(Algebraic) Proof Theory for Substructural Logics and Applications

Agata Ciabattoni

Vienna University of Technology

agata@logic.at

Substructural logics

- for reasoning, e.g., about natural language, vagueness, resources, algebraic varieties ...
- include intuitionistic logic, linear logic, fuzzy logics, ...
- defined by adding Hilbert axioms to Full Lambek calculus $\mathbf{FL}$ or equations to residuated lattices

  Example: Gödel logic is obtained by adding

  - the Hilbert axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$ to intuitionistic logic ($\mathbf{FL}$ + exchange, weakening and contraction), or
  - prelinearity $1 \leq (x \to y) \vee (y \to x)$ to Heyting algebras

Why proof theory?

- The applicability/usefulness of these logics, however, strongly depends on the availability of analytic calculi. Analytic calculi are
  - useful for establishing various properties of logics
  - key for developing automated reasoning methods.
- Gentzen sequent calculus has always been the favourite framework.

Sequent Calculus

Sequents

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$$

Axioms

E.g., $A \Rightarrow A$

Rules

- Logical (left and right)
- Structural
  E.g.

$$\frac{\Gamma, A, A \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (c, l)$$

- Cut

Sequent Calculus: the rule Cut

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\; Cut$$

- key to prove completeness w.r.t. Hilbert system

$$\text{modus ponens} \quad \frac{A \quad A \to B}{B}$$

- bad for proof search

Cut-elimination theorem

Each proof using Cut can be transformed into a proof without Cut.

The system FLe

FLe = commutative Lambek calculus (= intuitionistic Linear Logic without exponentials

Algebraic semantics:
A *(bounded pointed) commutative residuated lattice* is

$$\mathbf{P} = \langle P, \wedge, \vee, \otimes, \rightarrow, \top, \mathbf{0}, \mathbf{1}, \bot \rangle$$

1. $\langle P, \wedge, \vee \rangle$ is a lattice with $\top$ greatest and $\bot$ least
2. $\langle P, \otimes, \mathbf{1} \rangle$ is a commutative monoid.
3. For any $x, y, z \in P$, $x \otimes y \leq z \Longleftrightarrow y \leq x \rightarrow z$
4. $\mathbf{0} \in P$.

The system FLe

$$\frac{A, B, \Gamma \Rightarrow \Pi}{A \otimes B, \Gamma \Rightarrow \Pi} \otimes l \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B} \otimes r$$

$$\frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi}{\Gamma, A \rightarrow B, \Delta \Rightarrow \Pi} \rightarrow l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \rightarrow r$$

$$\frac{A, \Gamma \Rightarrow \Pi \quad B, \Gamma \Rightarrow \Pi}{A \vee B, \Gamma \Rightarrow \Pi} \vee l \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \vee r \qquad \frac{}{\mathbf{0} \Rightarrow} \mathbf{0}l$$

$$\frac{A_i, \Gamma \Rightarrow \Pi}{A_1 \wedge A_2, \Gamma \Rightarrow \Pi} \wedge l \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \wedge r \qquad \frac{}{\Gamma \Rightarrow \top} \top r$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathbf{0}} \mathbf{0}r \qquad \frac{}{\Rightarrow \mathbf{1}} \mathbf{1}r \qquad \frac{}{\bot, \Gamma \Rightarrow \Pi} \bot l \qquad \frac{\Gamma \Rightarrow \Pi}{\mathbf{1}, \Gamma \Rightarrow \Pi} \mathbf{1}l$$

Beyond sequent calculus

- Many useful and interesting logics seem do not fit comfortably into the sequent framework, however.

- A large range of variants and extensions have been indeed introduced.
  E.g.

  Hypersequent Calculi,

  Display calculi,

  Labelled Deductive Systems,

  Nested Calculi,

  Bunched Calculi,

  Calculus of Structures

  . . .

State of the art

The definition of analytic calculi is logic-tailored.

(Step i) choose (or define) a framework

(Step ii) find the “right” inference rule(s)

(Step iii) prove (soundness, completeness and) cut-elimination

This talk

(1) Define analytic calculi for large classes of substructural logics in a systematic and algorithmic way

(2) Characterize the expressive power of sequent and hypersequent structural rules

(3) Applications: use the introduced calculi for
   - uniform proofs of closure under order theoretic completions
   - uniform (and automated) proofs of *standard completeness*

Order Theoretic Completions

- A completion of an algebra $\mathbf{A}$ is a complete algebra $\mathbf{B}$ (i.e. it has arbitrary $\bigvee$ and $\bigwedge$) such that $\mathbf{A} \subseteq \mathbf{B}$.
- Completions are not unique: filter/ideal extensions, canonical extensions, Dedekind-MacNeille completions, ...

Dedekind MacNeille Completion

Dedekind Completion of Rationals

- For any $X \subseteq \mathbb{Q}$,

$$
\begin{aligned}
X^{\triangleright} &= \{y \in \mathbb{Q} : \forall x \in X . x \leq y\} \\
X^{\triangleleft} &= \{y \in \mathbb{Q} : \forall x \in X . y \leq x\}
\end{aligned}
$$

- X is closed if $X = X^{\triangleright\triangleleft}$
- $(\mathbb{Q}, +, \cdot)$ can be embedded into $(\mathcal{C}(\mathbb{Q}), +, \cdot)$ with

$$\mathcal{C}(\mathbb{Q}) = \{X \subseteq \mathbb{Q} : X \text{ is closed}\}$$

Dedekind completion extends to various ordered algebras (MacNeille).

Algebraic Proof Theory: motivating facts

Algebra The variety of Heyting algebras satisfying prelinearity $1 \leq (x \to y) \vee (y \to x)$ is not closed under Dedekind-MacNeille completions DM (cf. *Bezhanishvill& Harding '04*), but it is closed under DM when applied to s.i. algebras.

Proof Th. IL + prelinearity (= Gödel logic) does not admit a cut-free sequent calculus extending FLe but it does admit a cut-free hypersequent calculus.

Algebra The variety of MV algebras is not closed under any completion (cf. *Kowalski & Litak '08*).

Proof Th. Łukasiewicz logic does not admit any cut-free sequent or hypersequent calculus extending FLe.

Axioms vs Rules for Substructural Logics

(Commutative) Substructural Logics= $\mathbf{FLe}$ + axioms

E.g.
Contraction: $\alpha \to \alpha \otimes \alpha$ or Weakening: $\alpha \to 1$.

Cut-elimination is not preserved when axioms are added

Axioms vs Rules

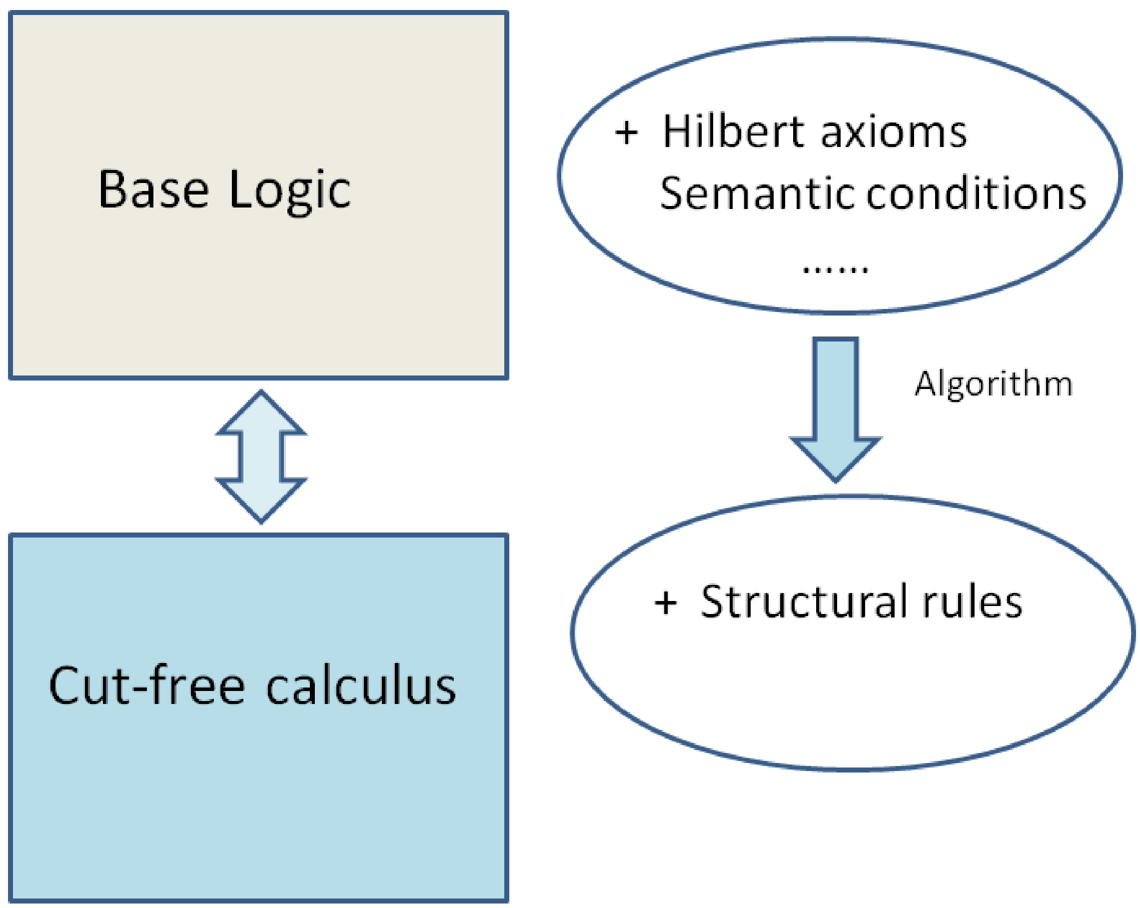
Example

- Contraction: $\alpha \to \alpha \otimes \alpha$ $\qquad \dfrac{A, A, \Gamma \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}\ (c)$
- Weakening l: $\alpha \to 1$ $\qquad \dfrac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (w, l)$
- Weakening r: $0 \to \alpha$ $\qquad \dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ (w, r)$

They are *equivalent*, i.e.

$$\vdash_{FLe+(axiom)} \quad = \quad \vdash_{FLe+(rule)}$$

From axioms to rules: the idea

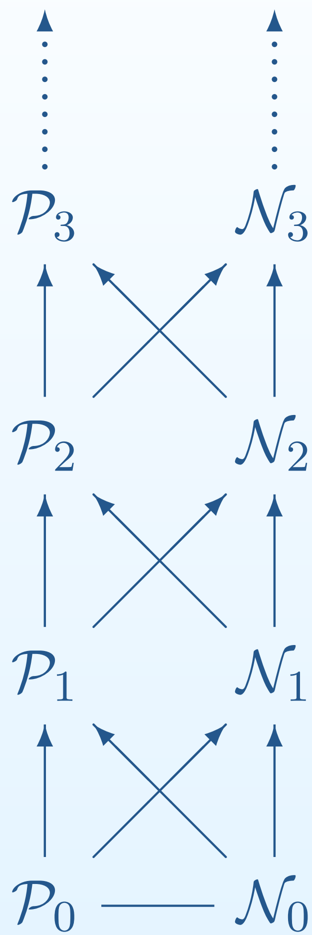


From axioms to rules: the ingredients

Starting point: a suitable classification of the properties

- Use of the invertible rules of the base calculus
- Use of the Ackermann Lemma
  An algebraic equation $t \leq u$ is equivalent to a quasiequation $u \leq x \Longrightarrow t \leq x$, and also to $x \leq t \Longrightarrow x \leq u$, where x is a fresh variable not occurring in t, u.

Classification



The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas (equations) defined by:

$\mathcal{P}_0,\ \mathcal{N}_0$:= Atomic formulas

$$\mathcal{P}_{n+1} := \mathcal{N}_n \mid \mathcal{P}_{n+1} \otimes \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid 1 \mid \bot$$

$$\mathcal{N}_{n+1} := \mathcal{P}_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0 \mid \top$$

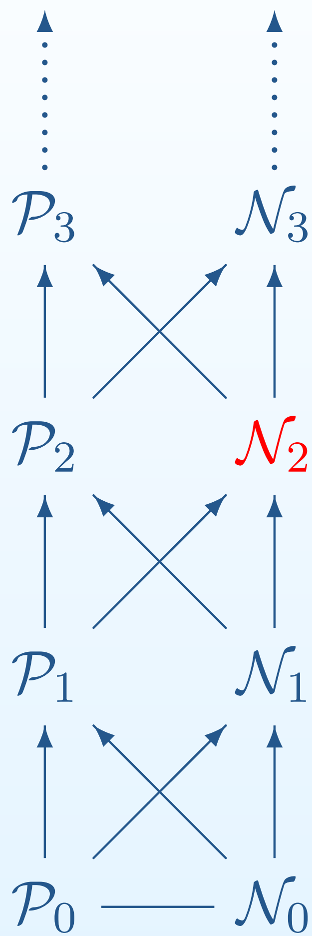
$\mathcal{P}$ and $\mathcal{N}$

- Positive connectives $\mathbf{1}, \bot, \otimes, \vee$ have *invertible left rules*:
- Negative connectives $\top, \mathbf{0}, \wedge, \to$ have *invertible right rules*:

Examples

Class	Axiom	Name
$\mathcal{N}_2$	$\alpha \to \mathbf{1},\ \bot \to \alpha$	weakening
	$\alpha \to \alpha \otimes \alpha$	contraction
	$\alpha \otimes \alpha \to \alpha$	expansion
	$\otimes\alpha^n \to \otimes\alpha^m$	knotted axioms ($n, m \geq 0$)
	$\neg(\alpha \wedge \neg\alpha)$	weak contraction
$\mathcal{P}_2$	$\alpha \vee \neg\alpha$	excluded middle
	$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\mathcal{P}_3$	$\neg\alpha \vee \neg\neg\alpha$	weak excluded middle
	$\neg(\alpha \otimes \beta) \vee (\alpha \wedge \beta \to \alpha \otimes \beta)$	(wnm)
$\mathcal{N}_3$	$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$	Łukasiewicz axiom

Our preliminary results



Algorithm to transform:

- axioms up to the class $\mathcal{N}_2$ into "good" structural rules in sequent calculus
- equations up to $\mathcal{N}_2$ into "good" quasiequations

Moreover

- analytic calculi iff DM completion
- in presence of weakening/integrality all axioms/equations up to $\mathcal{N}_2$ are tamed

(-, N. Galatos and K. Terui). LICS 2008 and APAL 2012

Bad $\mathcal{N}_2$ axioms/equations

in absence of weakening/integrality, e.g.,

$$\frac{\Gamma, B \Rightarrow A \quad A, \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow}$$

A concrete example

The subvariety of FL defined by

$$x \backslash x \leq x / x$$

does not admit any completion

Expressive power of structural sequent rules

Consider e.g.

$$(\alpha \to \beta) \vee (\beta \to \alpha) \in \mathcal{P}_3$$

- Can we find equivalent *good* structural sequent rules?

NO! Theorem
Each good (i.e. analytic) structural sequent rule is equivalent to an equation which is preserved by Dedekind MacNeille completions in presence of integrality.

(-, N. Galatos and K. Terui. APAL 2012)

Hypersequent calculus

It is obtained embedding sequents into hypersequents

$$\Gamma_1 \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Pi_n$$

where for all $i = 1, \ldots n$, $\Gamma_i \Rightarrow \Pi_i$ is a sequent.

$$\frac{G|\Gamma \Rightarrow A \quad G|A, \Delta \Rightarrow \Pi}{G|\Gamma, \Delta \Rightarrow \Pi}\ Cut \qquad \frac{}{G|A \Rightarrow A}\ Identity$$

$$\frac{G|\Gamma \Rightarrow A \quad G|B, \Delta \Rightarrow \Pi}{G|\Gamma, A \rightarrow B, \Delta \Rightarrow \Pi} \rightarrow l \qquad \frac{G|A, \Gamma \Rightarrow B}{G|\Gamma \Rightarrow A \rightarrow B} \rightarrow r$$

and adding suitable rules to manipulate the additional layer of structure.

$$\frac{G}{G \,|\, \Gamma \Rightarrow A}\ (\mathrm{ew}) \qquad \frac{G \,|\, \Gamma \Rightarrow A \,|\, \Gamma \Rightarrow A}{G \,|\, \Gamma \Rightarrow A}\ (\mathrm{ec})$$

Structural rules: an example

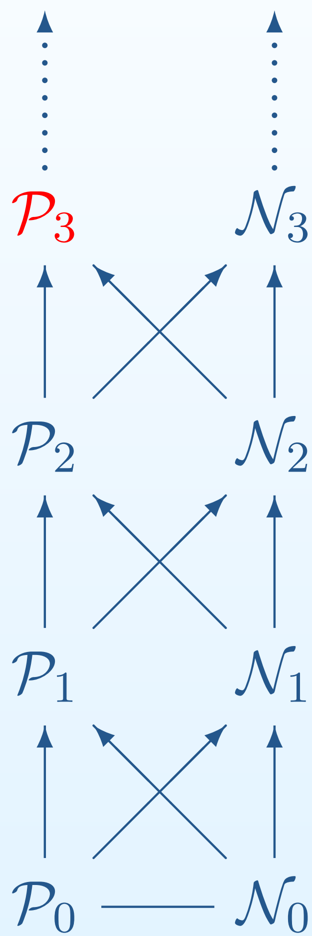
$$\frac{G \mid \Gamma, \Sigma' \Rightarrow \Delta' \quad G \mid \Gamma', \Sigma \Rightarrow \Delta}{G \mid \Gamma, \Sigma \Rightarrow \Delta \mid \Gamma', \Sigma' \Rightarrow \Delta'}\ (com)$$

(Avron, Annals of Math and art. Intell. 1991)

Gödel logic = **IL** + $(\alpha \to \beta) \vee (\beta \to \alpha)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta \qquad \alpha \Rightarrow \alpha}{\alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha}\ (\mathrm{com})}{\alpha \Rightarrow \beta \mid\ \Rightarrow \beta \to \alpha}\ (\to,\mathrm{r})}{\Rightarrow \alpha \to \beta \mid\ \Rightarrow \beta \to \alpha}\ (\to,\mathrm{r})}{\Rightarrow \alpha \to \beta \mid\ \Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}\ (\vee_{\mathrm{i}},\mathrm{r})}{\Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha) \mid\ \Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}\ (\vee_{\mathrm{i}},\mathrm{r})}{\Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}\ (\mathrm{EC})$$

Climbing up the hierarchy



Algorithm to transform:

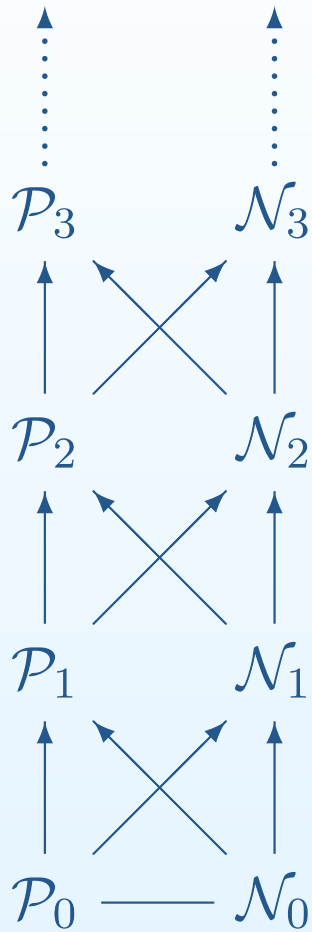
- axioms up to the class $\mathcal{P}_3'$ into "good" structural rules in hypersequent calculus
- equations up to $\mathcal{P}_3'$ into "good" analytic clauses

Moreover

- equations up to $\mathcal{P}_3'$ preserved by DM completions when applied to s.i. algebras
- analytic calculi iff HyperDM completion
- axioms/equations up to $\mathcal{P}_3$ are tamed in presence of integrality

(-, N. Galatos and K. Terui). Algebra Universalis, 2011, and Submitted 2014.

Expressive power of hypersequent rules



Sequent structural rules: only equations

- closed under DM completion, with integrality
- that hold in Heyting algebras (IL)

Hypersequent structural rules: only equations

- closed under HyperDM completions, with integrality
- that hold in Heyting algebras generated by the 3-element algebras or derive $1 \leq x \vee \neg x^n$ in FLew

(-, N. Galatos and K. Terui. Submitted 2014)

From axioms to good analytic clauses

$$(\alpha \to \beta) \vee (\beta \to \alpha)$$

is equivalent to

$$\frac{G \,|\, \Gamma, \Sigma' \Rightarrow \Delta' \quad G \,|\, \Gamma', \Sigma \Rightarrow \Delta}{G \,|\, \Gamma, \Sigma \Rightarrow \Delta \,|\, \Gamma', \Sigma' \Rightarrow \Delta'} \ (com)$$

whose algebraic reformulation is:

$$1 \leq (x \to y) \vee (y \to x)$$

is equivalent to

$$z \leq x \text{ and } w \leq y \Longrightarrow w \leq x \text{ or } z \leq y$$

To sum up

- systematic generation of good (hyper)sequent rules equivalent to axioms up to $\mathcal{P}_3'$ ($\mathcal{P}_3$ in presence of weakening)
- identification/introduction of appropriate completions that work for equations up to the level $\mathcal{P}_3'$ ($\mathcal{P}_3$ in presence of weakening)

http://www.logic.at/staff/lara/tinc/webaxiomcalc/

An application

Completeness of axiomatic systems with respect to algebras whose lattice reduct is the real unit interval $[0, 1]$.

(Hajek 1998) Formalizations of *Fuzzy Logic*

Some standard complete logics

T-norm based logics

- conjunction interpreted as a *t-norm*, i.e. a function $* : [0,1]^2 \to [0,1]$ satisfying, $\forall x, y, z \in [0,1]$: $x * y = y * x$ (Commutativity), $(x * y) * z = x * (y * z)$ (Associativity), $x \leq y$ implies $x * z \leq y * z$ (Monotonicity), $1 * x = x$ (Identity).
- implication interpreted as its *residuum*, i.e. a function $\Rightarrow_*: [0,1]^2 \to [0,1]$ where $x \Rightarrow_* y = max\{z \mid x * z \leq y\}$.

Example: Gödel logic

$$v : \text{Propositions} \to [0,1]$$

$$v(A \wedge B) = \min\{v(A), v(B)\} \qquad v(\bot) = 0$$

$$v(A \vee B) = \max\{v(A), v(B)\}$$

$$v(A \to B) = 1 \text{ if } v(A) \leq v(B), \text{ and } v(B) \text{ otherwise}$$

Some standard complete logics

T-norm based logics

- conjunction interpreted as a *t-norm*, i.e. a function $* : [0,1]^2 \to [0,1]$ satisfying, $\forall x, y, z \in [0,1]$: $x * y = y * x$ (Commutativity), $(x * y) * z = x * (y * z)$ (Associativity), $x \leq y$ implies $x * z \leq y * z$ (Monotonicity), $1 * x = x$ (Identity).
- implication interpreted as its *residuum*, i.e. a function $\Rightarrow_*: [0,1]^2 \to [0,1]$ where $x \Rightarrow_* y = max\{z \mid x * z \leq y\}$.

Monoidal T-norm based logic MTL (FLew + $(\alpha \to \beta) \vee (\beta \to \alpha)$) *(Godo, Esteva, FSS 2001)*

$v(A \otimes B) = v(A) * v(B)$, $\quad *$ left continous t-norm

$v(A \vee B) = \max\{v(A), v(B)\}$

$v(A \to B) = v(A) \Rightarrow_* v(B)$

$v(\bot) = 0$

Standard Completeness?

Question Given a logic $\mathcal{L}$ obtained by extending MTL with

- $A \vee \neg A$ (excluded middle)?
- $A^{n-1} \to A^n$ (n-contraction)?
- $\neg(A \otimes B) \vee (A \wedge B \to A \otimes B)$ (weak nilpotent minimum)?
- ....

Is $\mathcal{L}$ standard complete? (*is it a formalization of Fuzzy Logic?*)

case-by-case answer

Standard Completeness: usual approach

Given a logic $\mathcal{L}$:

1. Identify the algebraic semantics of $\mathcal{L}$ ($\mathcal{L}$-algebras)
2. Show completeness of $\mathcal{L}$ w.r.t. linear, countable $\mathcal{L}$-algebras
3. Find an embedding of countable $\mathcal{L}$-algebras into dense countable $\mathcal{L}$-algebras
4. Dedekind-MacNeille style completion (embedding into $\mathcal{L}$-algebras with lattice reduct $[0, 1]$)

- Step 3 (rational completeness): problematic (only ad hoc solutions)

Standard Completeness via proof theory

(Metcalfe, Montagna JSL 2007) $\mathcal{L}$ + (*density*) is rational complete:

$$\frac{(\Phi \to p) \vee (p \to \Psi) \vee \Xi}{(\Phi \to \Psi) \vee \Xi}\ (density)$$

where $p \notin \Phi, \Psi, \Xi$

(Step 1) Define a suitable calculus for $\mathcal{L}$ + (density)

(Step 2) Show that density produces no new theorems (Rational completeness)

(Step 3) Dedekind-MacNeille style completion

Density vs Cut in hypersequent calculi

$$\frac{(\Phi \to p) \lor (p \to \Psi) \lor \Xi}{(\Phi \to \Psi) \lor \Xi}\ (density)$$

- $$\frac{G \,|\, \Gamma \Rightarrow p \,|\, p \Rightarrow \Delta}{G \,|\, \Gamma \Rightarrow \Delta}\ (density)$$

  where p is does not occur in the conclusion.

- $$\frac{G \,|\, \Gamma \Rightarrow A \quad G \,|\, A \Rightarrow \Delta}{G \,|\, \Gamma \Rightarrow \Delta}\ (cut)$$

Density elimination

- Similar to cut-elimination
- Proof by induction on the length of derivations

(-, Metcalfe TCS 2008) Given a density-free derivation, ending in

$$\dfrac{\begin{array}{c}\vdots\, d' \\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\ (\mathrm{EC})$$

- Asymmetric substitution: p is replaced
  - With Δ when occuring on the right
  - With Γ when occuring on the left

Density elimination: problem with (com)

$$\dfrac{\dfrac{p \Rightarrow p \qquad \overset{\vdots}{\Pi \Rightarrow \Psi}}{\Pi \Rightarrow p \mid p \Rightarrow \Psi}\,(com)}{\dfrac{\vdots\, d}{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}}$$

$$\dfrac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\,(\text{D})$$

$$\dfrac{\Gamma \Rightarrow \Delta \qquad \overset{\vdots}{\Pi \Rightarrow \Psi}}{\Pi \Rightarrow \Delta \mid \Gamma \Rightarrow \Psi}\,(com) \quad \vdots\, d^*$$

$$\dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\,(\text{EC})$$

- $p \Rightarrow p$ axiom
- $\Gamma \Rightarrow \Delta$ not an axiom

Solution (with weakening)

(AC, Metcalfe 2008)

$$\dfrac{p \Rightarrow p \qquad \begin{array}{c}\vdots \\ \Pi \Rightarrow \Psi\end{array}}{\Pi \Rightarrow p \mid p \Rightarrow \Psi}\ (com)$$

$$\vdots\ d$$

$$\dfrac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ \text{(D)}$$

$$\dfrac{\begin{array}{c}\vdots \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\vdots \\ \Pi \Rightarrow \Psi\end{array}}{\Pi \Rightarrow \Delta \mid \Gamma \Rightarrow \Psi}\ (cut)$$

$$\vdots\ d^*$$

$$\dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ \text{(EC)}$$

Step 2: general conditions for density elimination

in presence of weakening

Theorem (AC, Baldi TCS to appear)
The hypersequent calculus for MTL + **convergent** rules admits density elimination

i.e. rules equivalent to axioms within the class $\mathcal{P}_3$ and whose premises do not mix "too much" the conclusion

Example :

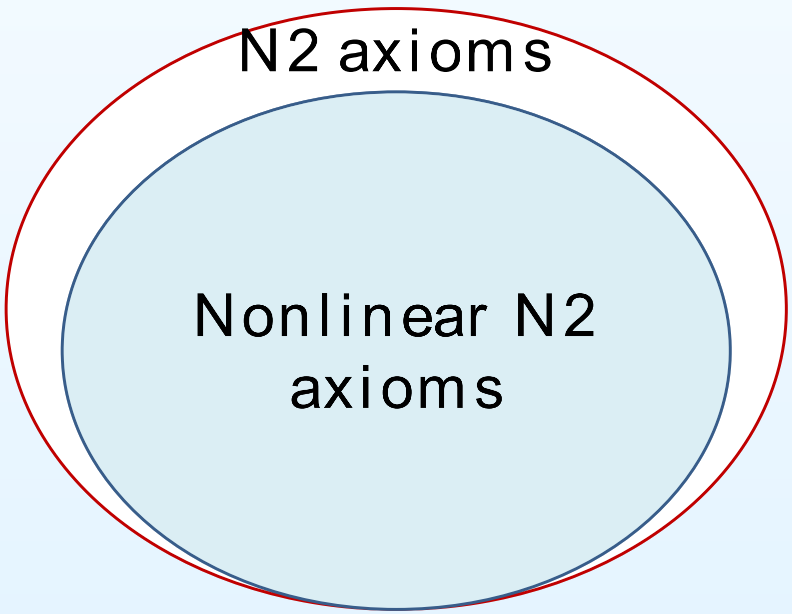
$$\frac{\begin{array}{cc} G \,|\, \Gamma_2, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \,|\, \Gamma_1, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \\ G \,|\, \Gamma_1, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \,|\, \Gamma_2, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \end{array}}{G \,|\, \Gamma_2, \Gamma_3 \Rightarrow \;|\, \Gamma_1, \Delta_1 \Rightarrow \Pi_1} \; \textit{(wnm)}$$

Axiom: $\neg(\alpha \otimes \beta) \vee (\alpha \wedge \beta \rightarrow \alpha \otimes \beta)$

Step 2: general conditions for density elimination

in absence of weakening

Theorem (AC, Baldi ISMVL 2015)
The hypersequent calculus for UL + nonlinear axioms (and/or mingle) admits density elimination



Recall: Standard Completeness via proof theory

(Metcalfe, Montagna JSL 2007) Given a logic $\mathcal{L}$:

(Step 1) Define a suitable calculus for $\mathcal{L}$ + (density)

(Step 2) Show that density produces no new theorems

(Step 3) Dedekind-MacNeille style completion

Example

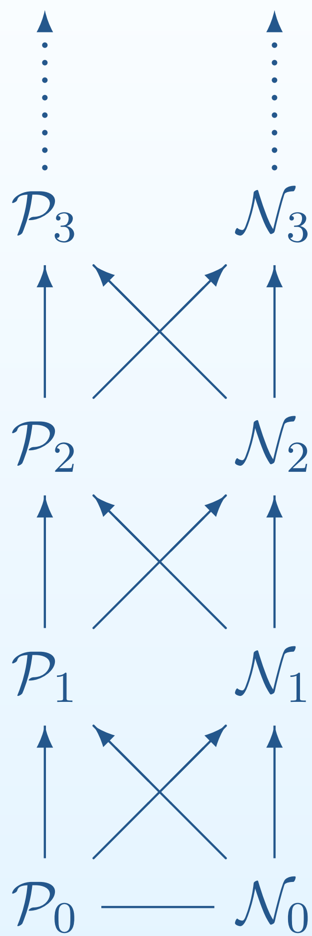
Known Logics

- MTL + $\neg(\alpha \cdot \beta) \vee ((\alpha \wedge \beta) \rightarrow (\alpha \cdot \beta))$
- MTL + $\neg\alpha \vee \neg\neg\alpha$
- MTL + $\alpha^{n-1} \rightarrow \alpha^n$
- UL + $\alpha^{n-1} \rightarrow \alpha^n$
- ...

New Fuzzy Logics

- MTL + $\neg(\alpha \cdot \beta)^n \vee ((\alpha \wedge \beta)^{n-1} \rightarrow (\alpha \cdot \beta)^n)$, for all $n > 1$
- UL + $\neg\alpha \vee \neg\neg\alpha$
- UL + $\alpha^m \rightarrow \alpha^n$
- ...

Open problems I



Uniform treatment of axioms in $\mathcal{N}_3$ and behond

Remark on $\mathcal{N}_3$: it contains (a) all (axiomatizable) intermediate logics (via canonical formulas), (b) equations that are not preserved under completions.

Partial answers:

- generation of logical rules
- adopting formalisms more complex than the (hyper)sequent calculus

Open problems II

- First-order, modal logics, ...
- ”Applications”:

  E.g.
  - new semantic foundations (e.g. non-deterministic matrices)
  - automated deduction procedures
  - decidability proofs
  - admissibility of rules (e.g. standard completeness)
  - ...

"Non-classical Proofs: Theory, Applications and Tools", research project 2012-2017