Coalgebraic Correspondence Theory and Gaifman Locality

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Modal logic is invariant under bisimulation.

Modal logic is a fragment of FOL:

\[ \square \phi \equiv \forall y. xRy \rightarrow \phi(y) \]

- **Van Benthem:** Modal logic is the bisimulation-invariant fragment of FOL.

- **Rosen:** This remains true over finite structures.
Modal Logic beyond □ and ◇

- **Probabilistic modal logic**
  - Frames: Markov chains \((X, (P_x)_{x \in X})\)
  - Operators: \(L_p\) ‘with probability at least \(p\)’

- **Graded modal logic**
  - Frames: Multigraphs \((X, f : X \times X \rightarrow \mathbb{N} \cup \{\infty\})\)
  - Operators: \(\Diamond_k\) ‘in more than \(k\) successors’

- **Conditional logic**
  - Frames: e.g. selection function frames \((X, f : X \times \mathcal{P}(X) \rightarrow \mathcal{P}(X))\)
  - Operators: \(\Rightarrow\) ‘if . . . then normally . . .’

- **Neighbourhood logic**
  - Frames: Neighbourhood frames \((X, R \subseteq X \times \mathcal{P}(X))\)
  - Operators: □

What about FO correspondence theory for these?

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Coalgebraic Modal Logic

Similarity type \( \Lambda \)

\[
\phi, \psi ::= \bot \mid \phi \land \psi \mid \neg \phi \mid \Diamond \phi \quad (\Diamond \in \Lambda).
\]

Interpret over functor \( T : \textbf{Set} \to \textbf{Set} \) by predicate liftings

\[
[\Diamond]_X : \mathcal{P}(X) \to \mathcal{P}(TX).
\]

Semantics: satisfaction relation \( \models \) over \( T \)-coalgebras \( \xi : X \to TX \),

\[
x \models \Diamond \phi : \iff \xi(x) \in [\Diamond]_X [\phi]
\]

where \( [\phi] = \{ y \in X \mid y \models \phi \} \).

- This covers all examples above, and more.
Coalgebraic Predicate Logic

Generalize Chang’s modal FO language (1973) to coalgebraic modalities:

\[
\phi ::= \bot | \neg \phi | \phi_1 \land \phi_2 | x = y | P(\bar{x}) | \forall x. \phi | x \heartsuit [y : \phi]
\]

- Model = FO-model + $T$-coalgebra
- Pure CPL: without $P(\bar{x})$

- $M, \nu \models x \heartsuit [y : \phi]$ iff $\xi(\nu(x)) \in \llbracket \heartsuit \rrbracket \{ c \in X | M, \nu[y \mapsto c] \models \phi \}$
The Standard Translation

\[ ST_x(\bigvee \phi) = x \bigvee [x : ST_x \phi] . \]

CML = Single-variable quantifier-free CPL
Examples

- **Kripke semantics** \((TX = \mathcal{P}X \times \mathcal{P}V)\):
  Standard FO correspondence language
  \[
  xRy \overset{\hat{\cdot}}{=} x \Diamond [z : z = y]
  \]

- **Neighbourhoods** \((T = Q \circ Q^{op})\): Chang’s modal FO language

- **Graded ML** \((T = \text{bags})\): local counting quantifiers
  \[
  \exists^x k y. \phi \overset{\hat{\cdot}}{=} x \Diamond_{k-1} [y : \phi]
  \]
  (Axiomatize FO with counting: \(\neg \exists^x 2 y. y = z\))

- **Similarly for probabilistic ML** \((T = \text{distributions})\),
  \[
  w^x_y(\phi) \geq p \overset{\hat{\cdot}}{=} x L_p [y : \phi]
  \]
Outline of Otto’s Proof of Rosen’s Theorem

- Assume w.l.o.g. finitely many propositional variables.

- Note that invariance of $\phi$ under disjoint sums implies locality, via Gaifman locality.

- Use local unravellings to reduce to locally tree-like structures.

- Combine this to prove that $\phi$ is already $\sim_k$-invariant.

- Conclude that $\phi$ is equivalent to a (finite) modal formula of depth $k$. 
Recall: Gaifman’s Theorem

**Gaifman graph** of a FO structure:

\[ x \sim y \text{ iff } x \text{ and } y \text{ are in some basic relation} \]

\[ \rightsquigglyeq \text{ Gaifman distance, Neighbourhoods } N^m_d(u). \]

**Definition:** A formula \( \phi(x) \) is **Gaifman d-local** if for \( u, w \in M \),

\[
N^m_d(u) \cong N^m_d(w) \implies (M, u \models \phi(x) \iff M, w \models \phi(x))
\]

**Gaifman’s theorem:** Every \( \phi(x) \in FOL \) is Gaifman local.
Wrong idea: “$x \sim y$ if $x \Diamond \lceil y : \phi \rceil$ and $\phi(z)$”

E.g. in probabilistic logic

$$x L_1 \lceil y : \top \rceil \quad \text{and} \quad \top(z),$$

so

$$x \sim z$$

for all $x, z$. 
Solution: Support

▸ $A \subseteq X$ is a support of $t \in TX$ iff $t \in TA \subseteq TX$.

▸ Then by naturality of predicate liftings,

$$t \in [[\Diamond]]_X [\phi] \quad \text{iff} \quad t \in [[\Diamond]]_A ([\phi] \cap A)$$

▸ Supporting Kripke frame $R$ for $\xi : X \to TX$:

$$R(x) = \{ y \mid xRy \} \quad \text{support of } \xi(x)$$
Gaifman Locality for Support CPL

- Pure support CPL = Pure CPL plus binary predicate supp interpreted by supporting Kripke frame
- Inherit Gaifman theorem by translating into multisorted FO language

\[ \Diamond \subseteq s \times n \]
\[ \subseteq \subseteq s \times n \]
\[ \text{supp} \subseteq s \times s. \]

Neighbourhood compatibility:
Isomorphic nbhds (nearly) remain isomorphic

Theorem (Gaifman theorem for pure support-CPL):
Pure support-CPL is Gaifman local
The Coalgebraic van Benthem/Rosen Theorem

Infinitary version:
Λ separating, \( \phi(x) \in FOL(\Lambda) \approx \)-invariant (over finite models) \( \implies \)
\( \phi(x) \) equivalent (over finite models) to some
infinitary finite-rank modal formula \( \psi(x) \).

Finitary version:
Same with \( \psi(x) \) finitary for finite \( \Lambda \).

- The finitary version is immediate from the infinitary version.

Does the finitary van Benthem/Rosen theorem hold for infinite \( \Lambda \)?
Known Instances

- The classical van Benthem/Rosen theorem

- The van Benthem theorem for neighbourhood logic (Hansen/Kupke/Pacuit 2009)
Conclusion

- Coalgebraic predicate logic: FOL over $T$-coalgebras.

- Have proved a coalgebraic van Benthem/Rosen theorem.

- Nagging open problem: for infinite signatures, want to improve to finitary formulas.

- Key ingredient: Gaifman locality for CPL
  - Measure distance via support
  - Inherit from standard FOL by making neighbourhoods explicit
Future Work

- Investigation of CPL:
  - Model theory
  - Decidable fragments

- Sahlqvist theory (working from Dahlqvist/Pattinson 2013)
The Classical Correspondence Language

- One unary predicate $p(x)$ for each propositional variable $p$
- Binary relation $R(x, y)$
- No axioms or restrictions on models
- Standard translation:
  \[
  ST_x(p) = p(x) \\
  ST_x(\Box \phi) = \forall y. R(x, y) \rightarrow ST_y(\phi).
  \]

- Van Benthem/Rosen: for all $\phi(x) \in FOL$, TFAE:
  1. $\phi(x)$ bisimulation-invariant (over finite structures)
  2. $\phi(x) \leftrightarrow ST_x(\psi)$ for some modal $\psi$ (over finite structures)

- Janin/Walukiewicz:
  the bisimulation-invariant fragment of MSOL is the $\mu$-calculus.
Coalgebraic Unravelling

Recall: Coalgebraic modal logic captures behavioural equivalence

- defined via cospans of morphisms $X \rightarrow \bullet \leftarrow Y$
- in general weaker than bisimilarity (via spans $X \leftarrow \bullet \rightarrow Y$).

Require bounded behavioural equivalence $\approx_k$, defined via the terminal sequence

$X \xleftarrow{\xi_0} T_1 \leftarrow \cdots \leftarrow T_n \leftarrow \cdots$

Key facts:

**Lemma:** For $A, B$ trees of depth $k$, $A, a \approx B, b$ iff $A, a \approx_k B, b$.

**Unravelling Lemma:** For $A, a$ ex. $A, a \approx B, b$ s.t. $N_{3k}^B(b)$ tree of depth $k$. 