Unified Correspondence as a Proof-Theoretic Tool

Apostolos Tzimoulis

6 May 2015
Delft University of Technology

joint work with:
Giuseppe Greco, Minghui Ma,
Alessandra Palmigiano, Zhiguang Zhao
http://www.appliedlogictudelft.nl

Motivation

Main question: which axioms give rise to analytic rules?

Correspondence theory can help in answering this question!

- Formal connections between **correspondence theory** and **display calculi**.
- **Primitive formulas** [Kracht '96] for classical modal logic **K** generalised to **primitive inequalities** for general **DLE-logics**.

Display Calculi

Natural generalization of sequent calculi.
Sequents $X \vdash Y$, where X, Y are structures:
$A, \quad A ; B, \quad ... \quad X > Y, \quad ...$

structural symbols assemble and disassemble structures
operational symbols assemble formulas.

Main feature: display property

$$\frac{\frac{\frac{Y \vdash X > Z}{X ; Y \vdash Z}}{Y ; X \vdash Z}}{X \vdash Y > Z}$$

display property: **adjunction** at the structural level.
Canonical proof of cut elimination

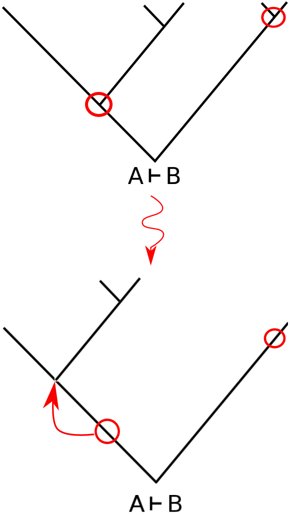
Complexity of the cut formula

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ Z \vdash \circ A\end{array}}{Z \vdash \Box A} \qquad \dfrac{\begin{array}{c}\vdots\ \pi_2\\ A \vdash Y\end{array}}{\Box A \vdash \circ Y}}{Z \vdash \circ Y}\ Cut$$

$$\Downarrow$$

$$\text{Display}\ \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ Z \vdash \circ A\end{array}}{\bullet Z \vdash A} \qquad \begin{array}{c}\vdots\ \pi_2\\ A \vdash Y\end{array}}{\bullet Z \vdash Y}\ Cut}{Z \vdash \circ Y}$$

Height of the cut



Proper Display Calculi

Theorem (Canonical cut elimination)

If a calculus satisfies the properties below, then it enjoys cut elimination.

- C1: structures can disappear, formulas are **forever**;
- **tree-traceable** formula-occurrences, via suitably defined congruence:
  - C2: same shape, C3: non-proliferation, C4: same position;
- C5: **principal = displayed**;
- C6, C7: rules are closed under **uniform substitution** of congruent parameters;
- C8: **reduction strategy** exists when cut formulas are both principal.

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\overline{\varphi}) \mid g(\overline{\varphi})$$

where $p \in \mathsf{PROP}$, $f \in \mathcal{F}$, $g \in \mathcal{G}$.

Str.	I		;		>		H		K	
Op.	$\top$	$\bot$	$\wedge$	$\vee$	$(\succ\!-)$	$(\rightarrow)$	f			g

Str.	H_i		K_h	
Op.		$(f_i^\sharp)$	$(g_h^\flat)$	

  for $\varepsilon_f(i) = \varepsilon_g(h) = 1$

Str.	H_i		K_h	
Op.	$(f_i^\sharp)$			$(g_h^\flat)$

  for $\varepsilon_f(i) = \varepsilon_g(h) = \partial$

Introduction rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

$$f_L \; \frac{H(A_1, \ldots, A_{n_f}) \vdash X}{f(A_1, \ldots, A_{n_f}) \vdash X} \qquad \frac{X \vdash K(A_1, \ldots, A_{n_g})}{X \vdash g(A_1, \ldots, A_{n_g})} \; g_R$$

$$f_R \; \frac{\left(X_i \vdash A_i \quad A_j \vdash X_j \quad \mid \quad \varepsilon_f(i) = 1 \quad \varepsilon_f(j) = \partial\right)}{H(X_1, \ldots, X_{n_f}) \vdash f(A_1, \ldots, A_{n_f})}$$

$$g_L \; \frac{\left(A_i \vdash X_i \quad X_j \vdash A_j \quad \mid \quad \varepsilon_g(i) = 1 \quad \varepsilon_g(j) = \partial\right)}{g(A_1, \ldots, A_{n_g}) \vdash K(X_1, \ldots, X_{n_g})}$$

Display postulates for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

- If $\varepsilon_f(i) = \varepsilon_g(h) = 1$

$$\frac{H(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y}{X_i \vdash H_i(X_1, \ldots, Y, \ldots, X_{n_f})} \qquad \frac{Y \vdash K(X_1 \ldots, X_h, \ldots X_{n_g})}{K_h(X_1, \ldots, Y, \ldots, X_{n_g}) \vdash X_h}$$

- If $\varepsilon_f(i) = \varepsilon_g(h) = \partial$

$$\frac{H(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y}{H_i(X_1, \ldots, Y, \ldots, X_{n_f}) \vdash X_i} \qquad \frac{Y \vdash K(X_1, \ldots, X_h, \ldots, X_{n_g})}{X_h \vdash K_h(X_1, \ldots, Y, \ldots, X_{n_g})}$$

Unified correspondence

DLE-logics
[CP12, CPS14]

Substructural logics
[CP14]

Mu-calculi
[CFPS14, CGP14]

Regular DLE-logics
Kripke frames with impossible worlds
[PSZ14a]

Display calculi
[GMPTZ14]

Jónsson-style vs Sambin-style canonicity
[PSZ14b]

Finite lattices and monotone ML
[FPS14]

Canonicity via pseudo-correspondence
[CPSZ14]

Algorithmic correspondence for DLE

Ackermann Lemma Based Algorithm

- engined by the Ackermann lemma.
- Reduction rules leading to the Ackermann elimination step.
- Residuation and approximation rules.
- Soundness on **perfect DLEs**:
  - approximation: both $\bigvee$-generated by the c. $\vee$-primes and $\bigwedge$-generated by the c. $\wedge$-primes;
  - residuation: all the operations are either right or left adjoints or residuals.

Perfect DLEs: the natural semantic environment both for ALBA and for display calculi for DLE.

Primitive inequalities

Primitive formulas: [Kracht 1996]

$$\begin{array}{rl} \text{Left-primitive} & \varphi := p \mid \top \mid \vee \mid \wedge \mid f(\vec{\varphi}/\vec{p}, \vec{\psi}/\vec{q}) \\ \text{Right-primitive} & \psi := p \mid \bot \mid \wedge \mid \vee \mid g(\vec{\psi}/\vec{p}, \vec{\varphi}/\vec{q}) \end{array}$$

Primitive inequalities:

$$\begin{array}{rl} \text{Left-primitive} & \varphi_1 \leq \varphi_2 \text{ with } \varphi_1 \text{ scattered} \\ \text{Right-primitive} & \psi_1 \leq \psi_2 \text{ with } \psi_2 \text{ scattered} \end{array}$$

Example:

$$\Diamond q \to \Box p \leq \Box(q \to p) \quad \rightsquigarrow \quad \frac{x \vdash \Diamond q \to \Box p}{x \vdash \Box(q \to p)} \quad \rightsquigarrow \quad \frac{X \vdash \circ Z > \circ Y}{X \vdash \circ(Z > Y)}.$$

First Attempt

Crucial observation: **same** structural connectives for the **basic** and for the **expanded** DLE.

Main strategy: transform **non-primitive** DLE inequalities into (conjunctions of) **primitive** DLE inequalities in the **expanded** language:

$$s(\vec{p},\vec{q}) \leq s'(\vec{p},\vec{q}) \qquad \& \left\{ \varphi_i^*(\vec{p},\vec{q}) \leq \varphi'^*_i(\vec{p},\vec{q}) \mid i \in I \right\}$$

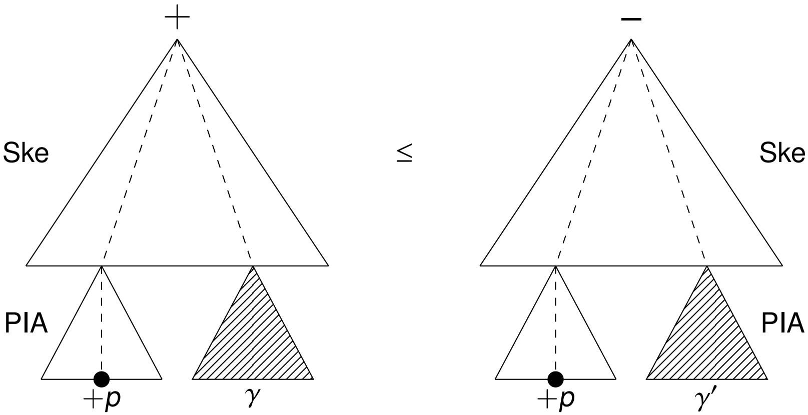
$\Updownarrow$ *ALBA* $\qquad$ $\Updownarrow$ ALBA on primitives

$$\& \left\{ \varphi_i^*(\vec{\mathbf{i}},\vec{\mathbf{m}}) \leq \varphi'^*_i(\vec{\mathbf{i}},\vec{\mathbf{m}}) \mid i \in I \right\} \quad = \quad \& \left\{ \varphi_i^*(\vec{\mathbf{i}},\vec{\mathbf{m}}) \leq \varphi'^*_i(\vec{\mathbf{i}},\vec{\mathbf{m}}) \mid i \in I \right\}$$

Inductive but not analytic

$$
\begin{array}{ll}
 & \forall[\Diamond p \leq \Diamond\Box p] \\
\text{iff} & \forall[(\mathbf{i} \leq \Diamond p \ \&\ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{i} \leq \Diamond \mathbf{j} \ \&\ \mathbf{j} \leq p \ \&\ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{i} \leq \Diamond \mathbf{j} \ \&\ \Diamond\Box \mathbf{j} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \forall \mathbf{m}[\Diamond\Box \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]] \\
\text{iff} & \forall[\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \mathbf{i} \leq \Diamond\Box \mathbf{j}] \\
\text{iff} & \forall[\Diamond \mathbf{j} \leq \Diamond\Box \mathbf{j}]
\end{array}
$$

Analytic inductive inequalities



Type 2: allowing multiple occurrences of var's in heads of inequalities

Let $\mathcal{G} = \varnothing$, $\mathcal{F} = \{\Diamond, \cdot\}$ where $\cdot$ binary and of order type $(1, 1)$

$$
\begin{array}{ll}
 & \forall[\Diamond\Diamond p \cdot \Diamond p \leq \Diamond p] \\
\text{iff} & \forall[(\mathbf{j} \leq \Diamond\Diamond p \cdot \Diamond p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond p \ \&\ \mathbf{i} \leq p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \ \&\ \mathbf{i} \leq p \ \&\ \mathbf{h} \leq p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \ \&\ \mathbf{i} \vee \mathbf{h} \leq p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \ \&\ \Diamond(\mathbf{i} \vee \mathbf{h}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \Rightarrow \forall \mathbf{m}[\Diamond(\mathbf{i} \vee \mathbf{h}) \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}]] \\
\text{iff} & \forall[\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \Rightarrow \mathbf{j} \leq \Diamond(\mathbf{i} \vee \mathbf{h})] \\
\text{iff} & \forall[\Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \leq \Diamond(\mathbf{i} \vee \mathbf{h})] \\
\hline
\text{iff} & \forall[\Diamond\Diamond p_1 \cdot \Diamond p_2 \leq \Diamond p_1 \vee \Diamond p_2] \text{ (ALBA for primitive)}
\end{array}
$$

$$
\cdots \quad \rightsquigarrow \quad \frac{\Diamond p_1 \vdash q \qquad \Diamond p_2 \vdash q}{\Diamond\Diamond p_1 \cdot \Diamond p_2 \vdash z} \quad \rightsquigarrow \quad \frac{\circ X \vdash Z \qquad \circ Y \vdash Z}{\circ \circ X \odot \circ Y \vdash Z}
$$

Frege axiom: a first reduction

$$
\begin{array}{ll}
& \forall[p \rightharpoonup (q \rightharpoonup r) \leq (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r)] \\
\text{iff} & \forall[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup r) \;\&\; (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup r) \;\&\; (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup \mathbf{n}) \leq \mathbf{m} \;\&\; r \leq \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup \mathbf{n}) \;\&\; (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup \mathbf{n}) \;\&\; (p \rightharpoonup q) \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \;\&\; \mathbf{i} \leq p) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \mathbf{i} \rightharpoonup (q \rightharpoonup \mathbf{n}) \;\&\; (\mathbf{i} \rightharpoonup q) \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \mathbf{i} \rightharpoonup (q \rightharpoonup \mathbf{n}) \;\&\; \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \;\&\; \mathbf{h} \leq \mathbf{i} \rightharpoonup q) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \mathbf{i} \rightharpoonup (q \rightharpoonup \mathbf{n}) \;\&\; \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \;\&\; \mathbf{i} \bullet \mathbf{h} \leq q) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \;\&\; \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{j} \leq \mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \Rightarrow \forall \mathbf{m}[\mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}]] \\
\text{iff} & \forall[\mathbf{j} \leq \mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n})] \\
\text{iff} & \forall[\mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \leq \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n})] \\
\hline
\text{iff} & \forall[p \rightharpoonup ((p \bullet q) \rightharpoonup r) \leq q \rightharpoonup (p \rightharpoonup r)] \text{ (ALBA for primitive)}
\end{array}
$$

$$\vdots$$

$$\frac{\text{iff} \quad \forall[\boldsymbol{i} \rightharpoonup ((\boldsymbol{i} \bullet \boldsymbol{h}) \rightharpoonup \boldsymbol{n}) \leq \boldsymbol{h} \rightharpoonup (\boldsymbol{i} \rightharpoonup \boldsymbol{n})]}{\text{iff} \quad \forall[p \rightharpoonup ((p \bullet q) \rightharpoonup r) \leq q \rightharpoonup (p \rightharpoonup r)] \text{ (ALBA for primitive)}}$$

by applying the usual procedure, we obtain the following rule:

$$\cdots \quad \rightsquigarrow \quad \frac{s \vdash p \rightharpoonup ((p \bullet q) \rightharpoonup r)}{s \vdash q \rightharpoonup (p \rightharpoonup r)} \quad \rightsquigarrow \quad \frac{W \vdash X \succ ((X \odot Y) \succ Z)}{W \vdash Y \succ (X \succ Z)}$$

Type 4

Frege axiom: a second reduction

$$
\begin{array}{ll}
& \forall[p \rightharpoonup (q \rightharpoonup r) \leq (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r)] \\
\text{iff} & \forall[(p \rightharpoonup (q \rightharpoonup r)) \bullet (p \rightharpoonup q) \leq p \rightharpoonup r] \\
\text{iff} & \forall[((p \rightharpoonup (q \rightharpoonup r)) \bullet (p \rightharpoonup q)) \bullet p \leq r] \\
\text{iff} & \forall[\mathbf{i} \leq ((p \rightharpoonup (q \rightharpoonup r)) \bullet (p \rightharpoonup q) \bullet p \;\&\; r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq (\mathbf{h} \bullet \mathbf{k}) \bullet \mathbf{j} \;\&\; \mathbf{h} \leq p \rightharpoonup (q \rightharpoonup r) \;\&\; \\
& \qquad \mathbf{k} \leq p \rightharpoonup q \;\&\; \mathbf{j} \leq p \;\&\; r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq (\mathbf{h} \bullet \mathbf{k}) \bullet \mathbf{j} \;\&\; (\mathbf{h} \bullet p) \bullet q \leq r \;\&\; \\
& \qquad \mathbf{k} \bullet p \leq q \;\&\; \mathbf{j} \leq p \;\&\; r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq (\mathbf{h} \bullet \mathbf{k}) \bullet \mathbf{j} \;\&\; (\mathbf{h} \bullet \mathbf{j}) \bullet q \leq r \;\&\; \mathbf{k} \bullet \mathbf{j} \leq q \;\&\; r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq (\mathbf{h} \bullet \mathbf{k}) \bullet \mathbf{j} \;\&\; (\mathbf{h} \bullet \mathbf{j}) \bullet (\mathbf{k} \bullet \mathbf{j}) \leq r \;\&\; r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq (\mathbf{h} \bullet \mathbf{k}) \bullet \mathbf{j} \;\&\; (\mathbf{h} \bullet \mathbf{j}) \bullet (\mathbf{k} \bullet \mathbf{j}) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{h} \bullet \mathbf{k}) \bullet \mathbf{j} \leq (\mathbf{h} \bullet \mathbf{j}) \bullet (\mathbf{k} \bullet \mathbf{j})] \\
\hline
\text{iff} & \forall[(r \bullet q) \bullet p \leq (r \bullet p) \bullet (q \bullet p)] \text{ (ALBA for primitive)}
\end{array}
$$

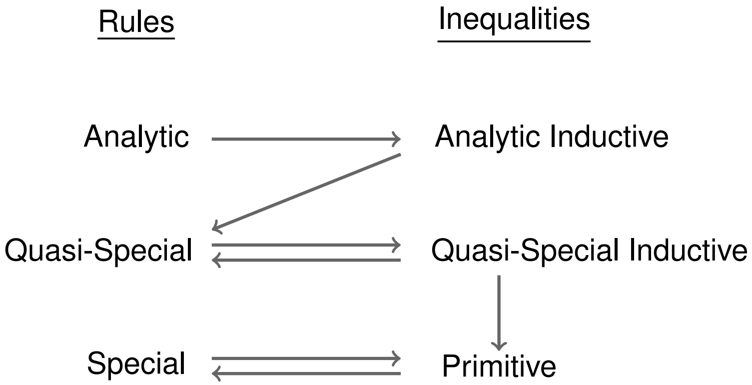
$$\vdots$$

$$\frac{\text{iff} \quad \forall[(\boldsymbol{h} \bullet \boldsymbol{k}) \bullet \boldsymbol{j} \leq (\boldsymbol{h} \circ \boldsymbol{j}) \bullet (\boldsymbol{k} \bullet \boldsymbol{j})]}{\text{iff} \quad \forall[(r \bullet q) \bullet p \leq (r \bullet p) \bullet (q \bullet p)] \text{ (ALBA for primitive)}}$$

by applying the usual procedure, we obtain the following rule:

$$\ldots \quad \rightsquigarrow \quad \frac{(r \bullet p) \bullet (q \bullet p) \vdash s}{(r \bullet q) \bullet p \vdash s} \quad \rightsquigarrow \quad \frac{(Z \odot X) \odot (Y \odot X) \vdash W}{(Z \odot Y) \odot X \vdash W}$$

Overview of main results



[Conradie Palmigiano 2012] Algorithmic Correspondence and Canonicity for Distributive Modal Logic, *APAL*, 163:338-376.

[Conradie Ghilardi Palmigiano] Unified Correspondence, in *Johan van Benthem on Logic and Information Dynamics*, Springer, 2014.

[Conradie Palmigiano 2014] Algorithmic correspondence and canonicity for non-distributive logics, *JLC*, to appear.

[Kracht 1996] Power and Weakness of the Modal Display Calculus, in *Proof Theory of Modal Logic*, 93-121, Kluwer.

[Conradie Palmigiano Sourabh] Algebraic modal correspondence: Sahlqvist and beyond, submitted, 2014.

[Conradie Palmigiano Sourabh Zhao] Canonicity and relativized canonicity via pseudo-correspondence, submitted, 2014.

[Greco Ma Palmigiano T. Zhao] Unified correspondence as a proof-theoretic tool, submitted, 2015.