A Computational Approach to Finite MTL-chains

Félix Bou

Artificial Intelligence Research Institute (IIIA - CSIC)
Barcelona (Spain)
fbou@iiia.csic.es

May 8th, 2015
ALCOP 2015 (Delft)

Outline

A Really Quick Overview of the Framework

- language: $\cdot, \vee, \wedge, 0, e, \to$
  - MTL: FL_{ew} + $(x \to y) \vee (y \to x) = e$ (prelinearity)
  - BL: MTL + $x \wedge y = x \cdot (x \to y)$
  - IMTL: MTL + $(x \to 0) \to 0 = x$
  - IBL (MV): BL + $(x \to 0) \to 0 = x$

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  - IMTL: MTL + $(x \to 0) \to 0 = x$
  - IBL (MV): BL + $(x \to 0) \to 0 = x$

- ℓ-monoid (positive) language: $\cdot, \vee, \wedge, 0, e$

What are the ℓ-reducts of MTL-chains?

- $\vee, \wedge$ is a linear partial order with bounds 0 and *e*,
- $\cdot$ is associative with neutral element *e* (i.e., monoid),
- $\cdot$ is commutative,
- $\cdot$ is monotone with respect to the partial order.

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When finite, w.l.o.g., it is enough to give the monoidal operation (under the assumption that the order is $0 < 1 < 2 \ldots < n$).

Are these ℓ-reducts of MTL-chains?

	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	2	2
3	0	0	2	3	3
4	0	1	2	3	4

	0	1	2	3	4
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Associativity is the only non-trivial property to check.

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An Embarrassing Question

Is the equation

$$x_1x_4x_7 \;\wedge\; x_2x_5x_8 \;\wedge\; x_3x_6x_9 \quad \leq \quad x_1x_2x_3 \;\vee\; x_4x_5x_6 \;\vee\; x_7x_8x_9$$

valid in MTL?

Main Problem to Consider

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- No?
- Yes?

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- No? (by HSP Theorem we would get a representation description for MTL-algebras)
- Yes?

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Main Problem to Consider

Is there some ℓ-monoid equation that distinguishes MTL from BL?

- No? (by HSP Theorem we would get a representation description for MTL-algebras)
- Yes? (requires a better understanding of (finite) MTL-chains)

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Why considering the ℓ-monoid reduct?

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1. a better understanding of the ℓ-monoid fragment of MTL-algebras will enlighten us with a better understanding of the full language (including residuum).

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Why considering the ℓ-monoid reduct?

1. a better understanding of the ℓ-monoid fragment of MTL-algebras will enlighten us with a better understanding of the full language (including residuum).
2. in some contexts it is easier to deal with the ℓ-monoid fragment than with the full language.

“Balance” between Pros and Cons

- full language $(\cdot, \vee, \wedge, 0, e, \to)$
  - Pro:
  - Con:
- ℓ-monoid $(\cdot, \vee, \wedge, 0, e)$
  - Pro:
  - Con:

“Balance” between Pros and Cons

- full language $(\cdot, \vee, \wedge, 0, e, \rightarrow)$
  - Pro: Congruences are nicely characterized
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  - Pro: Congruences are difficult
  - Con: Free abelian monoid is easy, it is $\bigoplus_{i \in \kappa}(\mathbb{N}, +, 0)$

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$$x \wedge y = y \wedge x \qquad x \vee y = y \vee x$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \qquad x \vee (y \vee z) = (x \vee y) \vee z$$

$$x \wedge (x \vee y) = x \qquad x \vee (x \wedge y) = x$$

$$x \wedge e = x \qquad x \vee 0 = x$$

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$$x \wedge e = x$$
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$x \cdot y = y \cdot x$$
$$x \cdot e = x$$
$$x \cdot 0 = 0$$

$$x \vee y = y \vee x$$
$$x \vee (y \vee z) = (x \vee y) \vee z$$
$$x \vee (x \wedge y) = x$$
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$$x \wedge e = x \qquad x \vee 0 = x$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

$$x \cdot y = y \cdot x$$

$$x \cdot e = x$$

$$x \cdot 0 = 0$$

$$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z) \qquad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z)$$

When $\mathbf{A} = (A, \cdot, \vee, \wedge, 0, e)$ is subdirectly irreducible?

1. **A** is a chain
2. **A** has a coatom
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Involutive MTL-algebras

- The residuum operation can be replaced, in the signature, with negation because the equation

$$x \to y = \neg(x \cdot \neg y)$$

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- Duality: if in an equation which is valid in an IMTL-chain $\boldsymbol{A}$ and which only uses the symbols $\cdot, +, \neg, \wedge, \vee, 0, e$ we simultaneously interchange

$$\cdot \text{ and } +, \qquad \wedge \text{ and } \vee, \qquad \leq \text{ and } \geq, \qquad 0 \text{ and } e,$$

then the resultant equation is also valid in the same $\boldsymbol{A}$.

Remark (Duality)

For every IMTL-algebra ***A***, the following two conditions are equivalent.

- The equation

$$x_1x_4x_7 \ \wedge \ x_2x_5x_8 \ \wedge \ x_3x_6x_9 \quad \leq \quad x_1x_2x_3 \ \vee \ x_4x_5x_6 \ \vee \ x_7x_8x_9$$

  is valid in ***A***.
- The equation

$$(x_1 + x_4 + x_7) \vee (x_2 + x_5 + x_8) \vee (x_3 + x_6 + x_9) \ \geq$$
$$\geq \ (x_1 + x_2 + x_3) \wedge (x_4 + x_5 + x_6) \wedge (x_7 + x_8 + x_9)$$

  is valid in ***A***.

How many-chains of cardinality *n*?

	1	2	3	4	5	6	7	8	9
BL	1	1	2	2^2	2^3	2^4	2^5	2^6	2^7

How many-chains of cardinality n?

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BL	1	1	2	2^2	2^3	2^4	2^5	2^6	2^7
IBL (MV)	1	1	1	1	1	1	1	1	1

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MTL	1	1	2	6	22	94	451	2386	13775

A030453 Number of linearly ordered Abelian monoids of size n (semi-groups with greatest element of the corresponding chain as neutral element); triangular norms on an n-chain. 1

1, 1, 2, 6, 22, 94, 451, 2386, 13775, 86417, 590489, 4446029, 37869449, 382549464 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

LINKS Table of n, a(n) for n=1..14.
Bernard de Baets, Home page
Index entries for sequences related to monoids

CROSSREFS Sequence in context: A150274 A109317 A109153 * A001861 A049526 A187251
Adjacent sequences: A030450 A030451 A030452 * A030454 A030455 A030456

KEYWORD nonn,hard,nice

AUTHOR Bernard De Baets (Bernard.DeBaets(AT)rug.ac.be)

STATUS approved

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MTL	1	1	2	6	22	94	451	2386	13775
IMTL	1	1	1	2	3	7	12	31	59


A034786 Number of linearly ordered Girard monoids of size n; number of t-norms on an n-chain inducing an involutive residual negator. 0

1, 1, 1, 2, 3, 7, 12, 31, 59, 161, 329, 944, 2067, 6148, 14558, 44483, 116372 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,4

REFERENCES M. Nachtegael, The Dizzy Number of Fuzzy Implication Operators on Finite Chains, in "Fuzzy Logic and Intelligent Technologies for Nuclear Science and Industry", ed. Ruan D., Abderrahim H., D'hondt P., Kerre E., 1998, pp. 29-35.

LINKS Table of n, a(n) for n=1..17.
Index entries for sequences related to monoids

CROSSREFS Cf. A030453.
Sequence in context: A134565 A100982 A186009 * A080107 A056156 A112837
Adjacent sequences: A034783 A034784 A034785 * A034787 A034788 A034789

KEYWORD nonn

AUTHOR Bernard De Baets (Bernard.DeBaets(AT)rug.ac.be), Mike Nachtegael (mike.nachtegael(AT)rug.ac.be)

STATUS approved

List of IMTL chains of cardinal 5

$*(5,0)$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	2	2
3	0	0	2	3	3
4	0	1	2	3	4

$*(5,1)$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	2
3	0	0	1	2	3
4	0	1	2	3	4

$*(5,2)$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	2
3	0	0	1	1	3
4	0	1	2	3	4

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0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	2	2
3	0	0	2	3	3
4	0	1	2	3	4

$*(5,1)$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	2
3	0	0	1	2	3
4	0	1	2	3	4

$*(5,2)$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	2
3	0	0	1	1	3
4	0	1	2	3	4

$+(5,0)$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	4	4
2	2	2	4	4	4
3	3	4	4	4	4
4	4	4	4	4	4

$+(5,1)$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	4
2	2	3	4	4	4
3	3	4	4	4	4
4	4	4	4	4	4

$+(5,2)$	0	1	2	3	4
0	0	1	2	3	4
1	1	3	3	4	4
2	2	3	4	4	4
3	3	4	4	4	4
4	4	4	4	4	4

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0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	2	2
3	0	0	2	3	3
4	0	1	2	3	4

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1	0	0	0	0	1
2	0	0	0	1	2
3	0	0	1	2	3
4	0	1	2	3	4

$*(5,2)$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	2
3	0	0	1	1	3
4	0	1	2	3	4

$+(5,0)$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	4	4
2	2	2	4	4	4
3	3	4	4	4	4
4	4	4	4	4	4

$+(5,1)$	0	1	2	3	4
0	0	1	2	3	4
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2	2	3	4	4	4
3	3	4	4	4	4
4	4	4	4	4	4

$+(5,2)$	0	1	2	3	4
0	0	1	2	3	4
1	1	3	3	4	4
2	2	3	4	4	4
3	3	4	4	4	4
4	4	4	4	4	4

Strong Advice: Use the sage package created by Peter Jipsen

List of IMTL chains of cardinal 6

$*(6,0)$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	2	2
3	0	0	0	2	3	3
4	0	0	2	3	4	4
5	0	1	2	3	4	5

$*(6,1)$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	1	2
3	0	0	0	1	1	3
4	0	0	1	1	1	4
5	0	1	2	3	4	5

$*(6,2)$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	2	2
3	0	0	0	3	3	3
4	0	0	2	3	4	4
5	0	1	2	3	4	5

$*(6,3)$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	2	2
3	0	0	0	1	2	3
4	0	0	2	2	4	4
5	0	1	2	3	4	5

$*(6,4)$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	1	2
3	0	0	0	3	3	3
4	0	0	1	3	3	4
5	0	1	2	3	4	5

$*(6,5)$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	1	2
3	0	0	0	1	1	3
4	0	0	1	1	2	4
5	0	1	2	3	4	5

$*(6,6)$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	1	2
3	0	0	0	1	2	3
4	0	0	1	2	3	4
5	0	1	2	3	4	5

List of IMTL chains of cardinal 6

$+(6,0)$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	3	5	5
2	2	2	3	5	5	5
3	3	3	5	5	5	5
4	4	5	5	5	5	5
5	5	5	5	5	5	5

$+(6,1)$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	4	4	4	5	5
2	2	4	4	5	5	5
3	3	4	5	5	5	5
4	4	5	5	5	5	5
5	5	5	5	5	5	5

$+(6,2)$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	3	5	5
2	2	2	2	5	5	5
3	3	3	5	5	5	5
4	4	5	5	5	5	5
5	5	5	5	5	5	5

$+(6,3)$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	3	3	5	5
2	2	3	4	5	5	5
3	3	3	5	5	5	5
4	4	5	5	5	5	5
5	5	5	5	5	5	5

$+(6,4)$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	2	4	5	5
2	2	2	2	5	5	5
3	3	4	5	5	5	5
4	4	5	5	5	5	5
5	5	5	5	5	5	5

$+(6,5)$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	3	4	4	5	5
2	2	4	4	5	5	5
3	3	4	5	5	5	5
4	4	5	5	5	5	5
5	5	5	5	5	5	5

$+(6,6)$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	5
2	2	3	4	5	5	5
3	3	4	5	5	5	5
4	4	5	5	5	5	5
5	5	5	5	5	5	5

List of IMTL chains of cardinal 7

$*(7,0)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	2	2
3	0	0	0	0	3	3	3
4	0	0	0	3	4	4	4
5	0	0	2	3	4	5	5
6	0	1	2	3	4	5	6

$*(7,1)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	2	2
3	0	0	0	0	2	3	3
4	0	0	0	2	3	4	4
5	0	0	2	3	4	5	5
6	0	1	2	3	4	5	6

$*(7,2)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	2	2
3	0	0	0	0	2	3	3
4	0	0	0	2	2	4	4
5	0	0	2	3	4	5	5
6	0	1	2	3	4	5	6

$*(7,3)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	1	2
3	0	0	0	0	3	3	3
4	0	0	0	3	4	4	4
5	0	0	1	3	4	4	5
6	0	1	2	3	4	5	6

$*(7,4)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	1	2
3	0	0	0	0	1	1	3
4	0	0	0	1	3	3	4
5	0	0	1	1	3	3	5
6	0	1	2	3	4	5	6

$*(7,5)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	1	2
3	0	0	0	0	1	1	3
4	0	0	0	1	2	3	4
5	0	0	1	1	3	3	5
6	0	1	2	3	4	5	6

$*(7,6)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	1	2
3	0	0	0	0	1	1	3
4	0	0	0	1	1	2	4
5	0	0	1	1	2	3	5
6	0	1	2	3	4	5	6

$*(7,7)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	1	2
3	0	0	0	0	1	1	3
4	0	0	0	1	1	1	4
5	0	0	1	1	1	1	5
6	0	1	2	3	4	5	6

$*(7,8)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	1	2
3	0	0	0	0	1	1	3
4	0	0	0	1	1	1	4
5	0	0	1	1	1	2	5
6	0	1	2	3	4	5	6

$*(7,9)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	1	2
3	0	0	0	0	1	2	3
4	0	0	0	1	2	3	4
5	0	0	1	2	3	4	5
6	0	1	2	3	4	5	6

$*(7,10)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	1	2
3	0	0	0	0	1	2	3
4	0	0	0	1	1	2	4
5	0	0	1	2	2	4	5
6	0	1	2	3	4	5	6

$*(7,11)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	2	2
3	0	0	0	0	1	2	3
4	0	0	0	1	1	2	4
5	0	0	2	2	2	5	5
6	0	1	2	3	4	5	6

List of IMTL chains of cardinal 7

+(7,0)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	2	3	4	6	6
2	2	2	2	3	6	6	6
3	3	3	3	6	6	6	6
4	4	4	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

+(7,1)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	2	3	4	6	6
2	2	2	3	4	6	6	6
3	3	3	4	6	6	6	6
4	4	4	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

+(7,2)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	2	3	4	6	6
2	2	2	4	4	6	6	6
3	3	3	4	6	6	6	6
4	4	4	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

+(7,3)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	2	3	5	6	6
2	2	2	2	3	6	6	6
3	3	3	3	6	6	6	6
4	4	5	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

+(7,4)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	3	3	5	5	6	6
2	2	3	3	5	6	6	6
3	3	5	5	6	6	6	6
4	4	5	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

+(7,5)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	3	3	5	5	6	6
2	2	3	4	5	6	6	6
3	3	5	5	6	6	6	6
4	4	5	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

+(7,6)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	3	4	5	5	6	6
2	2	4	5	5	6	6	6
3	3	5	5	6	6	6	6
4	4	5	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

+(7,7)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	5	5	5	5	6	6
2	2	5	5	5	6	6	6
3	3	5	5	6	6	6	6
4	4	5	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

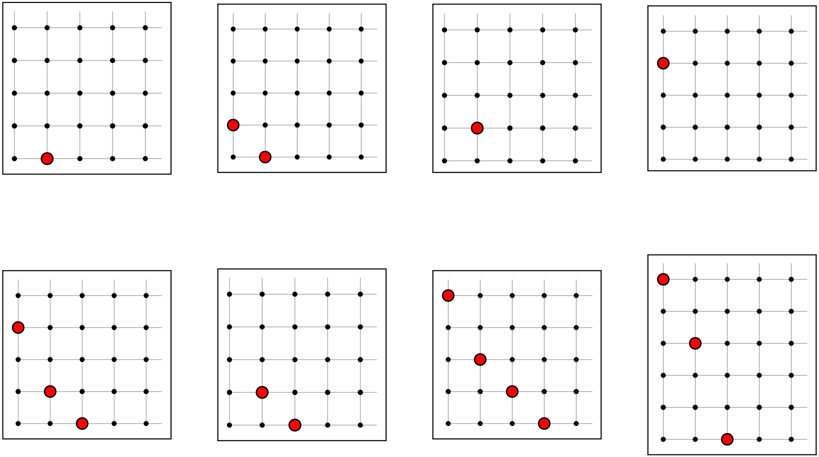
+(7,8)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	4	5	5	5	6	6
2	2	5	5	5	6	6	6
3	3	5	5	6	6	6	6
4	4	5	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

+(7,9)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	6
2	2	3	4	5	6	6	6
3	3	4	5	6	6	6	6
4	4	5	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

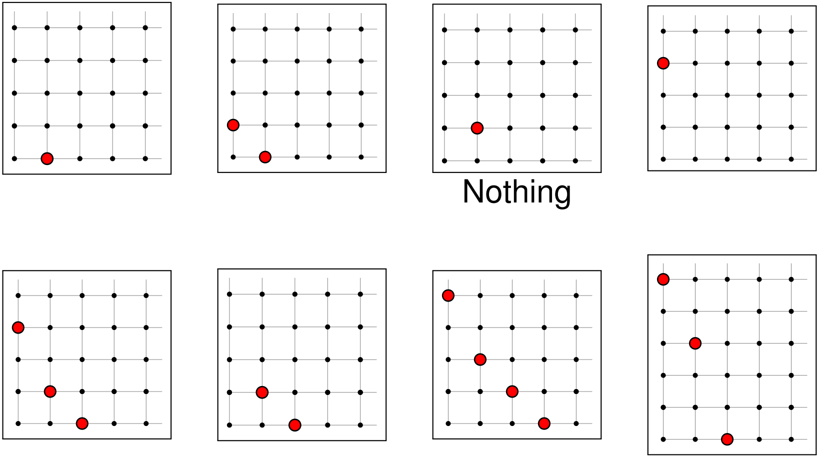
+(7,10)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	4	4	5	6	6
2	2	4	5	5	6	6	6
3	3	4	5	6	6	6	6
4	4	5	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

+(7,11)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	4	4	4	6	6
2	2	4	5	5	6	6	6
3	3	4	5	6	6	6	6
4	4	4	6	6	6	6	6
5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6

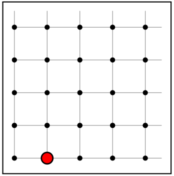
Describing 2-generated IMTL-chains through Token Configurations in $(\mathbb{N}^2, +, 0)$

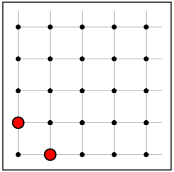


Describing 2-generated IMTL-chains through Token Configurations in $(\mathbb{N}^2, +, 0)$



Describing 2-generated IMTL-chains through Token Configurations in $(\mathbb{N}^2, +, 0)$


IMTL-chain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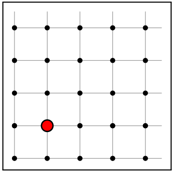
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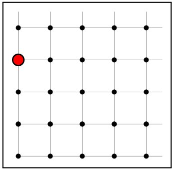
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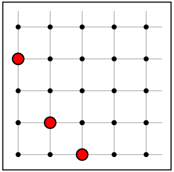
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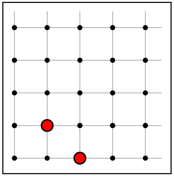
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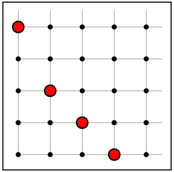
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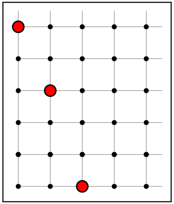
IMTL-chain

"Admissible" Configurations in $(\mathbb{N}^2, +, 0)$

1. antichain

“Admissible” Configurations in $(\mathbb{N}^2, +, 0)$

1. antichain (the “monomial ideal” generated is denoted *I*)

“Admissible” Configurations in $(\mathbb{N}^2, +, 0)$

1. antichain (the “monomial ideal” generated is denoted I)
2. any of the following equivalent conditions hold:

"Admissible" Configurations in $(\mathbb{N}^2, +, 0)$

1. antichain (the "monomial ideal" generated is denoted I)
2. any of the following equivalent conditions hold:
   - $(\{I - \{a\} : a \in \mathbb{N}^2\}, \subseteq)$ is a total order
   - for all $a, b \in \mathbb{N}^2$, either $I - \{a\} \subseteq I - \{b\}$ or $I - \{b\} \subseteq I - \{a\}$

“Admissible” Configurations in $(\mathbb{N}^2, +, 0)$

1. antichain (the “monomial ideal” generated is denoted I)
2. any of the following equivalent conditions hold:
   - $(\{I - \{a\} : a \in \mathbb{N}^2\}, \subseteq)$ is a total order
   - for all $a, b \in \mathbb{N}^2$, either $I - \{a\} \subseteq I - \{b\}$ or $I - \{b\} \subseteq I - \{a\}$
   - for all $a, b, c, d \in \mathbb{N}^2$, if $a + b \in I$ and $c + d \in I$, then either $a + d \in I$ or $b + c \in I$
   - for all $a, b, c, d \in \mathbb{N}^2$, if $a + b \in I$ and $c + d \in I$, then either $a + c \in I$ or $b + d \in I$

“Admissible” Configurations in $(\mathbb{N}^2, +, 0)$

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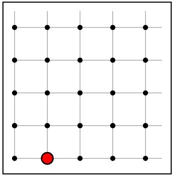
   - for all $a, b, c, d \in \mathbb{N}^2$, if $a + b \in I$ and $c + d \in I$, then either $a + d \in I$ or $b + c \in I$
   - for all $a, b, c, d \in \mathbb{N}^2$, if $a + b \in I$ and $c + d \in I$, then either $a + c \in I$ or $b + d \in I$
   - for all $a, b, c, d \in \mathbb{N}^2$, if $a + b \in \min(I)$ and $c + d \in \min(I)$, then either $a + d \in I$ or $b + c \in I$ [computational condition]

"Admissible" Configurations in $(\mathbb{N}^2, +, 0)$

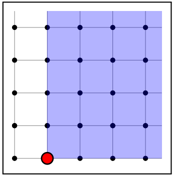
1. antichain (the "monomial ideal" generated is denoted I)
2. any of the following equivalent conditions hold:
   - $(\{I - \{a\} : a \in \mathbb{N}^2\}, \subseteq)$ is a total order
   - for all $a, b \in \mathbb{N}^2$, either $I - \{a\} \subseteq I - \{b\}$ or $I - \{b\} \subseteq I - \{a\}$
   - for all $a, b, c, d \in \mathbb{N}^2$, if $a + b \in I$ and $c + d \in I$, then either $a + d \in I$ or $b + c \in I$
   - for all $a, b, c, d \in \mathbb{N}^2$, if $a + b \in I$ and $c + d \in I$, then either $a + c \in I$ or $b + d \in I$
   - for all $a, b, c, d \in \mathbb{N}^2$, if $a + b \in \min(I)$ and $c + d \in \min(I)$, then either $a + d \in I$ or $b + c \in I$ [computational condition]

Communication Ideal: Any I with these conditions

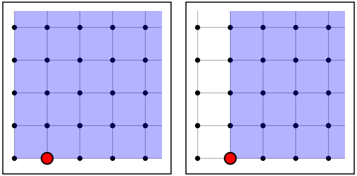
The IMTL-chain given by ...



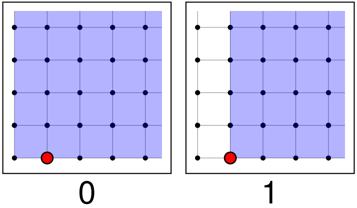
The IMTL-chain given by . . .



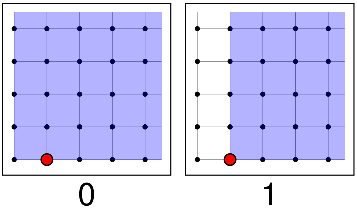
The IMTL-chain given by . . .



The IMTL-chain given by . . .

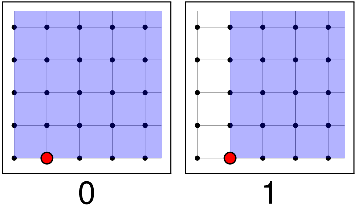


The IMTL-chain given by ...



$*(2,0)$	0	1
0	0	0
1	0	1

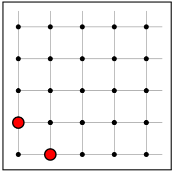
The IMTL-chain given by . . .



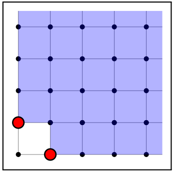
$*(2,0)$	0	1
0	0	0
1	0	1

2-element Boolean Algebra

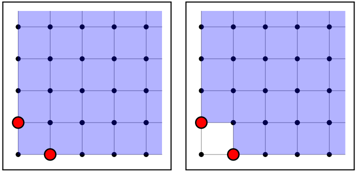
The IMTL-chain given by . . .



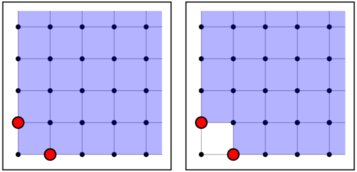
The IMTL-chain given by . . .



The IMTL-chain given by . . .

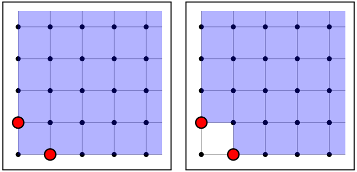


The IMTL-chain given by . . .



0 1

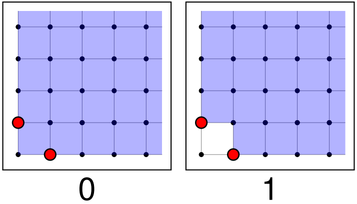
The IMTL-chain given by . . .



0 1

$*(2,0)$	0	1
0	0	0
1	0	1

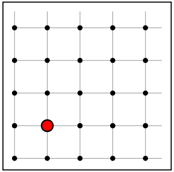
The IMTL-chain given by . . .



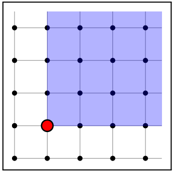
$*(2,0)$	0	1
0	0	0
1	0	1

2-element Boolean Algebra

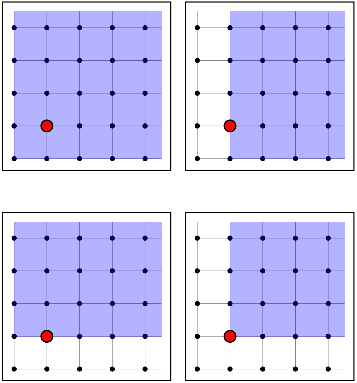
The IMTL-chain given by . . .



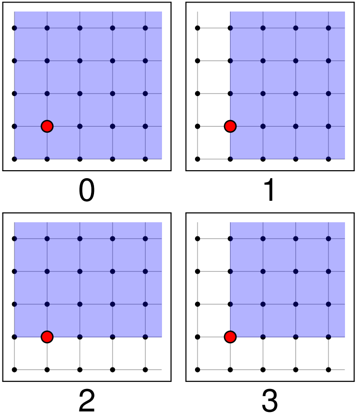
The IMTL-chain given by . . .



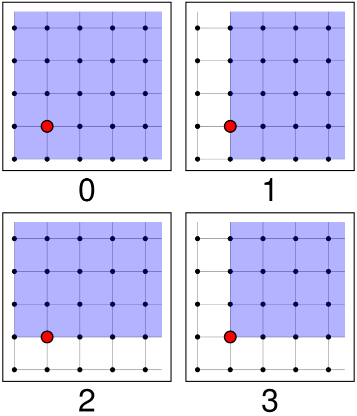
The IMTL-chain given by . . .



The IMTL-chain given by . . .

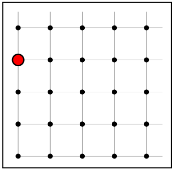


The IMTL-chain given by . . .

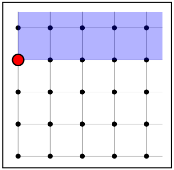


None (1 and 2 are not comparable)

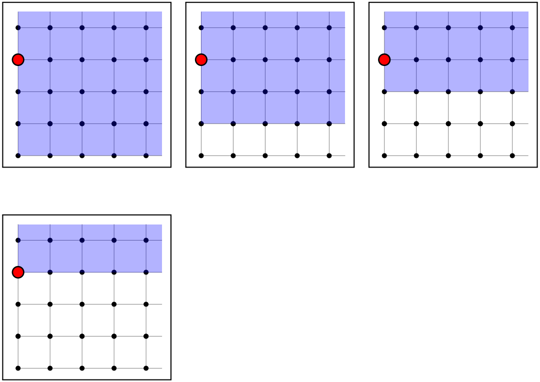
The IMTL-chain given by . . .



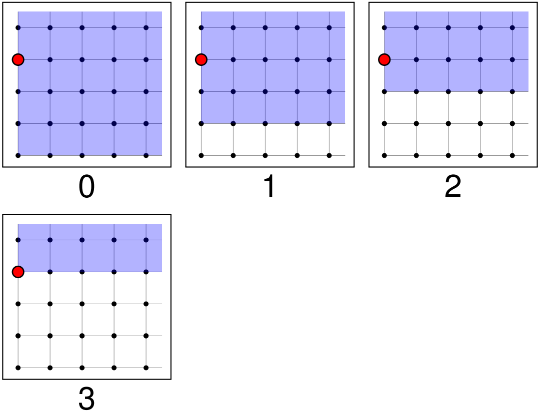
The IMTL-chain given by . . .



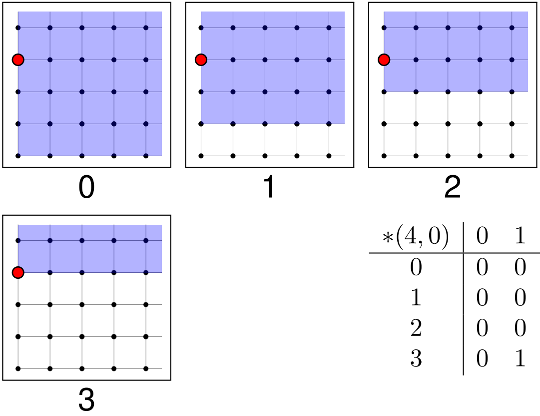
The IMTL-chain given by . . .



The IMTL-chain given by . . .

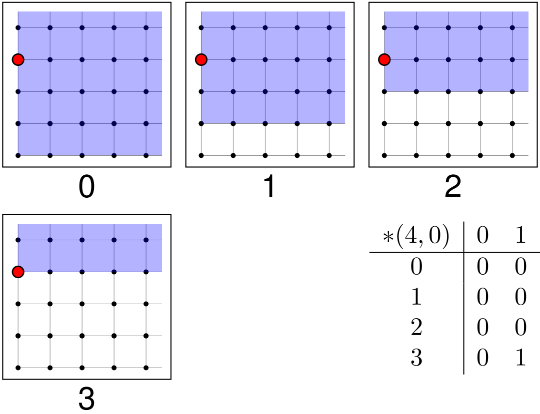


The IMTL-chain given by . . .



$*(4,0)$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

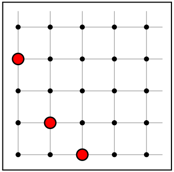
The IMTL-chain given by . . .



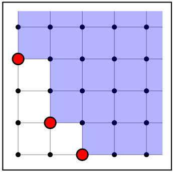
$*(4,0)$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

4-element MV chain

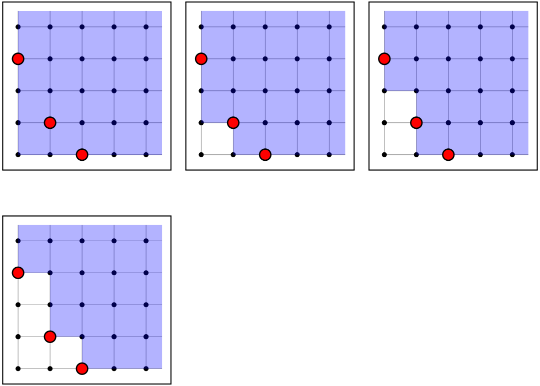
The IMTL-chain given by . . .



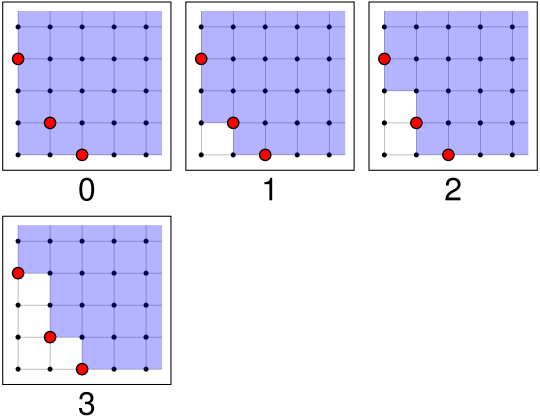
The IMTL-chain given by . . .



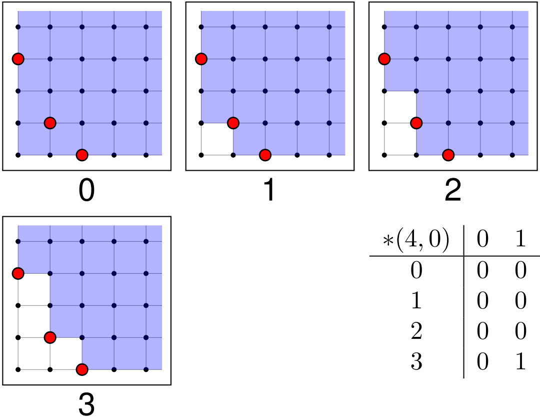
The IMTL-chain given by . . .



The IMTL-chain given by ...

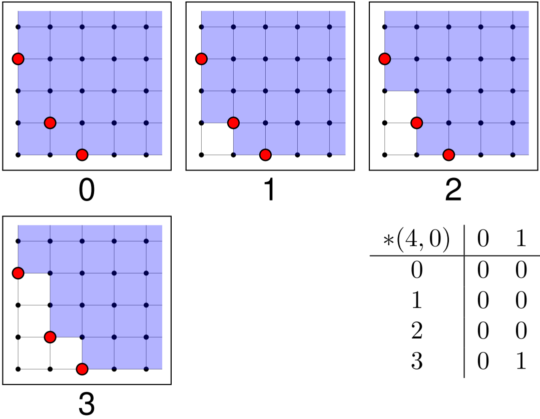


The IMTL-chain given by . . .



$*(4,0)$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

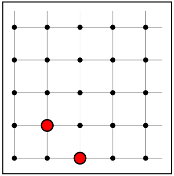
The IMTL-chain given by . . .



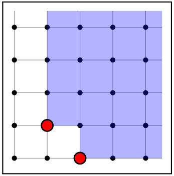
$*(4,0)$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

4-element MV chain

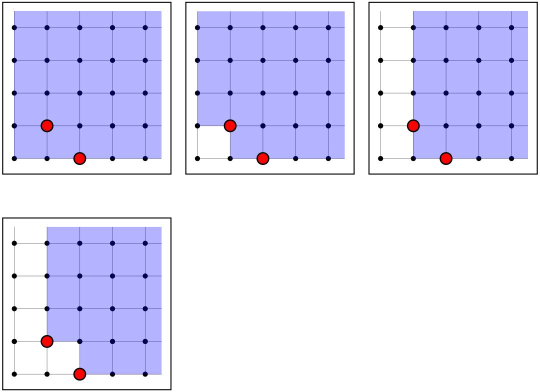
The IMTL-chain given by . . .



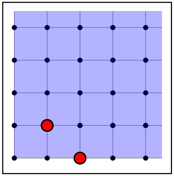
The IMTL-chain given by . . .



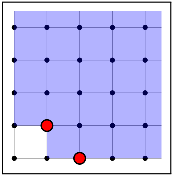
The IMTL-chain given by . . .



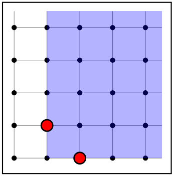
The IMTL-chain given by . . .



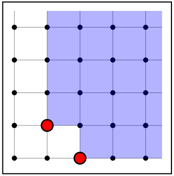
0



1

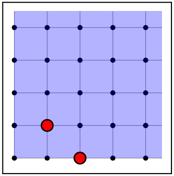


2

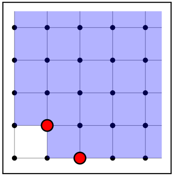


3

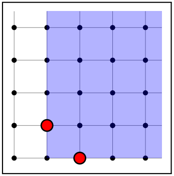
The IMTL-chain given by . . .



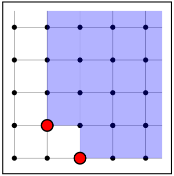
0



1



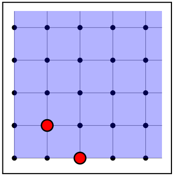
2



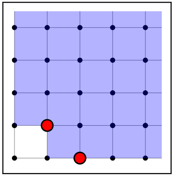
3

$*(4,1)$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	2	2
3	0	1	2	3

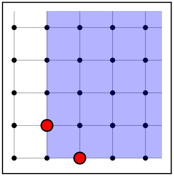
The IMTL-chain given by . . .



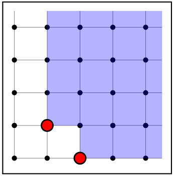
0



1



2

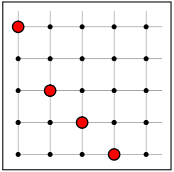


3

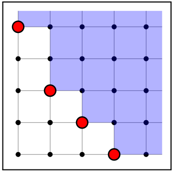
$*(4,1)$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	2	2
3	0	1	2	3

4-element Nilpotent Minimum chain

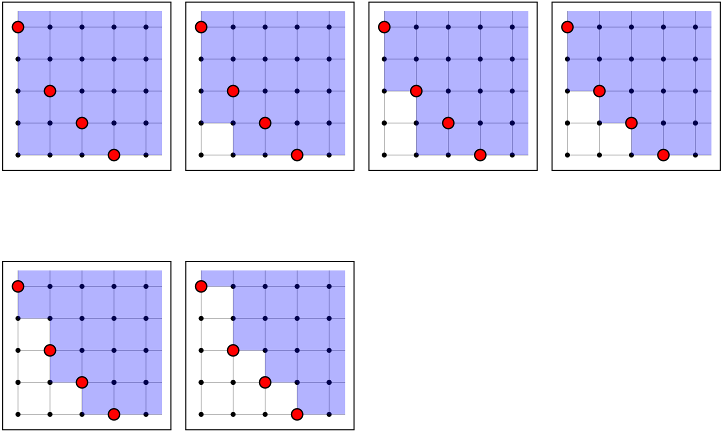
The IMTL-chain given by . . .



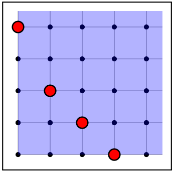
The IMTL-chain given by . . .

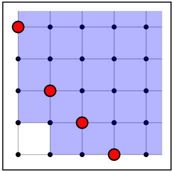


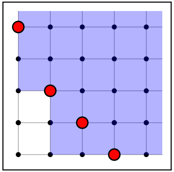
The IMTL-chain given by . . .



The IMTL-chain given by . . .


0


1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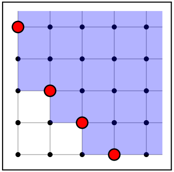
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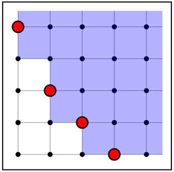
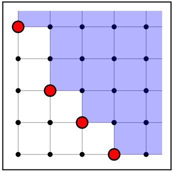
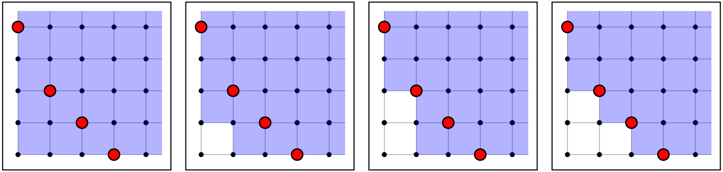
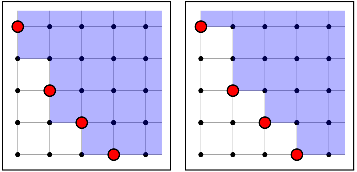

4


5

The IMTL-chain given by . . .



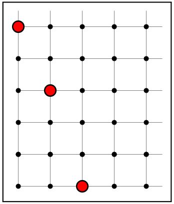
0 1 2 3



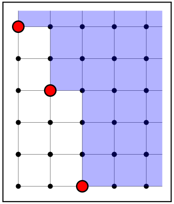
4 5

$*(6,5)$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	1	2
3	0	0	0	1	1	3
4	0	0	1	1	2	4
5	0	1	2	3	4	5

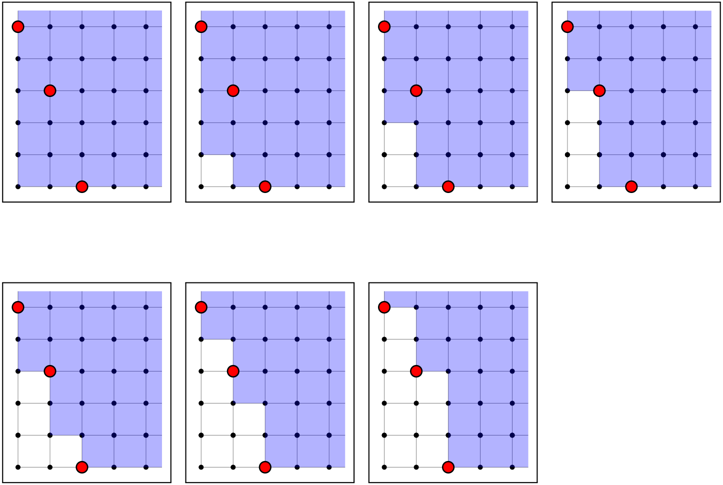
The IMTL-chain given by . . .



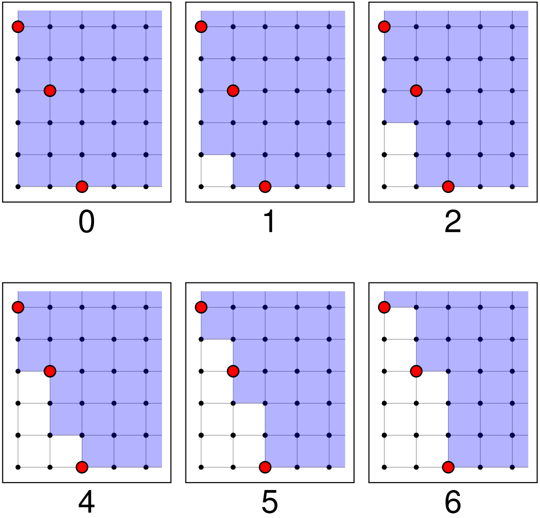
The IMTL-chain given by . . .



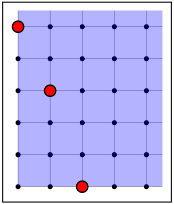
The IMTL-chain given by . . .

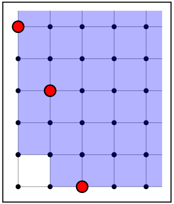


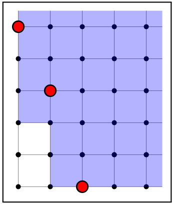
The IMTL-chain given by . . .

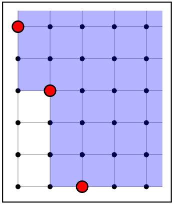


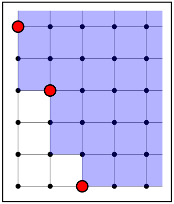
The IMTL-chain given by . . .

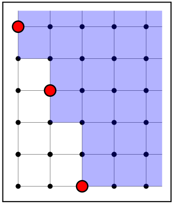

0

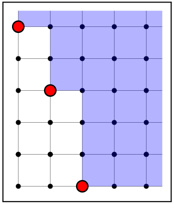

1


2


3


4


5


6

$*(7,10)$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	0	0	0	1	2
3	0	0	0	0	1	2	3
4	0	0	0	1	1	2	4
5	0	0	1	2	2	4	5
6	0	1	2	3	4	5	6

Representation of involutive chains

- Every algebra associated with a communication ideal of $(\mathbb{N}, +, 0)^2$ is a finite IMTL-chain.

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- The same can be said for arbitrary κ using the monoid $\bigoplus_{i \in \kappa} (\mathbb{N}, +, 0)$; but here it is crucial to remember that involutive refers to a notion in the ℓ-monoid fragment.

Representation of involutive chains

- Every algebra associated with a communication ideal of $(\mathbb{N}, +, 0)^2$ is a finite IMTL-chain.
- All IMTL-chains that are 2-generated (as ℓ-monoid) come from a communication ideal of $(\mathbb{N}, +, 0)^2$

- Every algebra associated with a communication ideal of $(\mathbb{N}, +, 0)^k$ is a finite IMTL-chain.
- All IMTL-chains that are k-generated (as ℓ-monoid) come from a communication ideal of $(\mathbb{N}, +, 0)^k$.

- The same can be said for arbitrary κ using the monoid $\bigoplus_{i \in \kappa}(\mathbb{N}, +, 0)$; but here it is crucial to remember that involutive refers to a notion in the ℓ-monoid fragment.
- If a chain is n-potent, then we can replace the monoid $(\mathbb{N}, +, 0)$ with the "truncated" one over $\{0, 1, 2, \ldots, n\}$.

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Some computational algebra notions

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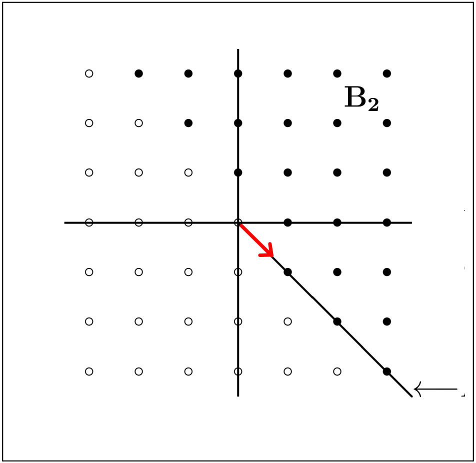
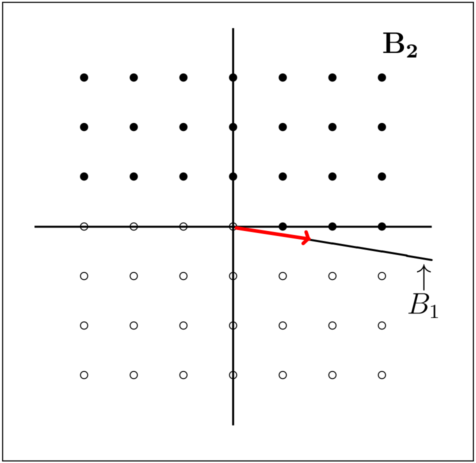
- Robbiano has classified all monomial orderings using invertible matrices of real numbers.
- There are very nice geometrical interpretations of what are monomial orderings.

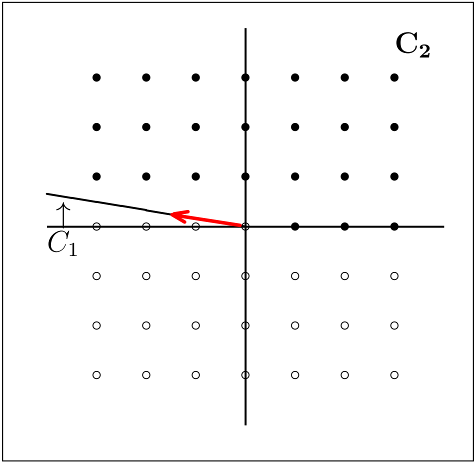
Some monomial orderings of $(\mathbb{N}, +, 0)^2$



Some monomial orderings of $(\mathbb{N}, +, 0)^2$



Some monomial orderings of $(\mathbb{N}, +, 0)^2$

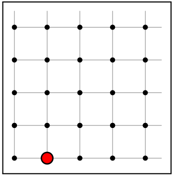


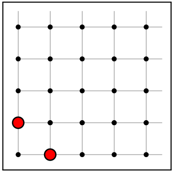
Two trivial ways to introduce communication ideals

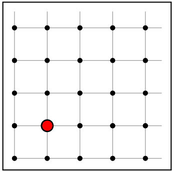
1. All upsets of admisible monomial orderings of $(\mathbb{N}^k, +, 0)$ are communication ideals.
   For $k = 2$, communications ideals coincide exactly with (principal) upsets of admissible monomial orderings.

2. The inverse image of a communication ideal under a monoid homomorphism is also a communication ideal.

Revisiting previous Token Configurations


IMTL-chain


IMTL-chain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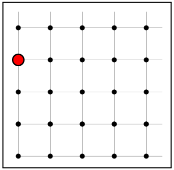
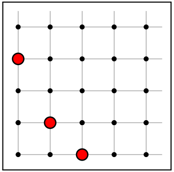
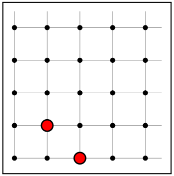
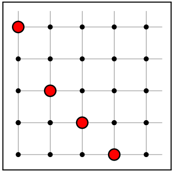
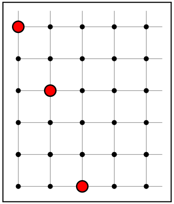

IMTL-chain


IMTL-chain


IMTL-chain


IMTL-chain


IMTL-chain

Application

The equation

$$x_1 x_4 x_7 \;\wedge\; x_2 x_5 x_8 \;\wedge\; x_3 x_6 x_9 \quad \leq \quad x_1 x_2 x_3 \;\vee\; x_4 x_5 x_6 \;\vee\; x_7 x_8 x_9$$

is valid in BL, but fails in MTL.

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Alternative presentation of the equation:

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- valid in BL [Proof Sketch: 1) It holds in the 1-generated infinite product algebra (by cancellativity), 2) It holds in finite MV-chains, 3) It holds in all BL-algebras]

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- valid in BL [Proof Sketch: 1) It holds in the 1-generated infinite product algebra (by cancellativity), 2) It holds in finite MV-chains, 3) It holds in all BL-algebras]
- fails in MTL [Proof Sketch: explicit 36-element chain ***E***]

```
#FUSION
fus_table = [
  # 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35      |
  #-------------------------------------------------------------------------------------------------------------|---
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  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3, 3], # |  3
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 3, 3, 4], # |  4
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5, 5, 5, 5, 5], # |  5
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 5, 5, 5, 5, 6], # |  6
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  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 5, 5, 5, 7, 7,18,18,18,18,18,18,18,18,18,18,18,20,20,23,23,24,24,24], # | 24
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]
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Claim

The algebra ***E*** is an exotic MTL-chain of dimension 9; the set of irreducible elements is $\{9, 15, 17, 22, 25, 28, 30, 32, 34\}$.

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$$\langle e(x_1), e(x_2), \ldots, e(x_9) \rangle = \langle 9, 28, 34, 30, 25, 15, 32, 17, 22 \rangle.$$

This is the unique counterexample up to symmetry

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This is the unique counterexample up to symmetry (and so there are 36 counterexamples)

```
#RESIDUUM
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  #-----------------------------------------------------------------------------------------------------------------|---
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  [21,22,22,22,22,25,25,25,25,25,30,30,30,34,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35], # | 14
  [20,21,22,22,22,25,25,25,25,25,30,30,30,33,34,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35], # | 15
  [19,19,19,22,22,23,23,25,25,25,28,28,30,32,32,32,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35], # | 16
  [18,19,19,22,22,23,23,24,25,25,27,28,30,32,32,32,34,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35], # | 17
  [17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35], # | 18
  [16,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,34,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35], # | 19
  [15,15,15,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,32,32,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35], # | 20
  [14,15,15,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,32,32,34,35,35,35,35,35,35,35,35,35,35,35,35,35,35,35], # | 21
  [13,14,15,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,32,32,33,34,35,35,35,35,35,35,35,35,35,35,35,35,35,35], # | 22
  [12,12,12,12,12,17,17,17,17,17,17,17,17,17,17,17,17,17,30,30,30,30,30,35,35,35,35,35,35,35,35,35,35,35,35,35], # | 23
  [11,11,11,12,12,15,15,17,17,17,17,17,17,17,17,17,17,17,28,28,30,30,30,32,35,35,35,35,35,35,35,35,35,35,35,35], # | 24
  [10,11,11,12,12,15,15,16,17,17,17,17,17,17,17,17,17,17,27,28,30,30,30,32,34,35,35,35,35,35,35,35,35,35,35,35], # | 25
  [ 9, 9, 9, 9, 9, 9, 9, 9, 9, 9,17,17,17,17,17,17,17,17,25,25,25,25,25,25,25,25,35,35,35,35,35,35,35,35,35,35], # | 26
  [ 8, 9, 9, 9, 9, 9, 9, 9, 9, 9,17,17,17,17,17,17,17,17,25,25,25,25,25,25,25,25,34,35,35,35,35,35,35,35,35,35], # | 27
  [ 7, 9, 9, 9, 9, 9, 9, 9, 9, 9,16,17,17,17,17,17,17,17,24,25,25,25,25,25,25,25,34,34,35,35,35,35,35,35,35,35], # | 28
  [ 6, 6, 6, 9, 9, 9, 9, 9, 9, 9,15,15,17,17,17,17,17,17,23,23,25,25,25,25,25,25,32,32,32,35,35,35,35,35,35,35], # | 29
  [ 5, 6, 6, 8, 9, 9, 9, 9, 9, 9,15,15,17,17,17,17,17,17,23,23,25,25,25,25,25,25,31,32,32,34,35,35,35,35,35,35], # | 30
  [ 4, 4, 4, 4, 4, 9, 9, 9, 9, 9,12,12,12,17,17,17,17,17,22,22,22,22,22,25,25,25,30,30,30,30,30,35,35,35,35,35], # | 31
  [ 3, 4, 4, 4, 4, 8, 9, 9, 9, 9,12,12,12,17,17,17,17,17,22,22,22,22,22,25,25,25,29,30,30,30,30,34,35,35,35,35], # | 32
  [ 2, 2, 2, 4, 4, 6, 6, 9, 9, 9,11,11,12,15,15,15,17,17,19,19,22,22,22,23,25,25,28,28,28,30,30,32,32,35,35,35], # | 33
  [ 1, 2, 2, 4, 4, 6, 6, 9, 9, 9,11,11,12,14,15,15,17,17,19,19,21,22,22,23,25,25,28,28,28,30,30,32,32,34,35,35], # | 34
  [ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35], # | 35
]
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#ADDITION
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  #----------------------------------------------------------------------------------------------------------------|---
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  [ 1, 2, 2, 4, 4, 6, 6, 9, 9, 9,11,11,12,14,15,15,17,17,19,19,21,22,22,23,25,25,28,28,28,30,30,32,32,34,35,35], # |  1
  [ 2, 2, 2, 4, 4, 6, 6, 9, 9, 9,11,11,12,15,15,15,17,17,19,19,22,22,22,23,25,25,28,28,28,30,30,32,32,35,35,35], # |  2
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  [ 4, 4, 4, 4, 4, 9, 9, 9, 9, 9,12,12,12,17,17,17,17,17,22,22,22,22,22,25,25,25,30,30,30,30,30,35,35,35,35,35], # |  4
  [ 5, 6, 6, 8, 9, 9, 9, 9, 9, 9,15,15,17,17,17,17,17,17,23,23,25,25,25,25,25,25,31,32,32,34,35,35,35,35,35,35], # |  5
  [ 6, 6, 6, 9, 9, 9, 9, 9, 9, 9,15,15,17,17,17,17,17,17,23,23,25,25,25,25,25,25,32,32,32,35,35,35,35,35,35,35], # |  6
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  [12,12,12,12,12,17,17,17,17,17,17,17,17,17,17,17,17,17,30,30,30,30,30,35,35,35,35,35,35,35,35,35,35,35,35,35], # | 12
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]
```

How to obtain the previous exotic chain $\boldsymbol{E}$?

Remember we want to falsify the equation

$$(x_1 + x_4 + x_7) \vee (x_2 + x_5 + x_8) \vee (x_3 + x_6 + x_9) \geq$$
$$\geq (x_1 + x_2 + x_3) \wedge (x_4 + x_5 + x_6) \wedge (x_7 + x_8 + x_9)$$

- There is a communication ideal I of $(\mathbb{N}^9, +, 0)$ such that

$$e_1 + e_4 + e_7 \notin I,\ e_2 + e_5 + e_8 \notin I,\ e_3 + e_6 + e_9 \notin I$$
$$e_1 + e_2 + e_3 \in I,\ e_4 + e_5 + e_6 \in I,\ e_7 + e_8 + e_9 \in I$$

- $h : (\mathbb{N}, +, 0)^9 \longrightarrow (\mathbb{Z}, +, 0)^5$ is the monoid homomorphism

$$h(a_1, a_2, \ldots, a_9) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix}$$

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- h is length-preserving and
  $h(e_1 + e_2 + e_3) = h(e_4 + e_5 + e_6) = h(e_7 + e_8 + e_9) =$
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- $I := I' \cup \{e_1 + e_2 + e_3, e_4 + e_5 + e_6, e_7 + e_8 + e_9\}$ (small perturbation),
- Claim: I is a communication ideal of $(\mathbb{N}, +, 0)^9$ satisfying our requirements.

Additional Remarks

- The variety generated by ℓ-monoid reducts of BL-algebras has an explicit axiomatization which requires an infinite number of axioms (essentially [Repnitskii, 1983-1984])

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- Open: Is there some "very expressive" language that cannot distinguish MTL from BL? What about $\cdot, \vee, 0, 1$?

Can these ideas help in . . . ?

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  - involutive MTL-chains are “locally finite”.

Summary

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- an small perturbation method has been used to obtain a quite pathological example of communication ideal (its associated ITML-chain is exotic).