Algebraic and Kripke Semantics for Many-Valued Probabilistic Logics

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Outline

The language of Łukasiewicz logic consists in a set $V = \{p_1, p_2, \ldots\}$ of propositional variables, the binary connective $\to$, and the truth-constant $\bot$ (for falsity). Further connectives are defined as follows:

$$
\begin{array}{lll}
\neg\varphi & \text{is} & \varphi \to \bot \\
\varphi \& \psi & \text{is} & \neg(\varphi \to \neg\psi) \\
\varphi \leftrightarrow \psi & \text{is} & (\varphi \to \psi)\&(\psi \to \varphi) \\
\varphi \oplus \psi & \text{is} & \neg\varphi \to \psi \\
\varphi \ominus \psi & \text{is} & \neg(\varphi \to \psi) \\
\varphi \wedge \psi & \text{is} & \varphi\&(\varphi \to \psi) \\
\varphi \vee \psi & \text{is} & (\varphi \to \psi) \to \psi
\end{array}
$$

Axioms of Ł:

(Ł1) $\varphi \to (\psi \to \varphi)$,
(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,
(Ł4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

The only inference rule is *Modus Ponens*: from φ and $\varphi \to \psi$, derive ψ.

An MV-algebra is a system $A = (A, \oplus, \neg, 0, 1)$ satisfying the following conditions:

- $(A, \oplus, 0)$ is a commutative monoid,
- $\neg(\neg x) = x$ for all $x \in A$,
- $x \oplus 1 = 1$ for all $x \in A$,
- $\neg(x \oplus \neg y) \oplus x = \neg(y \oplus \neg x) \oplus y$ for all $x, y \in A$.

The class of MV-algebras forms a variety denoted by $\mathbb{MV}$.
In any MV-algebra one can define further operations as follows:

$$x \to y = (\neg x \oplus y),\ x \ominus y = \neg(x \to y),\ x \odot y = \neg(\neg x \oplus \neg y),$$
$$x \leftrightarrow y = (x \to y) \odot (y \to x),\ x \vee y = (x \to y) \to y, \text{ and}$$
$$x \wedge y = \neg(\neg x \vee y).$$

Any MV-algebra A can be equipped with a partial order relation: for all $x, y \in A$,

$$x \leq y \text{ iff } x \to y = 1.$$

An MV-algebra is said to be an MV-chain if $\leq$ is linear.

An MV-algebra is semisimple if it is isomorphic to an MV-algebra of $[0,1]$-valued functions on a compact Hausdorff space X.

An MV-algebra is simple if it is isomorphic to an MV-subalgebra of the standard MV-chain:

$$[0,1]_{MV} = ([0,1], \oplus, \neg, 0, 1)$$

where: $\forall x, y \in [0,1]$, $x \oplus y = \min\{1, x+y\}$, $\neg x = 1 - x$.

(Notice: $[0,1]_{MV}$ is generic for $\mathbb{MV}$).

States on MV-algebras

A state on an MV-algebra A is a map

$$s : A \to [0,1]$$

Satisfying:

- $s(1) = 1$,
- For all $x, y \in A$ s.t. $x \odot y = 0$, $s(x \oplus y) = s(x) + s(y)$.

A state s is said faithful if $s(x) = 0$, implies $x = 0$.

For every MV-algebra A and every state s, there exists a unique Borel regular probability measure μ on the space of MV-homomorphisms H of A in $[0,1]_{MV}$ such that, for every $a \in A$, $s(a) = \int_H f_a \, \mathrm{d}\mu$.

In other words states represent the expected values of the elements of an MV-algebra, which are regarded as (bounded) random variables.

The modal logic $FP(Ł, Ł)$

The language of $FP(Ł, Ł)$ is obtained by adding a unary modality Pr in the language of Łukasiewicz logic. Formulas are defined by the stipulations:

- Every Łukasiewicz formula is a formula,
- For every Łukasiewicz formula φ, $\Pr(\varphi)$ is an atomic modal formula.
- (Atomic) modal formulas are closed under $\oplus$, $\odot$, $\rightarrow$, $\neg$.

Axioms for $FP(Ł, Ł)$ are:

- All the axioms of Łukasiewicz logic,
- $\Pr(\neg\varphi) \leftrightarrow \neg\Pr(\varphi)$,
- $\Pr(\varphi \rightarrow \psi) \rightarrow (\Pr(\varphi) \rightarrow \Pr(\psi))$,
- $\Pr(\varphi \oplus \psi) \leftrightarrow [(\Pr(\varphi) \rightarrow \Pr(\varphi \odot \psi)) \rightarrow \Pr(\psi)]$.

Rules are modus ponens, and the necessitation for Pr: $\frac{\varphi}{\Pr(\varphi)}$.

Probabilistic Kripke models

A Probabilistic Kripke Model for $FP(Ł, Ł)$ is a system $\mathcal{K} = (X, s)$ where:

- X is a non empty set of evaluations of Łukasiewicz formulas into $[0, 1]$.
- $s : [0, 1]^X \to [0, 1]$ is a state of $[0, 1]^X$.

If ϕ is a formula of $FP(Ł, Ł)$, if K is a Kripke model, and $x \in X$, the truth-values of Φ in K at x is defined as:

- If Φ is a Łukasiewicz formula, then $\|\Phi\|_{K,x} = x(\Phi)$,
- If Φ is $\Pr(\psi)$ and ψ is Łukasiewicz. Then $\|\Pr(\psi)\|_{K,x} = s(f_\psi)$, where $f_\psi : x \in X \mapsto x(\psi) \in [0, 1]$.
- If Φ is compound, then use truth functions of Łukasiewicz connectives.

A probabilistic Kripke model $({}^*X, {}^*s)$ is a hyperreal-valued probabilistic Kripke model, if each evaluation $x \in X$ and the map ${}^*s$ ranges on a non-trivial ultrapower ${}^*[0, 1]$ of the real unit interval.

Hyperreal-completeness for $FP(Ł, Ł)$

The logic $FP(Ł, Ł)$ is (strongly) complete with respect to the class of hyperreal-valued probabilistic Kripke model.

The logic SFP(Ł,Ł)

The language of $SFP(Ł, Ł)$ is that of $FP(Ł, Ł)$. Formulas are defined in the usual way dropping the restriction on the modality Pr.
Axioms for $SFP(Ł, Ł)$ are:

- All the axioms of Łukasiewicz logic,
- $\mathrm{Pr}(\bot) \leftrightarrow \bot$,
- $\mathrm{Pr}(\neg\varphi) \leftrightarrow \neg\,\mathrm{Pr}(\varphi)$,
- $\mathrm{Pr}(\mathrm{Pr}(\varphi) \oplus \mathrm{Pr}(\psi)) \leftrightarrow (\mathrm{Pr}(\varphi) \oplus \mathrm{Pr}(\psi))$,
- $\mathrm{Pr}(\varphi \oplus \psi) \leftrightarrow \mathrm{Pr}(\varphi) \oplus \mathrm{Pr}(\psi \ominus (\varphi \& \psi))$.

Rules are modus ponens, and the necessitation for Pr: $\frac{\varphi}{\mathrm{Pr}(\varphi)}$.

Semantics for $SFP(Ł, Ł)$

There are two main semantics for $SFP(Ł, Ł)$:

Probabilistic Kripke models and SMV-algebras.

Probabilistic Kripke models

A Probabilistic Kripke Model for $SFP(Ł, Ł)$ is a system $\mathcal{K} = (X, s)$ where:

- X is a non empty set of evaluations of Łukasiewicz formulas into $[0, 1]$.
- $s : [0, 1]^X \to [0, 1]$ is a state.

If ϕ is a formula of $SFP(Ł, Ł)$, if K is a Kripke model, and $x \in X$, the truth-values of Φ in K at x is defined as in the case of $FP(Ł, Ł)$.

$KR1SAT$ denotes the set of all $SFP(Ł, Ł)$-1-satisfiable formulas.
$KR1TAUT$ denotes the set of $SFP(Ł, Ł)$-tautologies.

SMV-ALGEBRAS

An SMV-algebras is an algebra

$$\mathcal{A} = (A, \oplus, \neg, \sigma, 0, 1)$$

where:

- $(A, \oplus, \neg, 0, 1)$ is an MV-algebra,
- $\sigma : A \to A$ satisfies the following:
  - $\sigma(0) = 0$,
  - $\sigma(\neg x) = \neg\sigma(x)$,
  - $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$,
  - $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$.

An SMV-algebra is said faithful if $\sigma(x) = 0$ implies $x = 0$.

SMV1SAT denotes the set of SFP(Ł, Ł)-1-satisfiable formulas in SMV-algebras.
SMV1TAUT denotes the set of SFP(Ł, Ł)-tautologies in SMV-algebras.

An example

Let X he a non-empty Hausdorff space and let $A = \mathscr{C}(X)$ be the MV-algebra of continuous functions from X to $[0,1]$.
Let $\mu : \mathscr{B}(X) \to [0,1]$ be a regular Borel probability measure on the Borel subsets of X.
Define $\sigma : A \to A$ in the following manner: for every $f \in A$,

$$\sigma(f) = \int_X f \mathrm{d}\mu.$$

(where we identify every real number $\alpha \in [0,1]$ with the function in $\mathscr{C}(X)$ constantly equal to α).

Then (A, σ) is an SMV-algebra. Moreover (A, σ) is simple even though is not linearly ordered.

On the variety of SMV-algebras

Unlike the case of MV-algebras, the variety $\mathbb{SMV}$ is NOT generated by its linearly ordered members. For instance

$$\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$$

holds in every SMV-chain, but not in every SMV-algebra.

Theorem

The class of SMV-algebra is generated as a quasivariety, by its members (A, σ) such that $\sigma(A)$ is an MV-chain.

Standard SMV-algebras

We already noticed that $\mathbb{SMV}$ is not generated by SMV-chains. The following definition introduces a candidate for *standard* SMV-algebras.

Definition

An SMV-algebra (A, σ) is said to be σ-simple if A is semisimple (i.e. an algebra of continuous $[0, 1]$-valued functions), and $\sigma(A)$ is a simple algebra (i.e. an MV-subalgebra of $[0, 1]_{MV}$).

$ST1SAT$ denotes the set of $SFP(\text{Ł}, \text{Ł})$-1-satisfiable formulas in σ-simple SMV-algebras.
$ST1TAUT$ denotes the set of $SFP(\text{Ł}, \text{Ł})$-tautologies in σ-simple SMV-algebras.

Tensor SMV-algebras

An interesting subclass of SMV-algebras can be built from an MV-algebra and an (external) state s of A in the following manner:

Let A be an MV-algebra, and let $s : A \to [0, 1]$ be a state.

Let $\mathcal{T}$ be the MV-algebra defined as $[0, 1]_{MV} \otimes A$,

Let $\sigma_s : \mathcal{T} \to \mathcal{T}$ be the internal state defined by: for all $\alpha \otimes a \in \mathcal{T}$,

$$\sigma_s(\alpha \otimes a) = \alpha \cdot s(a) \otimes 1.$$

Any SMV-algebra of this kind is called tensor SMV-algebra.

Tensor1SAT denotes the set of 1- satisfiable $SFP(Ł, Ł)$-formulas in tensor SMV-algebras
Tensor1TAUT denotes the set of all $SFP(Ł, Ł)$-tautologies in tensor SMV-algebras.

1-Satisfiability

Let ϕ be a formula in SFP. The following are equivalent

1. $\phi \in ST1SAT$,
2. $\phi \in KR1SAT$,
3. $\phi \in SMV1SAT$.

1-Tautologies

Let ϕ be a formula in SFP. The following are equivalent

1. $\phi \in ST1TAUT$,
2. $\phi \in Tensor1TAUT$,
3. $\phi \in KR1TAUT$.

Open problems (1)

We know that the logic $SFP(Ł, Ł)$ is complete w.r.t. the class of SMV-algebra. In particular, the following holds

Theorem

Let ϕ be a formula of SFP, then

1. *ϕ is a theorem of $SFP(Ł, Ł)$,*
2. *$\phi \in SMV1TAUT$,*
3. *ϕ is a tautology for those SMV-algebras (A, σ) such that $\sigma(A)$ is an MV-chain.*

Q1: Is $SFP(Ł, Ł)$ complete w.r.t. σ-simple SMV-algebras? In other words, is $SMV1TAUT = ST1TAUT$?
equivalently, is $SMV1TAUT = KR1TAUT$?

Open problems (2)

We already noticed that $SFP(\text{Ł}, \text{Ł})$ extends $FP(\text{Ł}, \text{Ł})$. The latter is known to be complete w.r.t. hyperreal-valued probabilistic Kripke models.

Q2: Is $SFP(\text{Ł}, \text{Ł})$ a conservative extension of $FP(\text{Ł}, \text{Ł})$? In other words, if ϕ is an FP-formula and $\langle {}^*X, {}^*s \rangle$ is a hyperreal-valued Kripke model such that $\langle {}^*X, {}^*s \rangle \not\models \phi$, can we define an SMV-algebra (A, σ) such that $(A, \sigma) \not\models \phi$?

Remark (join work with Lluis Godo): If **Q2** is true, then we (*should*) have a proof for the standard completeness of $SFP(\text{Ł}, \text{Ł})$.
(SMV1TAUT=ST1TAUT=KR1TAUT).

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THANK YOU FOR YOUR ATTENTION!