Coalgebraic Correspondence Theory and Gaifman Locality

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Introduction

Modal logic is invariant under bisimulation.

Modal logic is a fragment of FOL:

$$\Box\phi \mathrel{\hat{=}} \forall y.\, xRy \to \phi(y)$$

- Van Benthem:
  Modal logic is the bisimulation-invariant fragment of FOL.

- Rosen: This remains true over finite structures.

Modal Logic beyond $\Box$ and $\Diamond$

- Probabilistic modal logic
  - Frames: Markov chains $(X, (P_x)_{x \in X})$
  - Operators: L_p 'with probability at least p'
- Graded modal logic
  - Frames: Multigraphs $(X, f : X \times X \to \mathbb{N} \cup \{\infty\})$
  - Operators: $\Diamond_k$ 'in more than k successors'
- Conditional logic
  - Frames: e.g. selection function frames $(X, f : X \times \mathcal{P}(X) \to \mathcal{P}(X))$
  - Operators: $\Rightarrow$ 'if ... then normally ...'
- Neighbourhood logic
  - Frames: Neighbourhood frames $(X, R \subseteq X \times \mathcal{P}(X))$
  - Operators: $\Box$

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What about FO correspondence theory for these?

Coalgebraic Modal Logic

Similarity type Λ

$$\phi, \psi ::= \bot \mid \phi \wedge \psi \mid \neg\phi \mid \heartsuit\phi \qquad (\heartsuit \in \Lambda).$$

Interpret over functor $T : \mathbf{Set} \to \mathbf{Set}$ by predicate liftings

$$[\![\heartsuit]\!]_X : \mathcal{P}(X) \to \mathcal{P}(TX).$$

Semantics: satisfaction relation $\models$ over T-coalgebras $\xi : X \to TX$,

$$x \models \heartsuit\phi :\iff \xi(x) \in [\![\heartsuit]\!]_X [\![\phi]\!]$$

where $[\![\phi]\!] = \{y \in X \mid y \models \phi\}$.

- This covers all examples above, and more.

Coalgebraic Predicate Logic

Generalize Chang's modal FO language (1973)
to coalgebraic modalities:

$$\phi ::= \bot \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid x = y \mid P(\vec{x}) \mid \forall x.\, \phi \mid x\heartsuit\lceil y : \phi \rceil$$

- Model = FO-model + T-coalgebra
- Pure CPL: without $P(\vec{x})$
- $\mathfrak{M}, v \models x\heartsuit\lceil y : \phi \rceil$ iff $\xi(v(x)) \in [\![\heartsuit]\!]\{c \in X \mid \mathfrak{M}, v[y \mapsto c] \models \phi\}$

The Standard Translation

$$ST_x(\heartsuit\phi) = x\heartsuit\lceil x : ST_x\phi\rceil.$$

CML $=$ Single-variable quantifier-free CPL

Examples

- Kripke semantics ($TX = \mathcal{P}X \times \mathcal{P}V$):
  Standard FO correspondence language

$$xRy \quad \hat{=} \quad x\Diamond\lceil z : z = y\rceil$$

- Neighbourhoods ($T = \mathcal{Q} \circ \mathcal{Q}^{op}$): Chang's modal FO language
- Graded ML ($T =$ bags): local counting quantifiers

$$\exists^x k\, y.\phi \quad \hat{=} \quad x\Diamond_{k-1}\lceil y : \phi\rceil$$

  (Axiomatize FO with counting: $\neg\exists^x 2\, y.\, y = z$)
- Similarly for probabilistic ML ($T =$ distributions),

$$w^x_y(\phi) \geq p \quad \hat{=} \quad x\, L_p\lceil y : \phi\rceil$$

Outline of Otto's Proof of Rosen's Theorem

- Assume w.l.o.g. finitely many propositional variables.
- Note that invariance of ϕ under disjoint sums implies locality, via Gaifman locality.
- Use local unravellings to reduce to locally tree-like structures.
- Combine this to prove that ϕ is already $\sim_k$-invariant.
- Conclude that ϕ is equivalent to a (finite) modal formula of depth k.

Recall: Gaifman's Theorem

Gaifman graph of a FO structure:

$$x \text{——} y \quad \text{iff} \quad x \text{ and } y \text{ are in some basic relation}$$

$\rightsquigarrow$ Gaifman distance, Neighbourhoods $\mathsf{N}_d^{\mathfrak{M}}(u)$.

Definition: A formula $\phi(x)$ is Gaifman d-local if for $u, w \in \mathfrak{M}$,

$$\mathsf{N}_d^{\mathfrak{M}}(u) \cong \mathsf{N}_d^{\mathfrak{M}}(w) \implies (\mathfrak{M}, u \models \phi(x) \iff \mathfrak{M}, w \models \phi(x))$$

Gaifman's theorem: Every $\phi(x) \in FOL$ is Gaifman local.

Gaifman distance in CPL

Wrong idea: "x —— y if $x\heartsuit\lceil y:\phi\rceil$ and $\phi(z)$"

E.g. in probabilistic logic

$$xL_1\lceil y:\top\rceil \qquad \text{and} \quad \top(z),$$

so

$$x \text{ —— } z$$

for all x, z.

Solution: Support

- $A \subseteq X$ is a support of $t \in TX$ iff $t \in TA \subseteq TX$.
- Then by naturality of predicate liftings,

$$t \in [\![\heartsuit]\!]_X [\![\phi]\!] \quad \text{iff} \quad t \in [\![\heartsuit]\!]_A ([\![\phi]\!] \cap A)$$

- Supporting Kripke frame R for $\xi : X \to TX$:

$$R(x) = \{y \mid xRy\} \quad \text{support of } \xi(x)$$

Gaifman Locality for Support CPL

- Pure support CPL = Pure CPL plus binary predicate supp
  interpreted by supporting Kripke frame
- Inherit Gaifman theorem by translating into multisorted FO language

$$\heartsuit \subseteq s \times n$$
$$\in \subseteq s \times n$$
$$\mathrm{supp} \subseteq s \times s.$$

Neighbourhood compatibility:
Isomorphic nbhds (nearly) remain isomorphic

Theorem (Gaifman theorem for pure support-CPL):
Pure support-CPL is Gaifman local

The Coalgebraic van Benthem/Rosen Theorem

Infinitary version:
Λ separating, $\phi(x) \in FOL(\Lambda)$ $\approx$-invariant (over finite models) $\implies$
$\phi(x)$ equivalent (over finite models) to some
infinitary finite-rank modal formula $\psi(x)$.

Finitary version:
Same with $\psi(x)$ finitary for finite Λ.

- The finitary version is immediate from the infinitary version.

  Does the finitary van Benthem/Rosen theorem hold for infinite Λ?

Known Instances

- The classical van Benthem/Rosen theorem
- The van Benthem theorem for neighbourhood logic (Hansen/Kupke/Pacuit 2009)

Conclusion

- Coalgebraic predicate logic: FOL over T-coalgebras.
- Have proved a coalgebraic van Benthem/Rosen theorem.
- Nagging open problem: for infinite signatures, want to improve to finitary formulas.
- Key ingredient: Gaifman locality for CPL
  - Measure distance via support
  - Inherit from standard FOL by making neighbourhoods explicit

Future Work

- Investigation of CPL:
  - Model theory
  - Decidable fragments
- Sahlqvist theory (working from Dahlqvist/Pattinson 2013)

The Classical Correspondence Language

- One unary predicate $p(x)$ for each propositional variable p
- Binary relation $R(x, y)$
- No axioms or restrictions on models
- Standard translation:

$$ST_x(p) = p(x)$$
$$ST_x(\Box\phi) = \forall y. R(x, y) \to ST_y(\phi).$$

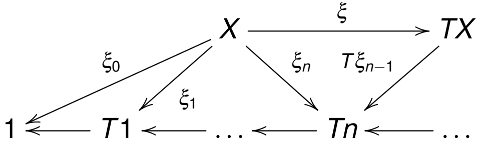
- Van Benthem/Rosen: for all $\phi(x) \in FOL$, TFAE:
  1. $\phi(x)$ bisimulation-invariant (over finite structures)
  2. $\phi(x) \leftrightarrow ST_x(\psi)$ for some modal ψ (over finite structures)
- Janin/Walukiewicz:
  the bisimulation-invariant fragment of MSOL is the μ-calculus.

Coalgebraic Unravelling

Recall: Coalgebraic modal logic captures behavioural equivalence

- defined via cospans of morphisms $X \to \bullet \leftarrow Y$
- in general weaker than bisimilarity (via spans $X \leftarrow \bullet \to Y$).

Require bounded behavioural equivalence $\approx_k$, defined via the terminal sequence



Key facts:

Lemma: For A, B trees of depth k, $A, a \approx B, b$ iff $A, a \approx_k B, b$.

Unravelling Lemma: For A, a ex. $A, a \approx B, b$ s.t. $N^B_{3k}(b)$ tree of depth k.