

### *1.8.3 Correspondence for propositional logics with fixed points*

In the generalized setting of subsection 1.8.1, fixed points have been added to the target language so as to be able to extend the correspondence methodology up to classes of formulas, pre-eminently exemplified by the Löb's formula, for which minimal valuations exist but are not elementarily definable. However, once fixed points are brought into the correspondence picture on the *target* side, it is natural to extend the correspondence program to settings in which fixed points belong also to the *source* language, like the modal mu-calculus; the extra expressivity of the source language will be safely accommodated by the expanded target language. In this vein, in [8], van Benthem and his collaborators syntactically characterized a certain class of formulas in the language of modal mu-calculus as the counterpart of the Sahlqvist class (hence named the class of *Sahlqvist mu-formulas*), on the basis of the minimal valuation methodology, via an extension of the classical model-theoretic proof. In [15], a correspondence result theoretically independent from [8] has been given for logics with fixed points on a weaker than classical base (thus applicable e.g. also to intuitionistic modal mu-calculus, or to certain substructural logics expanded with fixed points). In [15], the results in [8] are encompassed into the algebraic-algorithmic unified correspondence theory, and Sahlqvist mu-formulas are recognized in essence as Recursive formulas (see subsection 1.8.2) on the basis of the approach outlined in subsections 1.7.1 and 1.8.2. The paper [15] is rather technical; however, thanks to the insights developed so far in the present exposition, and particularly on the existing tight connection between the minimal valuation argument and (the recursive and non-recursive versions of) the Ackermann lemma, we are now in a position to give an informal account of these results, as well as of their relationship with results in [8].

Concretely, embedding the Sahlqvist-type theorem of [8] into the algebraic-algorithmic correspondence theory requires:

(a) extending (the distributive/intuitionistic/non-distributive versions of) the calculus for correspondence with dedicated approximation and adjunction/residuation rules (see footnote 9, page 21) capable of transforming systems of mu-inequalities into equivalent systems of mu-inequalities in Ackermann shape; [12]

(b) giving a (distributive/intuitionistic/non-distributive) counterpart of the class of Sahlqvist mu-formulas as defined in [8] in the style of definition 5;

(c) motivating the definitions in (b) by giving surjective projections from the non-classical languages involved to the classical, which preserve and reflect Sahlqvist status. An analogous projection has been given in [22] between DML and classical modal logic.

Due to space constrains we will only address (a) and (b). As to (a), notice preliminarily that the calculus for correspondence introduced in section 1.3 is already enough to perform the elimination of predicate variables on a *restricted* class of mu-formulas/ inequalities[13], as in the following example (cf. [8, example 5.3]):

[12] Notice that, thanks to the very general way in which the various versions of Ackemann's lemma have been stated, the corresponding Ackermann rules apply without changes to logical languages with fixed points.

[13] Namely, the one formed by those inequalities such that, for some order type $\epsilon$, all $\epsilon$-critical branches are excellent (cf. definition 4) or good (cf. subsection 1.8.2) according to the letter of these notions, and hence no fixed point binders occur in $\epsilon$-critical branches.

$$
\begin{array}{lll}
 & \forall p[\nu X.\Box(p \wedge X) \leq p] & \\
 & \text{iff } \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(p \wedge X) \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(ULA)} \\
(*) & \text{iff } \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(\mathbf{m} \wedge X) \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(LA)} \\
 & \text{iff } \forall \mathbf{m}[\nu X.\Box(\mathbf{m} \wedge X) \leq \mathbf{m}]. & \text{(FA inverse)}
\end{array}
$$

Indeed, the application of the rule (LA) is sound because the term function $\gamma(p) = \nu X.\Box(p \wedge X)$ is monotone in $p$. However, this calculus is certainly not powerful enough to be successful over the whole class of (intuitionistic counterparts of) Sahlqvist mu-formulas in [8]. In [15], an enhancement of the calculus has been defined by firstly adding approximation rules, of which the following are special instances (cf. [15] for the complete account):

$$
\frac{\mathbf{i} \leq \mu X.\varphi(X, \psi/!x, \overline{z})}{\exists \mathbf{j}[\mathbf{i} \leq \mu X.\varphi(X, \mathbf{j}/!x, \overline{z}) \ \&\ \mathbf{j} \leq \psi]}\ (\mu^+\text{-A}) \qquad \frac{\nu X.\varphi(X, \psi/!x, \overline{z}) \leq \mathbf{m}}{\exists \mathbf{n}[\nu X.\varphi(X, \mathbf{n}/!x, \overline{z}) \leq \mathbf{m} \ \&\ \psi \leq \mathbf{n}]}\ (\nu^+\text{-A})
$$

$$
\frac{\mathbf{i} \leq \mu X.\varphi(X, \psi/!x, \overline{z})}{\exists \mathbf{n}[\mathbf{i} \leq \mu X.\varphi(X, \mathbf{n}/!x, \overline{z}) \ \&\ \psi \leq \mathbf{n}]}\ (\mu^-\text{-A}) \qquad \frac{\nu X.\varphi(X, \psi/!x, \overline{z}) \leq \mathbf{m}}{\exists \mathbf{j}[\nu X.\varphi(X, \mathbf{j}/!x, \overline{z}) \leq \mathbf{m} \ \&\ \mathbf{j} \leq \psi]}\ (\nu^-\text{-A})
$$

where, in ($\mu^+$-A) (resp., ($\mu^-$-A)) the associated term function of $\varphi(X, x, \overline{z})$ is completely $\bigvee$-preserving in $(X, x) \in \mathbb{C} \times \mathbb{C}$ (resp., in $(X, x) \in \mathbb{C} \times \mathbb{C}^\partial$), and in ($\nu^+$-A) (resp., ($\nu^-$-A)) the associated term function of $\varphi(X, x, \overline{z})$ is completely $\bigwedge$-preserving in $(X, x) \in \mathbb{C} \times \mathbb{C}$ (resp., in $(X, x) \in \mathbb{C} \times \mathbb{C}^\partial$), for any perfect DLO $\mathbb{C}$ of the appropriate signature. Moreover, in each rule the variable $x$ is assumed not to occur in $\psi$. The notation $\varphi(!x)$ means that the variable $x$ has a unique occurrence in $\varphi$.

Some motivating intuitions and examples illustrating the functioning and applicability of these rules, as well as of the adjunction-rules below, are given in the ensuing discussion.

Secondly, adjunction rules for fixed point binders have been added, of which the following are special instances (cf. [15] for the complete account):

$$
\frac{\mu X.(A(X) \vee B(p)) \leq \chi}{p \leq \nu X.(E(X) \wedge D(\chi/p))}\ (\mu\text{-Adj}) \qquad \frac{\chi \leq \nu X.(E(X) \wedge D(p))}{\mu X.(A(X) \vee B(\chi/p)) \leq p}\ (\nu\text{-Adj})
$$

where, in each rule,

$$
A(X) = \bigvee_{i \in I} \delta_i(X), \quad B(p) = \bigvee_{j \in J} \delta'_j(p), \quad E(X) = \bigwedge_{i \in I} \beta_i(X) \ \text{ and } \ D(p) = \bigwedge_{j \in J} \beta'_j(p)
$$

with $I$ and $J$ finite sets of indexes, each $\delta_i$ and $\delta'_j$ interpreted as a unary left adjoint (typically, $\delta_i$ and $\delta'_j$ are concatenations of diamonds over a variable), and each $\beta_i$ and $\beta'_j$ interpreted as a unary right adjoint (typically, $\beta_i$ and $\beta'_j$ are boxed atoms). Finally, $\delta_i \dashv \beta_i$ and $\delta'_j \dashv \beta'_j$ for each $i$ and $j$.

Notice that, unlike the rules for propositional connectives, the rules above are *contextual*, i.e., dependent on assumptions on the formulas in the scope of the fixed point binder. This reflects the fact that the semantic interpretations of fixed point binders do not have intrinsic order-theoretic properties, but at most preserve those of the term functions associated with the formulas in their scope.

In [15], rules generalizing the ones above are proven to be sound w.r.t. the natural algebraic/relational semantics of (intuitionistic) modal mu-calculus. Thanks to these rules, inequalities we could previously not treat, such as $p \leq \nu X[\Box(X \wedge (q \to \bot)) \vee (\Diamond p \wedge \Diamond q)]$ (cf. [8, example 5.4]) can be reduced as follows:

$$\begin{array}{lr}
\forall p \forall q[p \leq \nu X[\Box(X \wedge (q \to \bot)) \vee (\Diamond p \wedge \Diamond q)]] & \\
\text{iff } \forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq p \ \&\ \nu X[\Box(X \wedge (q \to \bot)) \vee (\Diamond p \wedge \Diamond q)] \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(ULA)} \\
\text{iff } \forall q \forall \mathbf{i} \forall \mathbf{m}[\nu X[\Box(X \wedge (q \to \bot)) \vee (\Diamond \mathbf{i} \wedge \Diamond q)] \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(RA)} \\
\text{iff } \forall q \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{j} \leq q \ \&\ \nu X[\Box(X \wedge (\mathbf{j} \to \bot)) \vee (\Diamond \mathbf{i} \wedge \Diamond q)] \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & (\nu^{-}\text{-A}) \\
\text{iff } \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[\nu X[\Box(X \wedge (\mathbf{j} \to \bot)) \vee (\Diamond \mathbf{i} \wedge \Diamond \mathbf{j})] \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(RA)} \\
\text{iff } \forall \mathbf{i} \forall \mathbf{j}[\mathbf{i} \leq \nu X[\Box(X \wedge (\mathbf{j} \to \bot)) \vee (\Diamond \mathbf{i} \wedge \Diamond \mathbf{j})]]. & \text{(UA inverse)}
\end{array}$$

In the application of ($\nu^{-}$-A) above, $\varphi(X, !x, z)$ is $\Box(X \wedge (x \to \bot)) \vee z$, and $\psi$ is $q$.

Moreover, the following alternative reduction is now possible for the inequality $\nu X.\Box(p \wedge X) \leq p$, treated as the first example of the present subsection:

$$\begin{array}{ll}
\forall p[\nu X.\Box(p \wedge \Box X) \leq p] & \\
\text{iff } \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(p \wedge \Box X) \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(ULA)} \\
\text{iff } \forall p \forall \mathbf{i} \forall \mathbf{m}[\mu X.\blacklozenge(\blacklozenge X \vee \mathbf{i}) \leq p \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & (\nu\text{-Adj}) \\
\text{iff } \forall \mathbf{i} \forall \mathbf{m}[\mu X.\blacklozenge(\blacklozenge X \vee \mathbf{i}) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(RA)} \\
\text{iff } \forall \mathbf{i}[\mathbf{i} \leq \mu X.\blacklozenge(\blacklozenge X \vee \mathbf{i})]. & \text{(UA inverse)}
\end{array}$$

The application of ($\nu$-Adj) is performed modulo distributing modal connectives. Notice that, by unfolding the least fixed point $\mu X.\blacklozenge(\blacklozenge X \vee \mathbf{i})$, the clause $\forall \mathbf{i}[\mathbf{i} \leq \mu X.\blacklozenge(\blacklozenge X \vee \mathbf{i})]$ can be rewritten as $\forall \mathbf{i}[\mathbf{i} \leq \bigvee_{\kappa \geq 1} \blacklozenge^{\kappa} \mathbf{i}]$, which immediately translates on Kripke frames into the well known condition expressing the reflexivity of the transitive closure of the relation interpreting $\Box$.

As to (b), the class of $\epsilon$-*Recursive inequalities*, in the intuitionistic modal mu-language, has been syntactically defined in [15] closely following the approach of definition 5; this class is the intuitionistic counterpart of the class of Sahlqvist mu-formulas defined in [8]. Analogously to definition 5, the definition of $\epsilon$-Recursive inequalities is grounded on a classification of the nodes in the signed generation trees of formulas similar to the specification given in table 1.3. However, as was mentioned early on, the fixed point binders escape to some extent the order-theoretic classification, since their interpretation does not enjoy inherent order-theoretic properties, but rather preserves, in some cases, those of the term function in its scope. To take this fact into account, we firstly group nodes according to categories (we use the names *skeleton* and *PIA* for these categories, also appearing in [8], to explicitly establish a connection with the model-theoretic analysis conducted there), and secondly, we group nodes within each category according to their contextually relevant order-theoretic properties.

The shape of the $\epsilon$-Recursive inequalities is in essence the Sahlqvist / Inductive / Recursive shape introduced and discussed in subsection 1.7.1; as to the similarities, the outer skeleton is exactly the same as the outer part of a Sahlqvist formula; moreover the PIA part is defined in such a way that, when restricted to the binder-free fragment, it gives the inner part of the $\epsilon$-Recursive formulas (cf. subsection 1.8.2). The complete definition of the PIA part incorporates extra conditions regulating the relative positions of free fixed point variables and variables which we

| Outer Skeleton | Inner Skeleton | PIA |
|---|---|---|
| $\varDelta$-adjoints | Binders | Binders |
| $+$ $\vee$ $\wedge$ | $+$ $\mu$ | $+$ $\nu$ |
| $-$ $\wedge$ $\vee$ | $-$ $\nu$ | $-$ $\mu$ |
| SLR | SLA | SRA |
| $+$ $\Diamond$ $\lhd$ $\circ$ | $+$ $\Diamond$ $\lhd$ $\vee$ | $+$ $\Box$ $\rhd$ $\wedge$ |
| $-$ $\Box$ $\rhd$ $\star$ $\rightarrow$ | $-$ $\Box$ $\rhd$ $\wedge$ | $-$ $\Diamond$ $\lhd$ $\vee$ |
| | SLR | SRR |
| | $+$ $\wedge$ $\circ$ | $+$ $\vee$ $\star$ $\rightarrow$ |
| | $-$ $\vee$ $\star$ $\rightarrow$ | $-$ $\wedge$ $\circ$ |

Table 1.3: Skeleton and PIA nodes.

want to solve for; these conditions ensure that formulas in the scope of binders have the appropriate order-theoretic properties guaranteeing the applicability of the $\mu$- and $\nu$-adjunction rules. The inner skeleton essentially arises by the introduction of fixed point binders into the outer part of a Sahlqvist formula. As to the differences, this introduction blocks the application of $\varDelta$-rules (and more generally also the possibility of applying rules to single connectives), leaving us with only $\mu$- and $\nu$-approximation rules. Hence all the nodes are reclassified according to the properties which they enjoy and which are now relevant. Similar to the PIA formulas, inner skeletons incorporate extra conditions regulating the relative positions of free fixed point variables and variables which we want to solve for; these conditions ensure that formulas in the scope of binders have the appropriate order-theoretic properties guaranteeing the applicability of the $\mu$- and $\nu$-approximation rules. The shape of the inequalities in this class provides a winning strategy analogous to the one described for the Sahlqvist inequalities in subsection 1.7.1. Again, the order type $\epsilon$ tells us which occurrences of a given variable we need to 'display'. The $\epsilon$-Recursive shape guarantees that this is always possible. Indeed, going down a critical branch, we can surface the subtree containing the PIA part of the critical branch by applying *approximation rules* to the Skeleton nodes. Then adjunction/residuation rules such as ($\mu$-Adj) and ($\nu$-Adj) are applied to display the critical occurrences of variables in the subtrees containing the PIA parts, and to simultaneously calculate the minimal valuation for them. Finally, notice that the remaining occurrences of variables are of the opposite order type: this guarantees that they have the right polarity to receive the calculated minimal valuations, as prescribed by (LA), (RA) or their recursive counterparts.

The analysis of PIA-formulas conducted in [8] can be summarized in the slogan "PIA formulas provide minimal valuations". In this respect, the crucial model-theoretic property possessed by PIA-formulas is the *intersection property*, isolated by van Benthem in [6]. The order-theoretic import of this property is clear: if a formula has the intersection property then the term function associated with it is completely meet preserving. In the complete lattice setting in which we find ourselves, this is equivalent to it being a right adjoint; this is exactly the order-theoretic property guaranteeing the soundness of adjunction/residuation rules like ($\mu$-Adj) and ($\nu$-Adj).[14]

## 1.9 Correspondence across different semantics

In section 1.7 we saw that it is possible to uniformly implement the 'correspondence calculus' for different logics, and how, accordingly, the definitions of syntactic classes like the Sahlqvist class could be ported to these logics. In the current section we shift our focus to consider the related question of what happens when we keep the logical language and the order-theoretic properties of the connectives fixed, while varying the relational semantics. In terms of figure 1.1(b) this means that we maintain the algebraic interpretation of the logic while imposing different dualities. The key point we wish to illustrate is that the calculus of correspondence is sound in the setting of perfect distributive lattice expansions, and hence that the elimination of propositional variables can proceed largely independently of any considerations on the dual relational structures; the outcome of the reduction/elimination process can be then further translated so as to fit different relational environments.

We take as our running example Pierce's law $((p \to q) \to p) \to p$, which was considered in [5, Section 3.2] where correspondence is studied for this and other formulas belonging to the implicative fragment of intuitionistic logic. The calculus of section 1.3 gives us the following reduction, which is sound on perfect Heyting algebras (i.e., perfect distributive lattices expanded with the right residual $\to$ of $\wedge$):

$$\begin{array}{ll} & \forall p \forall q[(p \to q) \to p \leq p] \\ \text{iff} & \forall p[(p \to \bot) \to p \leq p] \\ \text{iff} & \forall p \forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq (p \to \bot) \to p \;\&\; p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\ \text{iff} & \forall \mathbf{j} \forall \mathbf{m}[\mathbf{j} \leq (\mathbf{m} \to \bot) \to \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}] \\ \text{iff} & \forall \mathbf{m}[(\mathbf{m} \to \bot) \to \mathbf{m} \leq \mathbf{m}]. \end{array} \tag{1.1}$$

Thus, the propositional variables have been eliminated, and we can interpret the result on intuitionistic frames (i.e. posets), via the well known duality between perfect Heyting algebras and intuitionistic frames. Recall that, in intuitionistic frames, variables (and consequently all formulas) are evaluated to upward-closed subsets (up-sets), and that in particular $[\![\varphi \to \psi]\!] = (((([\![\varphi]\!]^c \cup [\![\psi]\!])^c){\downarrow})^c = ([\![\varphi]\!] \cap [\![\psi]\!]^c){\downarrow}^c$, where $S^c$ and $S{\downarrow}$ denote the set theoretic complement and downward closure of the subset $S$, respectively. All this fits with the duality between perfect Heyting algebras and intuitionistic frames, according to which the algebra elements correspond to up-sets of frames, and in particular the meet prime elements correspond to the

[14] We must warn the reader that this account, and in particular the formulation of the additional rules, is slightly oversimplified. Complete details can be found in [15].

complements of principal down-sets, which we denote by $w{\downarrow}^c$. So, translating the outcome obtained above via this duality yields:

$$
\begin{array}{llll}
 & \forall \mathbf{m}[(\mathbf{m} \to \bot) \to \mathbf{m} \leq \mathbf{m}] & \text{iff} & \forall w[((w{\downarrow}^c \cap \varnothing^c){\downarrow}^c \cap w{\downarrow}^{cc}){\downarrow}^c \subseteq w{\downarrow}^c] \\
\text{iff} & \forall w[w{\downarrow} \subseteq ((w{\downarrow}^c \cap \varnothing^c){\downarrow}^c \cap w{\downarrow}){\downarrow}] & \text{iff} & \forall w[w{\downarrow} \subseteq ((w{\downarrow}^c){\downarrow}^c \cap w{\downarrow}){\downarrow}] \\
\text{iff} & \forall w[w \in ((w{\downarrow}^c){\downarrow}^c \cap w{\downarrow}){\downarrow}] & \text{iff} & \forall w \exists u[w \leq u \ \&\ u \in (w{\downarrow}^c){\downarrow}^c \cap w{\downarrow})] \\
\text{iff} & \forall w \exists u[w \leq u \ \&\ u \leq w \ \&\ u \in (w{\downarrow}^c){\downarrow}^c] & \text{iff} & \forall w[w \in (w{\downarrow}^c){\downarrow}^c] \\
\text{iff} & \forall w[w \notin (w{\downarrow}^c){\downarrow}] & \text{iff} & \forall w \forall v[v \notin w{\downarrow} \ \Rightarrow\ w \not\leq v] \\
\text{iff} & \forall w \forall v[v \not\leq w \ \Rightarrow\ w \not\leq v] & \text{iff} & \forall w \forall v[w \leq v \ \Rightarrow\ v \leq w].
\end{array}
$$

Thus, as discussed in [5, Example 78], we see that Pierce's law takes us to classical propositional logic, by constraining the ordering on intuitionistic frames to be discrete.

Pierce's law (as well as any other axiom in the implicative fragment of intuitionistic logic) can be alternatively interpreted on ternary frames, as they are defined e.g. in [42], where a Kripkean semantics is employed for the non-associative Lambek calculus, and a restricted Sahlqvist theorem is proven. A *ternary frame* (cf. [42, Definition 1]) is a structure $(W, R)$ such that $W$ is a nonempty set and $R$ is a ternary relation on $W$. For all $X, Y \subseteq W$, let $R[Y, X] = \{z \mid \exists x \exists y[x \in Y \ \&\ y \in X \ \&\ R(xyz)]\}$. Implication can be interpreted on ternary frames as follows: for all $X, Y \subseteq W$,

$$X \Longrightarrow Y = \{z \mid \forall x \forall y[(R(yxz) \ \&\ x \in X) \Rightarrow y \in Y]\} = R[Y^c, X]^c.$$

Valuations send proposition letters to arbitrary subsets of the universe of ternary frames. Thus, the complex algebra of the ternary frame $(W, R)$ can be defined as the perfect algebra $(\mathcal{P}(W), \cup, \cap, W, \varnothing, \Longrightarrow)$, and this assignment can be extended to a fully fledged discrete Stone-type duality for BAOs, in the style of e.g. [47]. In particular, $\Longrightarrow$ as defined above is order-reversing (in fact, completely join-reversing) in its first coordinate and order-preserving (in fact, completely meet-preserving) in its second coordinate[15] (see section 1.11 and references therein for more details). Thus, the very same reduction performed in (1.1) is sound also w.r.t. the complex algebras of ternary frames defined above, or equivalently, w.r.t. ternary frame semantics. Relying on this duality, the final clause of (1.1) can be interpreted on ternary frames as follows (we abuse notation and write $w^c$ for $\{w\}^c = W \setminus \{w\}$):

$$
\begin{array}{llll}
 & \forall \mathbf{m}[(\mathbf{m} \to \bot) \to \mathbf{m} \leq \mathbf{m}] & \text{iff} & \forall w[R[w^{cc}, R[\varnothing^c, w^c]^c]^c \subseteq w^c] \\
\text{iff} & \forall w[w \in R[\{w\}, R[W, w^c]^c]] & \text{iff} & \forall w \exists x \exists y[R(xyw) \ \&\ y \in R[W, w^c]^c \ \&\ x = w] \\
\text{iff} & \forall w \exists y[R(wyw) \ \&\ y \in R[W, w^c]^c] & \text{iff} & \forall w \exists y[R(wyw) \ \&\ \forall x \forall z[R(xzy) \Rightarrow z = w]].
\end{array}
$$

Notice that, in the more familiar case in which the operation $\bullet$, uniquely identifying $\Longrightarrow$, coincides with meet, the ternary relation which dually represents the binary map given by $(U, V) \mapsto U \cap V$ is $R = \{(x, x, x) \mid x \in W\}$; in this case, $X \Longrightarrow Y$ reduces to the classical $X^c \cup Y$, and the first-order clause above is always true.

---

[15] In fact, $\Longrightarrow$ can be uniquely identified as the right residual of $\bullet$ (*fusion*), given by $Y \bullet Z := \{x \mid \exists y \exists z[y \in Y \ \&\ z \in Z \ \&\ R(x, y, z)]\}$.

*A second example.* The inequality $(p \wedge q) \to r \leq (p \to r) \vee (q \to r)$, in the intuitionistic language, is reduced by the calculus as follows:

$$
\begin{array}{l}
\forall p \forall q \forall r[(p \wedge q) \to r \leq (p \to r) \vee (q \to r)] \\
\forall \mathbf{i} \forall \mathbf{m} \forall p \forall q \forall r[(\mathbf{i} \leq (p \wedge q) \to r \ \& \ (p \to r) \vee (q \to r) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\forall \mathbf{i} \forall \mathbf{m} \forall p \forall q \forall r[(\mathbf{i} \leq (p \wedge q) \to r \ \& \ (p \to r) \leq \mathbf{m} \ \& \ (q \to r) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{m} \forall p \forall q \forall r[(\mathbf{i} \leq (p \wedge q) \to r \ \& \ (\mathbf{j} \to r) \leq \mathbf{m} \ \& \ \mathbf{j} \leq p \ \& \ (\mathbf{k} \to r) \leq \mathbf{m} \ \& \ \mathbf{k} \leq q) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{m} \forall r[(\mathbf{i} \leq (\mathbf{j} \wedge \mathbf{k}) \to r \ \& \ (\mathbf{j} \to r) \leq \mathbf{m} \ \& \ (\mathbf{k} \to r) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{m} \forall r[(\mathbf{i} \wedge (\mathbf{j} \wedge \mathbf{k}) \leq r \ \& \ (\mathbf{j} \to r) \leq \mathbf{m} \ \& \ (\mathbf{k} \to r) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{m}[((\mathbf{j} \to (\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k})) \leq \mathbf{m} \ \& \ (\mathbf{k} \to (\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k})) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{m}[(\mathbf{j} \to (\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k})) \vee (\mathbf{k} \to (\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k})) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k}[\mathbf{i} \leq (\mathbf{j} \to (\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k})) \vee (\mathbf{k} \to (\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}))] \\
\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k}[\mathbf{i} \leq (\mathbf{j} \to (\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k})) \ \invamp \ \mathbf{i} \leq (\mathbf{k} \to (\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}))] \\
\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k}[\mathbf{i} \wedge \mathbf{j} \leq \mathbf{k} \ \invamp \ \mathbf{i} \wedge \mathbf{k} \leq \mathbf{j}].
\end{array}
$$

For reasons analogous to those discussed in the previous example, this reduction is sound w.r.t. several classes of algebras based on perfect distributive lattices (and hence w.r.t. the classes of set-based structures dual to each of these), which include, but are not limited to, perfect (i.e. complete and atomic) Boolean algebras (hence sets), perfect Heyting algebras (hence posets) and the perfect BAO of the previous example (hence ternary frames as in the previous example). When interpreted according to the first or third option, or equivalently on sets or ternary frames, the last line in the reduction above becomes:

$$\forall w \forall v \forall u[\{w\} \cap \{v\} \subseteq \{u\} \ \invamp \ \{w\} \cap \{u\} \subseteq \{v\}],$$

which always holds, as was expected, since the inequality treated above is classically (but not intuitionistically) valid. When interpreted in perfect Heyting algebras, or equivalently on posets, the last line in the reduction above can be further translated into

$$\forall w \forall v \forall u[w{\uparrow} \cap v{\uparrow} \subseteq u{\uparrow} \ \invamp \ w{\uparrow} \cap u{\uparrow} \subseteq v{\uparrow}],$$

which is equivalent to the condition that every principal up-set be linearly ordered. Indeed, it is clear that, if in a poset there are states $w, v, u$ such that $w \leq v$ and $w \leq u$ but $u \nleq v$ and $v \nleq u$, then neither inclusion in the condition above holds for these states; conversely, reasoning by cases should convince the reader that if in a poset every principal up-set is linearly ordered, then the displayed condition holds. For instance, if $u \nleq v$ and $v \nleq u$, and $w{\uparrow} \cap v{\uparrow} \neq \varnothing \neq w{\uparrow} \cap u{\uparrow}$, let us assume that $w{\uparrow} \cap v{\uparrow} \nsubseteq u{\uparrow}$, i.e. that there exists some $x \in w{\uparrow} \cap v{\uparrow}$ such that $u \nleq x$, and let $y \in w{\uparrow} \cap u{\uparrow}$. Then $y \nleq x$, but since $x, y \in w{\uparrow}$, the assumption implies that $x \leq y$, and hence $y \in v{\uparrow}$, as desired.

*Intuitionistic correspondence via Gödel translation.* So far in the present section, we have seen that the purely syntactic encoding of correspondence arguments is particularly advantageous in those situations (common to many nonclassical logics) in which a given logical language is interpreted on more than one type of set-based structures; indeed, the soundness of a given algorithmic reduction depends exclusively on the order-theoretic properties of the interpretation of the logical connectives, and, provided these properties are satisfied in each interpretation, the same reduction will yield first-order correspondents in relational structures of different types. Sometimes, as in the case of intuitionistic modal logic, the availability of different relational semantics for a given logic reflects itself in the fact that the category of perfect algebras naturally associated with the logic in question is dually equivalent to each category of relational structures supporting the interpretation of that logic. However, in some cases, the roles of algebras and relational structures might be reversed, in the sense that more than one category of algebras might be dually associated with one and the same category of relational structures. This is the case in e.g. the category of posets and p-morphisms, which is dually equivalent to both the category of perfect Heyting algebras and complete homomorphisms via Birkhoff duality, and to a suitable full subcategory of perfect modal algebras and complete homomorphisms via the Jónsson-Tarski duality. Notice that, for every poset $(W, \leq)$, the inclusion map $\mathcal{P}^{\uparrow}(W) \hookrightarrow (\mathcal{P}(W), [\leq])$ satisfies the clauses of the Gödel assignment, i.e. $U \mapsto U = [\leq]U$ and $(U \to V) \mapsto (U \to V) = [\leq](U^c \cup V)$ for every $U, V \in \mathcal{P}^{\uparrow}(W)$, which implies the well known fact that an intuitionistic formula is valid on a given poset $(W, \leq)$ if its Gödel translation is. On the syntactic side, the Sahlqvist/Inductive shape of formulas in the language of intuitionistic logic is preserved under the Gödel translation. In the light of these observations it is natural to ask to what extent intuitionistic correspondence arguments can be subsumed by classical correspondence arguments via the Gödel translation. This question, formulated as vaguely as we have, can be reformulated more concretely in ways which—more importantly for our purposes here—lend themselves to be investigated with the tools of the unified correspondence theory outlined in the present paper.

One such reformulation is: can the reduction steps for the intuitionistic language which are sound on perfect Heyting algebras be *simulated* by suitable reduction steps for the target modal language of the Gödel translation and which are sound on perfect BAOs? And is the Gödel translation itself, as it were, such a simulation? This would be the case, in a sense, if the minimal valuations calculated in performing the reduction steps on an intuitionistic inequality and on its Gödel translation were always semantically identical. In general, one cannot expect this to hold, as the minimal valuation provided by the calculus in the classical setting need not be *persistent*, as required by the intuitionistic notion of validity. However, running the calculus on the Gödel translation of the inequality in the example above proves instructive; below, $\Box$ stands for $[\leq]$ and $\blacklozenge$ for $\langle\geq\rangle$.

$$
\begin{aligned}
&\forall p \forall q \forall r[\Box((\Box p \wedge \Box q) \to \Box r) \leq \Box(\Box p \to \Box r) \vee \Box(\Box q \to \Box r)]\\
&\forall \mathbf{i} \forall \mathbf{m} \forall p \forall q \forall r[(\mathbf{i} \leq \Box((\Box p \wedge \Box q) \to \Box r) \mathbin{\&} \Box(\Box p \to \Box r) \vee \Box(\Box q \to \Box r) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\forall \mathbf{i} \forall \mathbf{m} \forall p \forall q \forall r[(\mathbf{i} \leq \Box((\Box p \wedge \Box q) \to \Box r) \mathbin{\&} \Box(\Box p \to \Box r) \leq \mathbf{m} \mathbin{\&} \Box(\Box q \to \Box r) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{m} \forall p \forall q \forall r[(\mathbf{i} \leq \Box((\Box p \wedge \Box q) \to \Box r) \mathbin{\&} \Box(\mathbf{j} \to r) \leq \mathbf{m}\\
&\qquad \mathbin{\&} \mathbf{j} \leq \Box p \mathbin{\&} \Box(\mathbf{k} \to r) \leq \mathbf{m} \mathbin{\&} \mathbf{k} \leq \Box q) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{m} \forall p \forall q \forall r[(\mathbf{i} \leq \Box((\Box p \wedge \Box q) \to \Box r) \mathbin{\&} \Box(\mathbf{j} \to r) \leq \mathbf{m}\\
&\qquad \mathbin{\&} \blacklozenge \mathbf{j} \leq p \mathbin{\&} \Box(\mathbf{k} \to r) \leq \mathbf{m} \mathbin{\&} \blacklozenge \mathbf{k} \leq q) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{m} \forall r[(\mathbf{i} \leq \Box((\Box \blacklozenge \mathbf{j} \wedge \Box \blacklozenge \mathbf{k}) \to \Box r) \mathbin{\&} \Box(\mathbf{j} \to r) \leq \mathbf{m} \mathbin{\&} \Box(\mathbf{k} \to r) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{m} \forall r[\blacklozenge(\blacklozenge \mathbf{i} \wedge (\Box \blacklozenge \mathbf{j} \wedge \Box \blacklozenge \mathbf{k})) \leq r \mathbin{\&} \Box(\mathbf{j} \to r) \leq \mathbf{m} \mathbin{\&} \Box(\mathbf{k} \to r) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{m}[(\Box(\mathbf{j} \to \blacklozenge(\blacklozenge \mathbf{i} \wedge \Box \blacklozenge \mathbf{j} \wedge \Box \blacklozenge \mathbf{k})) \leq \mathbf{m} \mathbin{\&} \Box(\mathbf{k} \to \blacklozenge(\blacklozenge \mathbf{i} \wedge \Box \blacklozenge \mathbf{j} \wedge \Box \blacklozenge \mathbf{k})) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
&\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{m}[(\Box(\mathbf{j} \to \blacklozenge(\blacklozenge \mathbf{i} \wedge \Box \blacklozenge \mathbf{j} \wedge \Box \blacklozenge \mathbf{k})) \vee \Box(\mathbf{k} \to \blacklozenge(\blacklozenge \mathbf{i} \wedge \Box \blacklozenge \mathbf{j} \wedge \Box \blacklozenge \mathbf{k})) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]
\end{aligned}
$$

$$\begin{array}{l}
\forall\mathbf{i}\forall\mathbf{j}\forall\mathbf{k}[\mathbf{i} \leq \Box(\mathbf{j} \to \blacklozenge(\blacklozenge\mathbf{i} \wedge \Box\blacklozenge\mathbf{j} \wedge \Box\blacklozenge\mathbf{k})) \vee \Box(\mathbf{k} \to \blacklozenge(\blacklozenge\mathbf{i} \wedge \Box\blacklozenge\mathbf{j} \wedge \Box\blacklozenge\mathbf{k}))]\\
\forall\mathbf{i}\forall\mathbf{j}\forall\mathbf{k}[\mathbf{i} \leq \Box(\mathbf{j} \to \blacklozenge(\blacklozenge\mathbf{i} \wedge \Box\blacklozenge\mathbf{j} \wedge \Box\blacklozenge\mathbf{k})) \parr \mathbf{i} \leq \Box(\mathbf{k} \to \blacklozenge(\blacklozenge\mathbf{i} \wedge \Box\blacklozenge\mathbf{j} \wedge \Box\blacklozenge\mathbf{k}))]\\
\forall\mathbf{i}\forall\mathbf{j}\forall\mathbf{k}[\blacklozenge\mathbf{i} \wedge \mathbf{j} \leq \blacklozenge(\blacklozenge\mathbf{i} \wedge \Box\blacklozenge\mathbf{j} \wedge \Box\blacklozenge\mathbf{k}) \parr \blacklozenge\mathbf{i} \wedge \mathbf{k} \leq \blacklozenge(\blacklozenge\mathbf{i} \wedge \Box\blacklozenge\mathbf{j} \wedge \Box\blacklozenge\mathbf{k})].
\end{array}$$

The minimal valuation computed above assigns $p$ to $\blacklozenge\mathbf{j} = \langle\geq\rangle\{w\} = w{\uparrow}$; analogously, $q$ is mapped by the same valuation to $\blacklozenge\mathbf{k} = u{\uparrow}$, and $r$ to $\blacklozenge(\blacklozenge\mathbf{i}\wedge(\Box\blacklozenge\mathbf{j}\wedge\Box\blacklozenge\mathbf{k}))$. The assignment for $r$ can be rewritten as follows:

$$\blacklozenge(\blacklozenge\mathbf{i}\wedge(\Box\blacklozenge\mathbf{j}\wedge\Box\blacklozenge\mathbf{k})) = (w{\uparrow}\cap((v{\uparrow}^c){\downarrow})^c\cap((u{\uparrow}^c){\downarrow})^c){\uparrow} = (w{\uparrow}\cap v{\uparrow}\cap u{\uparrow}){\uparrow} = w{\uparrow}\cap v{\uparrow}\cap u{\uparrow}.$$

So, in this case, the minimal valuation provided by the reduction of the Gödel translation in the boolean setting is *exactly the same* as that provided by the reduction of the original inequality in the intuitionistic setting. This example is of course not enough to justify any general claims, but it does suggest a line for further investigation, namely the identification of classes of intuitionistic formulas for which the correspondence arguments are subsumed by the correspondence arguments of their Gödel translations in the strongest sense, as discussed above. As an initial observation in this direction we note that whenever the algorithm solves for positive occurrences of variables (cf. discussion on signed generation trees before definition 5), as in the example above, these variable occurrences will surface, if at all, on the right-hand side of inequalities; this, together with the fact that the Gödel translation prefixes all variables with a $\Box$, implies that the minimal valuations provided by the algorithm will be (the extensions of) finite disjunctions of $\blacklozenge$-terms. The latter are always upward closed, as required by the intuitionistic semantics.

Things do not work out so nicely for all intuitionistic Sahlqvist formulas, as revisiting our first example in the current section, the Pierce inequality, will show. This inequality is $\epsilon$-Sahlqvist for $\epsilon(p) = \partial$ and $\epsilon(q) = 1$ and for no other order type $\epsilon$. Running the correspondence algorithm on its Gödel translation yields:

$$\begin{array}{l}
\forall p\forall q[\Box(\Box(\Box p \to \Box q) \to \Box p) \leq \Box p]\\
\forall p[\Box(\Box(\Box p \to \Box\bot) \to \Box p) \leq \Box p]\\
\forall\mathbf{i}\forall\mathbf{m}\forall p[(\mathbf{i} \leq \Box(\Box(\Box p \to \Box\bot) \to \Box p)\ \&\ \Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
\forall\mathbf{i}\forall\mathbf{m}\forall\mathbf{n}\forall p[(\mathbf{i} \leq \Box(\Box(\Box p \to \Box\bot) \to \Box p)\ \&\ \Box\mathbf{n} \leq \mathbf{m}\ \&\ p \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
\forall\mathbf{i}\forall\mathbf{m}\forall\mathbf{n}[(\mathbf{i} \leq \Box(\Box(\Box\mathbf{n} \to \Box\bot) \to \Box\mathbf{n})\ \&\ \Box\mathbf{n} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\
\forall\mathbf{i}\forall\mathbf{n}[\mathbf{i} \leq \Box(\Box(\Box\mathbf{n} \to \Box\bot) \to \Box\mathbf{n}) \Rightarrow \forall\mathbf{m}[\Box\mathbf{n} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]]\\
\forall\mathbf{i}\forall\mathbf{n}[\mathbf{i} \leq \Box(\Box(\Box\mathbf{n} \to \Box\bot) \to \Box\mathbf{n}) \Rightarrow \mathbf{i} \leq \Box\mathbf{n}]\\
\forall\mathbf{n}[\Box(\Box(\Box\mathbf{n} \to \Box\bot) \to \Box\mathbf{n}) \leq \Box\mathbf{n}].
\end{array}$$

The minimal valuation provided by the above reduction assigns $p$ to a co-atom, i.e. to a set of type $W \setminus \{w\}$, which need not be upward-closed. There are probably other, less naïve ways in which the intuitionistic correspondence argument for the Pierce axiom can be simulated classically via its Gödel translation, but we leave this question open.

Finally, notice that the preservation of the intuitionistic Sahlqvist or Inductive classes under the Gödel translation is a very restricted phenomenon. This preservation occurs thanks mainly to the lack of order-theoretic variety in the intuitionistic signature. Namely, the interpretation of each binary connective in the intuitionistic signature is either a right residual or a right adjoint; in other words, the intuitionistic signature does not include any ‘pure diamond-type’ connective. As soon as pure diamond-type connectives are added, this transfer breaks down: for instance, Sahlqvist inequalities in the language of intuitionistic modal logic are not preserved under the Gödel translation. Indeed, the Gödel-translation of the Sahlqvist inequality $\Box\Diamond p \leq \Diamond p$ yields $\Box\Diamond[\leq]p \leq \Diamond[\leq]p$, which is not Sahlqvist, and actually—by van Benthem’s classification of the modal reduction principles [3]—it does not even have a first-order frame correspondent.

## 1.10 Conclusions

**Unified correspondence.** As van Benthem has aptly remarked, our “algebraic analysis is a combinatorial formalization of essentials of correspondence reasoning.” Indeed, classical correspondence arguments have been mechanized, and transformed into chains of equivalent rewritings of quasi-inequalities in the extended language $\mathcal{L}^+$. The language $\mathcal{L}^+$ can be captured by the monadic second-order frame language. The chains of equivalent rewritings aim at transforming quasi-inequalities in $\mathcal{L}^+$ into equivalent quasi-inequalities in a *fragment* of $\mathcal{L}^+$ which can be captured by the first-order frame language. In this process, minimal valuation arguments, which are pivotal for *local* correspondence, are encoded as applications of the Ackermann rule. To support the claim that these rewritings encode correspondence arguments as desired, the soundness of the rewriting rules needs to be verified. This has been done, via *duality theory*, in an *algebraic* setting. This move to algebras, per se, is not indispensable as long as the classical setting is concerned. However, the algebraic setting brings about a crucial advantage: it makes it possible to identify the properties really underlying the correspondence mechanism. And it turns out that no property exclusive to the classical setting is needed. This observation paves the way for rolling out correspondence theory, in great uniformity, to a wide variety of logics, including e.g. classical and intuitionistic modal mu-calculus (see [15]), polyadic and hybrid modal logics (see [19] and [24]), monotone modal logic [29], modal logics with propositional quantifiers [13] or graded modalities [28], and substructural logics (see also below). This is what we understand as *unified correspondence*. In this setting, it is possible, e.g., to give a general purpose definition of Sahlqvist formulas (cf. section 1.7) simultaneously applicable to several languages, and purely based on the order-theoretic behaviour of the interpretations of logical connectives.

**Dropping distributivity.** We wish to stress that the soundness of the approximation rules introduced in section 4 depends on the perfect lattices being completely join-generated by the set of their completely join *prime* elements, which implies that the perfect lattices in which these rules are sound are necessarily *distributive*. However, more general approximation rules can be introduced, which are sound on (non-distributive) perfect lattices. Hence, correspondence theory in the style illustrated in the present paper covers also logics with algebraic semantics based on general lattices, for instance *substructural logics* (cf. [21] for complete details).

**Complexity.** While we hope that the reader is convinced that the calculus of correspondence facilitates simple, perspicuous and uniform derivations, we do not claim that it improves upon the computational complexity of other methods like the traditional Sahlqvist-van Benthem algorithm. Still, a few remarks on complexity are perhaps in order. When restricted to the class of Sahlqvist formulas, or to any other class of formulas on which it is guaranteed to succeed, the calculus of correspondence yields an algorithm for computing the first-order correspondents of the members of this class. It is not difficult to see that this algorithm's runtime complexity is polynomial in the size of the input formula. More sophisticated versions of the calculus could involve more costly computations like testing for monotonicity of terms (as opposed to mere syntactic positivity), and can take us to the full complexity of the underlying logic or beyond (see e.g., [16]). When applied to arbitrary formulas, the calculus of correspondence is only a semi-algorithm, as is to be expected, since the question whether a formulas has a first-order frame correspondent is undecidable [14]. Some considerations relevant to implementation and computational optimization are treated in [34] and chapter 13 of [30].

**Constructive canonicity.** Perhaps the most important classical applications of correspondence is its connection to canonicity. Indeed, it has been appropriately argued [46] that the correspondence machinery can be extended and made applicable also in the context of descriptive frames, where it leads to canonicity results. Such results are often stated as persistence results (validity can be moved from a descriptive frame to its underlying Kripke frame); however, when seen from the dual, algebraic side, they can be stated as *transfer* results, namely that validity transfers from an algebra to its canonical extension. Formulated in this way, canonicity requires a rich metatheory for which the ultrafilter theorem (depending on the axiom of choice) must be available. However, there is a method for building canonical extensions 'without ultrafilters' in a constructive way. The idea [36] is to exploit a Galois connections induced by an abstract 'containment' relation between filters and ideals, and to define the canonical extension as the resulting algebra of Galois-stable subsets. Indeed, in the presence of the axiom of choice, this construction is isomorphic to the canonical extension defined via duality. However, the canonical extension defined in [36] has an autonomous life also in a constructive (topos-theoretically valid) metatheory, and moreover, it has a rich enough internal structure that the transfer results for Sahlqvist-type equations can be proved in two steps, without relying on any correspondence result. Thus, canonicity (the alter ego of correspondence) is meaningful also in a purely constructive context.

**Inverse correspondence.** We focused on the question of finding first-order (or FO+LFP) correspondents for modal formulas. In this way we 'cover' only a fragment of the first-order (or FO+LFP) correspondence language, so it is natural to reverse direction and ask which first-order (or FO+LFP) frame conditions are modally definable. The more specific question of characterizing the first-order formulas which are frame correspondents of Sahlqvist formulas was answered by Marcus Kracht [41] and more generally for the inductive formulas by Stanislav Kikot [40]. Analogous questions for intuitionist modal logic or when correspondence with FO+LFP is sought are still open.

**Step-by-step construction of finitely generated free algebras, and correspondence methods.** The step-by-step construction of finitely generated free algebras is gaining more and more attention, viz. [1, 35, 12, 25]. The case of equations of rank 1 has been thoroughly investigated in connection with research issues relevant to coalgebraic logic; interestingly, preliminary results show that, in order to extend these results beyond rank 1, the correspondence machinery is needed in the setting of the so-called *step frames* [10], two-sorted Kripke frames modelling partially defined modalities. Possible developments of this line of investigation invest proof-theoretic questions related to the subformula property [9].

## 1.11 Appendix

### *1.11.1 Distributive complex algebras and frames*

An element $c \neq \bot$ of a complete lattice $\mathbb{C}$ is *completely join-irreducible* iff $c = \bigvee S$ implies $c \in S$ for every $S \subseteq \mathbb{C}$; moreover, $c$ is *completely join-prime* if $c \neq \bot$ and, for every subset $S$ of the lattice, $c \leq \bigvee S$ iff $c \leq s$ for some $s \in S$. An element $c \neq \bot$ of a complete lattice is an *atom* if there is no element $y$ in the lattice such that $\bot < y < c$. An element $c \neq \top$ of a complete lattice is *completely meet-irreducible* iff $c = \bigwedge S$ implies $c \in S$ for every $S \subseteq \mathbb{C}$; moreover, $c$ is *completely meet-prime* if $c \neq \top$ and, for every subset $S$ of the lattice, $c \geq \bigwedge S$ iff $c \geq s$ for some $s \in S$. An element $c \neq \top$ of a complete lattice is a *co-atom* if there is no element $y$ in the lattice such that $c < y < \top$.

If $c$ is an atom (resp. a co-atom), then $c$ is completely join-prime (resp. meet-prime), and if $c$ is completely join-prime (resp. meet-prime), then $c$ is completely join-irreducible (resp. meet-irreducible). If $\mathbb{C}$ is *frame distributive* (i.e. finite meets distribute over arbitrary joins) then the completely join-irreducible elements are completely join-prime, and if $\mathbb{C}$ is a complete Boolean lattice, then the completely join-prime elements are atoms. The collections of all completely join- and meet-irreducible elements of $\mathbb{C}$ are respectively denoted by $J^{\infty}(\mathbb{C})$ and $M^{\infty}(\mathbb{C})$.

**Definition 6.** A *perfect* lattice is a complete lattice $\mathbb{C}$ such that $J^{\infty}(\mathbb{C})$ join-generates $\mathbb{C}$ (i.e. every element of $\mathbb{C}$ is the join of elements in $J^{\infty}(\mathbb{C})$) and $M^{\infty}(\mathbb{C})$ meet-generates $\mathbb{C}$ (i.e. every element of $\mathbb{C}$ is the meet of elements in $M^{\infty}(\mathbb{C})$). A *perfect distributive lattice* is a perfect lattice such that $J^{\infty}(\mathbb{C})$ coincides with the set of all completely join-prime elements of $\mathbb{C}$ and $M^{\infty}(\mathbb{C})$ coincides with the set of all completely meet-prime elements of $\mathbb{C}$; a *perfect Boolean lattice* is a perfect lattice such that $J^\infty(\mathbb{C})$ coincides with the set of all the atoms of $\mathbb{C}$ (or $M^\infty(\mathbb{C})$ coincides with the set of all the co-atoms of $\mathbb{C}$).

*Complete atomic modal algebras* are those modal algebras $\mathbb{A}$ the lattice reducts of which is a perfect Boolean lattice and moreover, their $\Diamond$ operation preserves arbitrary joins, i.e. $\Diamond(\bigvee S) = \bigvee_{s\in S}\Diamond s$ for every $S \subseteq \mathbb{A}$. Discrete Stone duality between complete atomic modal algebras and their complete homomorphisms and Kripke frames and their bounded morphisms is defined on objects by mapping any Kripke frame $\mathcal{F} = (W, R)$ to its *complex algebra* $\mathcal{F}^+ = (\mathcal{P}(W), \langle R\rangle)$, where $\langle R\rangle X = R^{-1}[X] = \{w \in W : \exists x(x \in X \ \&\ wRx)\}$ for every $X \in \mathcal{P}(W)$, and every complete atomic modal algebra $\mathbb{A} = (\mathbb{B}, \Diamond)$ to its *atom structure* $\mathbb{A}_+ = (J^\infty(\mathbb{B}), R)$, where $xRy$ iff $x \leq \Diamond y$ for all atoms $x, y \in J^\infty(\mathbb{B})$. As a consequence of this duality, the Stone representation theorem holds for complete atomic modal algebras, which states that these can be equivalently characterized as the modal algebras each of which is isomorphic to the complex algebra of some Kripke frame.

Likewise, a Stone-type duality (extending the finite *Birkhoff* duality) holds between perfect distributive lattices and their complete homomorphisms and *posets* and monotone maps, which is defined on objects as follows: every poset $X$ is associated with the lattice $\mathcal{P}^\uparrow(X)$ of the upward-closed subsets of $X$, and every perfect lattice $\mathbb{C}$ is associated with $(J^\infty(\mathbb{C}), \geq)$ where $\geq$ is the reverse lattice order in $\mathbb{C}$, restricted to $J^\infty(\mathbb{C})$. As a consequence of this duality, perfect distributive lattices can be equivalently characterized (see e.g. [32]) as those lattices each of which is isomorphic to the lattice $\mathcal{P}^\uparrow(X)$ of the upward-closed subsets of some poset $X$.

As was mentioned early on, just in the same way in which the duality between complete atomic Boolean algebras and sets can be expanded to a duality between complete atomic modal algebras and Kripke frames, the duality between perfect distributive lattices and posets can be expanded to a duality between perfect DLOs and posets endowed with arrays of relations, each of which dualizes one additional operation in the usual way, i.e., $n$-ary operations give rise to $n + 1$-ary relations, and the assignments between operations and relations are defined as in the classical setting. We are not going to report on this duality in full detail (we refer e.g. to [47, 33, 22]), but we limit ourselves to mention that, for instance, the DLOs endowed with four unary operators as in (1.1) are dual to the relational structures $\mathcal{F} = (W, \leq, R_\Diamond, R_\Box, R_\lhd, R_\rhd)$ such that $(W, \leq)$ is a nonempty poset, $R_\Diamond, R_\Box, R_\lhd, R_\rhd$ are binary relations on $W$ and the following inclusions hold:

$$\begin{array}{ll} \geq \circ R_\Diamond \circ \geq \ \subseteq\ R_\Diamond & \leq \circ R_\rhd \circ \geq \ \subseteq\ R_\rhd \\ \leq \circ R_\Box \circ \leq \ \subseteq\ R_\Box & \geq \circ R_\lhd \circ \leq \ \subseteq\ R_\lhd. \end{array}$$

The *complex algebra* of any such relational structure $\mathcal{F}$ (cf. [33, Sec. 2.3]) is

$$\mathcal{F}^+ = (\mathcal{P}^\uparrow(W), \cup, \cap, \varnothing, W, \langle R_\Diamond\rangle, [R_\Box], \langle R_\lhd], [R_\rhd\rangle),$$

where, for every $X \subseteq W$,

$$\begin{array}{ll}[R_\Box]X := \{w \in W \mid R_\Box[w] \subseteq X\} & = (R_\Box^{-1}[X^c])^c \\ \langle R_\Diamond\rangle X := \{w \in W \mid R_\Diamond[w] \cap X \neq \varnothing\} & = R_\Diamond^{-1}[X] \\ [R_\rhd\rangle X := \{w \in W \mid R_\rhd[w] \subseteq X^c\} & = (R_\rhd^{-1}[X])^c \\ \langle R_\lhd]X := \{w \in W \mid R_\lhd[w] \cap X^c \neq \varnothing\} & = R_\lhd^{-1}[X^c].\end{array}$$

Here $(\cdot)^c$ denotes the complement relative to $W$, while $R[x] = \{w \mid w \in W$ and $xRw\}$ and $R^{-1}[x] = \{w \mid w \in W$ and $wRx\}$. Moreover, $R[X] = \bigcup\{R[x] \mid x \in X\}$ and $R^{-1}[X] = \bigcup\{R^{-1}[x] \mid x \in X\}$.

### 1.11.2 Adjunction and residuation

Let $P$ and $Q$ be partial orders. The maps $f : P \to Q$ and $g : Q \to P$ form an *adjoint pair* (notation: $f \dashv g$) iff for every $x \in P$ and $y \in Q$, $f(x) \leq y$ iff $x \leq g(y)$. Whenever $f \dashv g$, $f$ is the *left adjoint* of $g$ and $g$ is *the right adjoint* of $f$. Adjoint maps are order-preserving. If a map admits a left (resp. right) adjoint, the adjoint is unique and can be computed pointwise from the map itself and the order.

**Proposition 3** *1. Right adjoints (resp. left adjoints) between complete lattices are exactly the completely meet-preserving (resp. join-preserving) maps;*
*2. right (resp. left) adjoints on powerset algebras $\mathcal{P}(W)$ are exactly the maps defined by assignments of type $X \mapsto [R]X = (R^{-1}[X^c])^c$ (resp. $X \mapsto \langle R\rangle X = R^{-1}[X]$) for some binary relation $R$ on $W$.*
*3. For any binary relation $R$ on $W$, the left adjoint of $[R]$ is the map $\langle R^{-1}\rangle$, defined by the assignment $X \mapsto R[X]$.*

*Proof.* 1. See [26, Proposition 7.34].
2. For a left adjoint $f : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$, define $R$ as follows: for every $x, z \in W$, $xRz$ iff $x \in f(\{z\})$. For a right adjoint $g : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$, define $R$ as follows: for every $x, z \in W$, $xRz$ iff $x \notin g(W \setminus \{z\})$.

The notion of adjunction can be made parametric and generalized to $n$-ary maps in a component-wise fashion: an $n$-ary map $f : P^n \to P$ on a poset $P$ is *residuated* if there exists a collection of maps $\{g_i : P^n \to P \mid 1 \leq i \leq n\}$ s.t. for every $1 \leq i \leq n$ and for all $x_1, \ldots, x_n, y \in P$,

$$f(x_1, \ldots, x_n) \leq y \quad \text{iff} \quad x_i \leq g_i(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n).$$

The map $g_i$ is the *i-th residual* of $f$. Residuated maps are order preserving in each coordinate, and for each $1 \leq i \leq n$, the residual $g_i$ is order-preserving in its $i$th coordinate and order-reversing in all other coordinates. The facts stated in the following example and proposition are well known in the literature in their binary instance (cf. [31, Subsection 3.1.3]):

*Example 4.* For every $(n+1)$-ary relation $\mathcal{S}$ on $W$ and every $(X_1, \ldots, X_n) \in \mathcal{P}(W)^n$, let

$$\mathcal{S}[X_1, \ldots, X_n] := \{y \in W \mid \exists x_1 \cdots \exists x_n[\bigwedge_{i=1}^{n} x_i \in X_i \ \wedge \ \mathcal{S}(x_1, \ldots, x_n, y)]\}.$$

The $n$-ary operation on $\mathcal{P}(W)$ defined by the assignment $(X_1, \ldots, X_n) \mapsto \mathcal{S}[X_1, \ldots, X_n]$ is residuated and its $i$-th residual is the map $g_i : \mathcal{P}(W)^n \to \mathcal{P}(W)$ which maps every $n$-tuple $(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_n)$ to the set $\{w \in W \mid \alpha^i_{\mathcal{S}}(w)\}$, where $\alpha^i_{\mathcal{S}}(w)$ is the following first-order formula:

$$\forall x_1 \cdots \forall y \cdots \forall x_n[(\bigwedge_{k \in \mathbf{n}_i} x_k \in X_k \ \& \ \mathcal{S}(x_1, \ldots, w, \ldots, x_n, y)) \Rightarrow y \in Y],$$

and moreover $\mathbf{n}_i = \{1, \ldots, n\} \setminus \{i\}$.

**Proposition 4** *If $f : P^n \to P$ is residuated and $\{g_i : P^n \to P \mid 1 \leq i \leq n\}$ is the collection of its residuals, then:*

1. *if $P$ is a complete lattice, then $f$ preserves arbitrary joins in each coordinate;*
2. *if $P$ is a powerset algebra, $f$ coincides with the map defined by the assignment $\mathcal{S}[X_1, \ldots, X_n]$ as in example 4, for some $(n + 1)$-ary relation $\mathcal{S}$ on $W$.*

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